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Summer term 2016 — L. Nickel, C. Velasco

Exercises on Mathematical Statistical Physics
Math Sheet 1

Problem 1 (Probability and random variables). We make an experiment where secants of the unit circle $S^1 \subset \mathbb{R}^2$ are randomly chosen in the following ways:

a) Choose two (different) points on the circle at random (with uniform distribution), connect them with a straight line.

b) Choose one point $p \neq (0,0)$ inside the circle at random (with uniform distribution) and draw the secant which goes through p and is orthogonal to the line from $(0,0)$ to p.

Define suitable probability spaces $(\Omega_a, \mathcal{A}_a, \mathbb{P}_a)$ and $(\Omega_b, \mathcal{A}_b, \mathbb{P}_b)$ for both cases! Define random variables $X_j : \Omega_j \to \mathbb{R}$ $(j = a, b)$ that assign to a choice of points the length of the respective secant. Calculate the probability distribution of X_a and X_b and the respective expectations values!

Problem 2 (Sigma algebras). Let $\mathcal{A}_1, \mathcal{A}_2$ be σ-algebras over the same set of samples Ω.

a) Prove that the intersection $\mathcal{A} := \mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra over Ω.

b) Give an example where the union $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

Problem 3 (Borel set). In this exercise, $\mathcal{B}(\Omega)$ denotes the Borel set of Ω, and $\mathcal{B}$ without brackets denotes the Borel set of $\mathbb{R}$.

a) Let $\chi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\chi^{-1}(\mathcal{B}) \subset \mathcal{B}.$$

b) In this exercise, we consider the example $\Omega = [0, 1[$. Show that there exists a unique translation-invariant probability measure $\mathbb{P}$ on $\mathcal{B}(\Omega)$.

c) Think about how the result of b) implies that the Vitali set V constructed in the lecture is not an element of the Borel set.

The solutions to these exercises will be discussed on Friday, 22.04., and Monday, 25.04.