LMU Munich
Summer term 2016

Prof. Dr. B. Paredes, Prof. Dr. P. Pickl
L. Nickel, C. Velasco

Exercises on Mathematical Statistical Physics
Physics Sheet 1

Comment: This exercices correspond to the toric code model explained during the lectures on the 19th and 20th May. We also really recommend you reading the article given as a reference at the end of this sheet (at least the first three sections).

Problem 1 (The toric code in the dual lattice). Let us consider the toric code model in which we have a set of spins located at each edge of a square lattice, as shown in figure 1. The Hamiltonian of the system is

$$H = -\sum_v A_v - \sum_p B_p \tag{1}$$

running v and p over all vertex and plaquettes of the lattice, respectively, and being A_v and B_p the vertex and plaquette operators defined as follows

$$A_v = \prod_{j\in v} \sigma_j^z \tag{2}$$

$$B_p = \prod_{j\in \partial p} \sigma_j^x \tag{3}$$

with σ_j^α the Pauli operator of the j-th spin along the α direction.

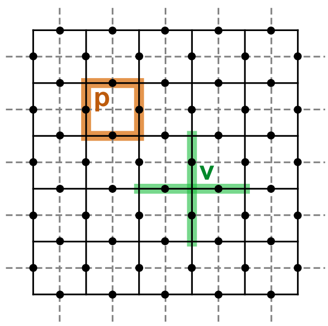

Figure 1: Toric code model.

Every state of the computational basis, that is, a product state in which every spin is in an eigenstate of σ^z, can be associated to a graph in the lattice. A spin in the state $|0\rangle$ corresponds to no line and a spin in the state $|1\rangle$ is associated with an orange line in the face of the lattice where it is located. Thus, any state of the system can be expressed as a linear combination of graphs and, as shown during the lectures, this representation is really convenient in order to find the ground states of the system. Figure 2**a**.

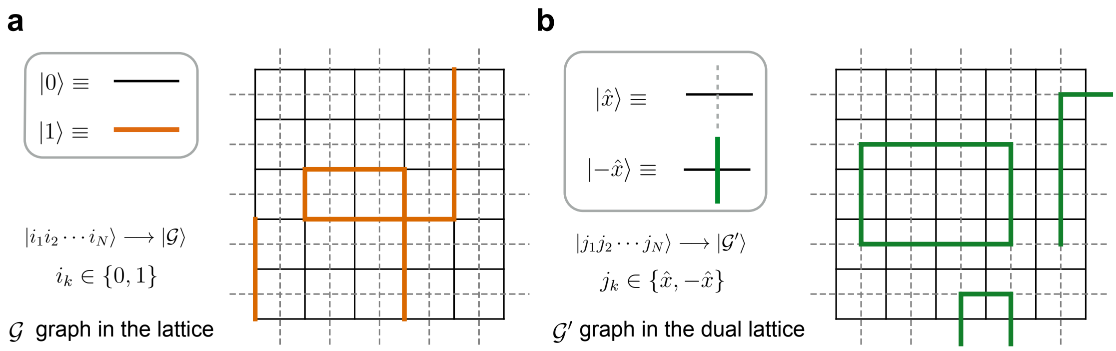

Figure 2: Graphs representations.

However, this is not the only way of solving such a problem. We can work in the basis of eigenstates of σ^x and associate a green line in the dual lattice to every spin in state $|-\hat{x}\rangle$ and no line to those in state $|\hat{x}\rangle$. So we have one graph in the dual lattice for each element of that basis and we can write any state of the system as a combination of such graphs. Figure 2**b**.

(a) Consider one state of the system that, using the first representation, corresponds to a graph with no lines. Consider also another state that, using the dual lattice representacion, corresponds to a graph with no lines. Are these two states the same?

(b) Find the ground state $|\psi\rangle$ of the system for no boundary conditions (i.e. no edge is connected with any other) using the dual lattice representation. For that, see how do the terms in the Hamiltonian act on graphs in the dual lattice and then impose the vertex and plaquette conditions

$$A_v |\psi\rangle = |\psi\rangle \ \forall\, v \tag{4}$$

$$B_p |\psi\rangle = |\psi\rangle \ \forall\, p \tag{5}$$

Which kind of dual lattice graphs contribute to the ground state?

(c) Consider now the system on the surface of a torus. Will the ground state have some degeneracy? If so, what structure does the eigensubspace of ground states have? Find some global string operators that generate all degenerated ground states and find other global string operators that allow you to characterize them. Are these states different from those we have derived during the lectures?

Problem 2 (Closed loops configurations). Consider the graph representation for the computational basis in the toric code model (as in figure 2**a**). How many different contractible closed loops configurations are there for no boundary conditions? How many are there in the torus situation? Are there any differences? If so, draw some expamples of contractible closed loops graphs that have to be considered in one case but not in the other.

Problem 3 (Local order parameter). Consider the ground state of the toric code (we do not care about the boundary conditions here). Which is the expectated value of the j-th spin along the direction given by an arbitrary unit vector $\hat{n}$? That is,

$$\langle\psi|\, \hat{n}\cdot\vec{\sigma}\, |\psi\rangle\,. \tag{6}$$

Problem 4 (Kitaev Hamiltonian). Consider the following Hamiltonian, which was introduced by Kitaev (see reference),

$$H_K = -\sum_v A'_v - \sum_p B'_p \tag{7}$$

with

$$A'_v = \prod_{j\in v} \sigma_j^x \tag{8}$$

$$B'_p = \prod_{j\in\partial p} \sigma_j^z \tag{9}$$

Is it the same Hamiltonian as in 1, the one we studied in the lectures? Do they have the same ground states? If not, what are their differences?

Problem 5 (Planar lattice with holes). Consider the toric code model on a plane with no boundary conditions but with some holes in it, as in figure 3. How many degenerated ground states does it have? Which string operators are needed in order to generate all of them? How many degenerated ground states are there for an arbitrary number of holes h?

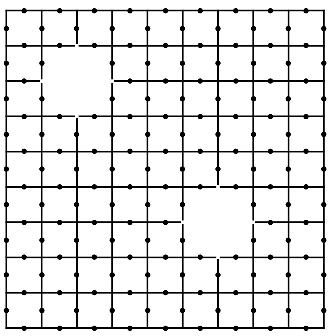

Figure 3: Toric code model with some holes in the lattice.

Problem 6 (Cylindrical lattice). Consider the toric code model on a surface with two edges connected so that it forms a cylinder, as in figure 4. How many degenerated ground states does it have? Which string operators are needed in order to generate all of them? And to characterize them?

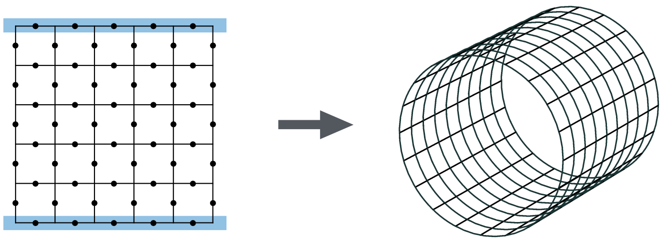

Figure 4: Toric code model on a cylinder.

References: Kitaev, A. Y. Fault-tolerant quantum computation by anyons. *Ann. Phys.* **303**, 2-30 (2003) (arXiv:quant-ph/9707021v1)

The solutions to these exercises will be discussed on Friday, 27th May, and Monday, 30th May.