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Summer term 2016 | L. Nickel, C. Velasco

Exercises on Mathematical Statistical Physics
Math Sheet 2

Problem 1 (Warm-up on probability theory). Let in the following $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

a) Let $T : \Omega \to \Omega$ be a map. Prove that the set $\mathcal{T} := \{ A \subset \Omega | \, T^{-1}(A) = A \}$ is a σ-algebra over Ω. (*Remark: This result will be used in ergodic theory.*) Prove that $\{ A \subset \Omega | \, T(A) = A \}$ is not in general a σ-algebra.

b) Let $\Theta \subset \Omega$. Construct a natural σ-algebra of Θ by using the structure provided by $\mathcal{A}$.

c) Let $A \in \mathcal{A}$. Show that the conditional probability $\mathbb{P}_A$ given as $\mathbb{P}_A(B) := \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$ defines a probability measure on $(\Omega, \mathcal{A})$.

Problem 2 (Gronwall's lemma). Gronwall's lemma is a very useful result from analysis which we will use in several derivations of effective equations. Prove the following versions:

a) Let f and c be real-valued functions defined on $[0, \infty)$ and let f be differentiable on $(0, \infty)$. If f satisfies $\frac{d}{dt} f(t) \leq c(t) f(t)$, then

$$f(t) \leq \exp\left(\int_0^t c(s) ds \right) f(0).$$

b) If f as above satisfies, with positive constants $c_1, c_2 \in \mathbb{R}$, $\frac{d}{dt} f(t) \leq c_1 f(t) + c_2$, then

$$f(t) \leq e^{c_1 t} f(0) + \left(e^{c_1 t} - 1 \right) \frac{c_2}{c_1}.$$

Problem 3 (Liouville's theorem). We are given a classical system of N particles with phase space coordinates $(q, p) = (q_1, ..., q_N, p_1, ..., p_N) \in \mathbb{R}^{6N}$, Hamiltonian function $\mathcal{H}(q, p)$ and equations of motion

$$\dot{q}(t) = \partial_p \mathcal{H}(q(t), p(t)), \quad \dot{p}(t) = -\partial_q \mathcal{H}(q(t), p(t)),$$

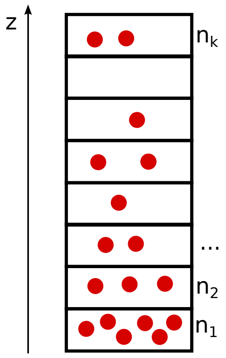
with initial values $(q(0), p(0)) = (q_0, p_0)$. This defines for $t \in \mathbb{R}$ the Hamiltonian flow $\Phi_t : \mathbb{R}^{6N} \to \mathbb{R}^{6N}, (q_0, p_0) \mapsto (q(t), p(t))$. The usual Lebesgue measure on phase

space is denoted by λ. Prove Liouville's theorem:
The phase space volume is preserved under the Hamiltonian flow, i.e.

$$\lambda \circ \Phi_t = \lambda \quad \forall t \in \mathbb{R}.$$

Hint: Use a Jacobian. (But there are several ways to prove this). You might need that matrices satisfy $\det(M) = e^{\operatorname{tr}\ln(M)}$ *and might need to take the time derivative of the Jacobian.*

Problem 4 (Equilibrium configurations by combinatorics).



In this exercise, we derive the typical configuration of a huge number N of particles that can be in different "states" $j = 1, ..., k$ such that a particle in state j has energy e_j. You might for example think of N air molecules in our atmosphere and the k boxes belonging to different heights above the ground, as depicted on the left. The number of particles in box j is called n_j. We will see that the correct (i.e. typical) distribution of the particles to the boxes can be calculated simply by using combinatorics.

a) Derive the distribution for which the number of possibilities is maximal under the constraints
$$\sum_{j=1}^{k} e_j n_j = E, \quad \sum_{j=1}^{k} n_j = N.$$
You may use that N and the n_j are very large, so Stirling's formula $\ln N! \approx N \ln N$ is applicable.

b) What does your formula imply for the concrete situation of the atmosphere, where $e_j \propto j$?

Problem 5 (Large numbers). Assume we have $0.25 \cdot 10^{23}$ particles and 2 boxes (e.g. the right and the left half of a container of air) such that the probability for each particle to be in the right or in the left box is (independently of each other), $\frac{1}{2}$. Give an upper bound for the probability that the fraction of particles that are in the left box is $\geq \frac{1+10^{-8}}{2}$ and that it is larger than $\frac{3}{4}$. Find a good bound that shows that the probability in the latter case is at most 10^{-1000}.
Would you buy an insurance against spontaneous suffocation?

The solutions to these exercises will be discussed on Friday, 29.04., and Monday, 02.05.