LMU Munich
Summer term 2016

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Exercises on Mathematical Statistical Physics
Physics Sheet 2

Comment: We encourage you to read the article given as a reference at the end of this exercices sheet, which presents experimental schemes for the realization of the toric code model and for the detection and manipulation of anyons. Exercices 5 and 6 contain some simple questions related to that article.

Problem 1 (Spectrum of the toric code model). Consider the toric code model. Every eigenstate of the system must be also an eigenstate of every plaquette and vertex operator, since all of them commute with the Hamiltonian. So we can identify each eigenstate with a set of eigenvalues

$$\left\{\lambda_1, \lambda_2, ..., \lambda_{N_p}; \xi_1, \xi_2, ..., \xi_{N_v}\right\}, \quad \lambda_a,\, \xi_b \in \{1, -1\} \tag{1}$$

being N_p and N_v the number of plaquettes and vertices, respectively.

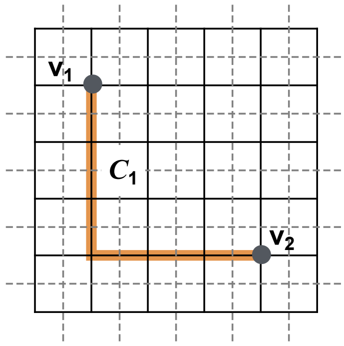
(a) Consider the system on a square planar lattice with the edges not identified. Is there an eigenstate of the system for any configuration of $\lambda's$ and $\xi's$? How many different configurations are allowed? Compare the result with the dimension of the Hilbert space of the system, what can you conclude?

(b) Answer the previous questions for the case in which the system is on a torus. Is there any difference?

(c) Can you use the association between eigenstates and the set of values (1) to construct the spectrum of the Hamiltonian? Which are the eigenenergies of the system? Are the eigenstates degenerate?

Problem 2 (Vertex excitations in the toric code). Consider the toric code model on the surface of a torus. We define the folowing string operators, along the strings $\mathcal{C}_1$ and $\mathcal{C}_2$ (see figure 1), as

$$S^x_{\mathcal{C}_1} = \prod_{j\in\mathcal{C}_1} \sigma^x_j, \quad S^x_{\mathcal{C}_2} = \prod_{j\in\mathcal{C}_2} \sigma^x_j, \tag{2}$$



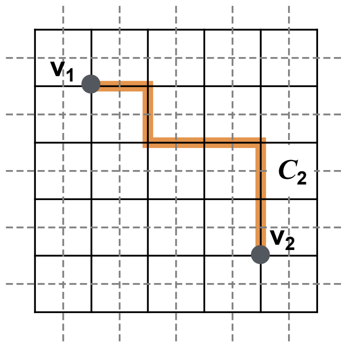


Figure 1: Strings $\mathcal{C}_1$ and $\mathcal{C}_2$ from vertex v_1 to v_2.

(a) Consider the resulting states of applying each of those string operators to a ground state of the system. Are these new states eigenstates of the Hamiltonian? If so, which are their energies? Are they the same state?

(b) If we consider each string operator $S^x_{\mathcal{C}}$ for all possible strings connecting two arbitrary vertices v_1 and v_2, do they generate the same state when applied to a ground state of the system?

(c) Consider the eigensubspace of having two vertex excitations (i.e. e-type quasiparticles) at some particular vertices v_1 and v_2. Which is the dimension of that subspace? How can the different eigenstates in that subspace be generated from the ground state?

Problem 3 (Plaquette excitations in the toric code). We consider now another kind of string operators, along strings in the dual lattice (see figure 2),

$$S^z_{\mathcal{C}'_1} = \prod_{j \in \mathcal{C}'_1} \sigma^z_j, \quad S^z_{\mathcal{C}'_2} = \prod_{j \in \mathcal{C}'_2} \sigma^z_j, \tag{3}$$

(a) Consider the resulting states of applying each of those string operators to a ground state of the system. Are these new states eigenstates of the Hamiltonian? If so, which are their energies? Are they the same state?

(b) If we consider each string operator $S^z_{\mathcal{C}'}$ for all possible strings in the dual lattice connecting two arbitrary plaquettes p_1 and p_2, do they generate the same state when applied to a ground state of the system?

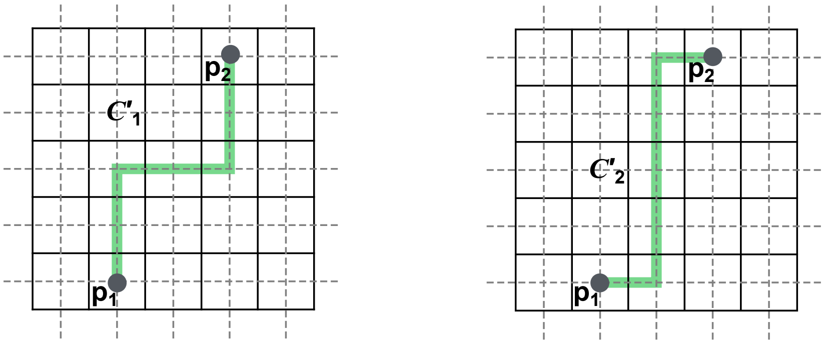


Figure 2: Strings $\mathcal{C}'_1$ and $\mathcal{C}'_2$ in the dual lattice from plaquette p_1 to p_2.

(c) Consider the eigensubspace of having two plaquette excitations (i.e. m-type quasiparticles) at some particular plaquettes p_1 and p_2. Which is the dimension of that subspace? How can the different eigenstates in that subspace be generated from the ground state?

Problem 4 (Quasiparticles and topology). The ground state of the toric code model has a degeneracy that reflects the topology of the surface where the system is on. When the system is on a torus there are four different ground states.

On the other hand, quasiparticles can be created on top of ground states, they can be moved and also destroyed. What quasiparticles must be created and what must be done before destroying them in order to go from one ground state to another?

Problem 5 (String operator). We have a set of spins in an optical lattice confined to a plane so that they realize the toric code model. The whole system is in an optical cavity whose mode can interact with a set of selected spins of the lattice.

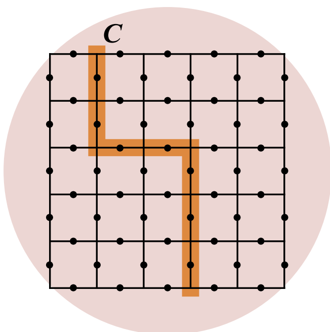


Figure 3: **Classification**

The Hamiltoian that drives this interaction is:

$$H = \chi a^\dagger a \sum_{j\in\mathcal{C}} \sigma_j^z, \tag{4}$$

being χ a certain parameter, $a^\dagger(a)$ the creation(annihilation) operator of a photon in the cavity mode and $\mathcal{C}$ a certain string in the lattice (for example, see figure 3). For a number of photons in the optical cavity n_a, which is the time evolution operator at a certain time t? Can you generate the following string operator

$$S_{\mathcal{C}}^z = \prod_{j\in\mathcal{C}} \sigma_j^z, \tag{5}$$

just by letting the system evolve in time? For which time and which number of photons in the cavity?

Problem 6 (Quantum memory). Suppose that we have a toric code model system whose eigensubspace of ground states has dimension 2. We can define two orthogonal ground states $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ which are locally indistinguishable but can be characterized through the global string operator Z,

$$Z\,|\tilde{0}\rangle = |\tilde{0}\rangle \tag{6}$$

$$Z\,|\tilde{1}\rangle = -\,|\tilde{1}\rangle\,. \tag{7}$$

There is also another global string operator X that changes from one state to the other, that is

$$X\,|\tilde{0}\rangle = |\tilde{1}\rangle \tag{8}$$

$$X\,|\tilde{1}\rangle = |\tilde{0}\rangle\,. \tag{9}$$

The elementary unit of information at the quantum level is a qubit, which is a two level system (for example, a 1/2-spin particle) that can be in any superposition of both logical states $|0\rangle$ and $|1\rangle$. If we store the information of a qubit in one single particle it can be easily destroyed by the interaction with the enviroment. However, we can use the subspace of ground states of the torus as an effective qubit, which is protected from local perturbations. To do that, we consider a probe spin which is in a state $|\phi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and the toric code system, which is our quantum memory, initialized in the ground state $|\tilde{0}\rangle$. We want to transfer the information from the probe spin to the quantum memory, that is performing the operation

$$|\phi\rangle_A \otimes |\tilde{0}\rangle_M \xrightarrow{\mathrm{SWAP_{in}}} |0\rangle_A \otimes |\phi\rangle_M \tag{10}$$

(a) Find a way of obtaining the operator $\mathrm{SWAP_{in}}$ using controlled string operators of the form

$$\Lambda[S] = |0\rangle_A\,\langle 0|_A \otimes \mathbb{I} + |1\rangle_A\,\langle 1|_A \otimes S, \tag{11}$$

being S any string operator, and combining them with local unitary operators on the probe qubit.

(b) With the same ingredients, find also a way of obtaining the operator $\mathrm{SWAP}_{\mathrm{out}}$, defined as

$$|0\rangle_A \otimes |\phi\rangle_M \xrightarrow{\mathrm{SWAP}_{\mathrm{out}}} |\phi\rangle_A \otimes |\tilde{0}\rangle_M \tag{12}$$

References: L. Jiang, G. K. Brennen, A. V. Gorshkov, K. Hammerer, M. Hafezi, E. Demler, M. D. Lukin, and P. Zoller Anyonic interferometry and protected memories in atomic spin lattices. *Nature Physics* **4**, 482 - 488 (2008) (arXiv:quant-ph/0711.1365v1)

The solutions to these exercises will be discussed on Friday, 17th June, and Monday, 20th June.