LMU Munich | Prof. Dr. B. Paredes, Prof. Dr. P. Pickl
Summer term 2016 | L. Nickel, C. Velasco

Exercises on Mathematical Statistical Physics
Physics Sheet 3

Problem 1 (Gauge transformations). The quantum Hamiltonian for a particle in the presence of the vector potential $\vec{A}$ is given by:

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2.$$

(a) Consider the two vector potentials

$$\vec{A}_L = xB\hat{y} \tag{1}$$
$$\vec{A}_S = \frac{B}{2}(x\hat{y} - y\hat{x}). \tag{2}$$

Show that they both correspond to the same magnetic field, $\vec{B} = B\hat{z}$. Find the gauge transformation between the corresponding Hamiltonians.

(b) The physics of a particle in a uniform magnetic field is invariant under translations. Is the Hamiltonian invariant under translations for the two vector potentials above? Try to solve this apparent contradiction by showing that the change in the Hamiltonian that occurs under translation is equivalent to a gauge change. Prove this for an arbitrary vector potential that corresponds to a uniform magnetic field.

Problem 2 (Landau gauge). The so called Landau gauge for a particle in two dimensions in a perpendicular uniform magnetic field corresponds to a choice of the vector potential $\vec{A} = xB\hat{y}$.

(a) Show that the eigenstates of the Hamiltonian in the Landau gauge have the form:

$$\psi_{nk}(x,y) = \frac{1}{\sqrt{L_y}} e^{iky}\varphi_n(x + k\ell^2),$$

where the φ_n are the eigenfunctions of a one-dimensional oscillator with frequency ω_c, $\hbar k$ is the momentum in the y direction, which is a conserved quantity, and L_y is the length of the system in the y direction.

(b) Show that the corresponding eigenenergies are

$$E_{nk} = \hbar\omega_c(n + \frac{1}{2}).$$

Note that the energy is independent of the quantum number k. Observe that the eigenenergies, the Landau levels, are the same as the ones we obtained in class, without needing to specify the gauge.

(c) Show that the degeneracy of each Landau level is equal to the number of flux quanta penetrating the sample, as we had obtained in class for the symmetric gauge. (Hint: suppose that the sample is rectangular with dimensions L_x and L_y. Assume periodic boundary conditions in the y direction and count how many momentum eigenstates fit within a length L_x.)

(d) What is the smallest size wave packet that can be constructed using Landau gauge eigenstates of the first Landau level?

Problem 3 (Symmetric gauge). The symmetric gauge for a particle in two dimensions in a perpendicular uniform magnetic field corresponds to a choice of the vector potential $\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x})$.

(a) Write down the expression for the Hamiltonian in the symmetric gauge. Show that the Hamiltonian can be written as

$$H = H_2 + \omega L,$$

where $H_2 = \hbar\omega(1 + a_x^\dagger a_x + a_y^\dagger a_y)$ is the Hamiltonian corresponding to a two dimensional harmonic oscillator with frequency $\omega = \omega_c/2$ and L is the angular momentum operator.

(b) What is the relation between the oscillator operators a_x, a_y and the ladder operators a and b that we defined in class?

(c) Consider the Hamiltonian above with an arbitrary ω. What are the eigenstates and eigenvalues in this case? Can you relate this Hamiltonian to the one of a particle in a rotating two dimensional harmonic potential? What is the rotation frequency?

Problem 4 (Lowest Landau level). In the symmetric gauge, the eigenstates of the lowest Landau level have the form

$$\psi_m(z, \bar{z}) = \alpha_m z^m e^{z\bar{z}/4\ell^2},$$

where α_m is a normalization factor and $z = x + iy$ is the complex coordinate in the plane.

(a) Calculate the normalization factor α_m.

(b) Calculate the expectation values $\langle m|z\bar{z}|m\rangle$ and $\langle m|z^2\bar{z}^2|m\rangle$. Conclude that for large m the function ψ_m is strongly localized within a distance ℓ in a ring of radius $r_m = \ell\sqrt{m+1}$.

(c) Consider a wave function completely localized at position z_0, $\psi(z) = \alpha\delta(z - z_0)$. What is the normalization factor α? What is the projection of this wave function onto the lowest Landau level? Conclude that the smallest wave packet that can be constructed within the lowest Landau level has size $\sim 2\pi\ell^2$. Compare this result with the one that you obtained using the Landau gauge and conclude that it is independent of the choice of gauge.

(d) What is the minimum uncertainty that we can have when determining the momentum of an electron confined to the lowest Landau level? (Hint: Find the smallest wave packet in momentum space that can be constructed with states within the lowest Landau level).

(e) Find the Fourier transform of the wave function ψ_m. What is the reason behind the result you obtain? On the light of this result interpret your answer to d).

(f) Calculate the electric current carried by an electron in the state ψ_m of the lowest Landau level. Remember that the velocity is proportional to the mechanical momentum and not to the canonical momentum.

(g) Compare the states ψ_m with the states of the lowest Landau level that you obtained in Problem 2 using the Landau gauge. Are they related by a gauge transformation?

The solutions to these exercises will be discussed on Friday, 24th June, and Monday, 11th July.