Mathematical QM - Lecture 2

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Contents

Literature

The following discussion is largely based on (but brutally abbreviated here):
F. Strocchi, Mathematical Structure of Quantum Mechanics
W. Thirring, Mathematical Physics III: Quantum Mechanics
J.v. Neumann: Mathematische Grundlagen der Quantenmechanik

1 The C^*-algebra approach to classical and quantum physics

In this section I will expose the single and only relevant difference between classical and quantum mechanics: quantum mechanics has non-commutative observables. The rest is only mathematical apparatus. In particular, the Hilbert space can be generated using two ingredients: the assumption that observables are algebraic objects and the concept of a "state" of a system.

1.1 Algebraic operations with lab-experiments

The properties of systems such as position, momentum and energy are abstractions from actual experiments. We can perform algebraic operations on experiments by adding and multiplying the numbers we find in them.

In that sense experiments can be considered elements from a (C^*-) algebra. Very loosely, a C^*-algebra is an algebra that has a $*$-operation analogous to complex or hermitian conjugation and where every element has some finite norm. The norm has the standard properties of a norm. A precise definition will be given after the motivation.

1.1.1 Observables

The idea of an "observable" is that is represents an actual experiment. In that sense, a classical observable is a function on phase space that *can be realized in one specific experimental setup.* Its value $A(\vec{r},\vec{p})$ is the number extracted from the measurement, given the system is at the phase-space point $\vec{r},\vec{p}\in\Gamma$, e.g. $\Gamma=\mathbb{R}^{2n}$ The requirement of realization in an experiment restricts observables in this sense to *bounded* functions:

$$||A|| := \sup_{\Gamma} |A(\vec{r},\vec{p})| < \infty, \tag{1}$$

which also defines our norm. We see that this operational definition excludes much of what we would consider "properties" of a system (position, momentum, energy, etc.), but it includes everything we could ever measure in an experiment, which could be considered the more scientific (in an empirical sense) statement. We will re-encounter this fundamental problem of abstraction, when we will deal with the unbounded operators of quantum mechanics.

The abstract "properties" of a system can be considered as limits of the observables. These limits may exist, but they will not in general be themselves observables (in particular not be bounded), because the series is not convergent in the supremum norm. *If* a sequence of measurement arrangement *is* convergent, then, by the completeness of the C^* algebra, the limit is also an bounded and may/should have an experimental realization.

1.1.2 States

As an idealization, the state of a classical systems is fully defined by a point in Γ. However, this is not what any scientist ever has been able to observe. If we restrict ourselves to what we can *really* do, we arrive at a slightly different "operational" definition of a state: It is a procedure that we can repeat many times with what we consider high precision. That "preparation procedure" defines what we call a state. Repeated observations of the same observable using the same preparation procedure will give a certain distribution of observed values. We can compute the expectation value of the observables for a given preparation procedure ω as the average over many measurement results a_i:

$$\langle A \rangle_\omega = \frac{1}{N} \sum_{i=1}^{N} a_i \in \mathbb{R} \tag{2}$$

The individual value of one measurement is of little significance because of the statistical uncertainty of our preparation procedure.

A "high precision" measurement (for the purpose of a given experiment) is one where the measurement values scatter little, e.g. when the variance is small and higher powers of the expectation values obey

$$\langle A^n \rangle \approx \langle A \rangle^n \tag{3}$$

A state is a *functional* on the observables, more generally on the C^*-algebra that contains the observables: it maps any element $A \in \mathcal{A}$ into $\langle A \rangle_\omega \in \mathbb{C}$, in physics terms: the state (=preparation) of the system determines the average value of a measurement (and also its variance etc.). We usually construct observables as to be real numbers.

In our specific example we see three properties that we will consider the defining properties of a state, linearity, normalization $\langle 1 \rangle_\omega = 1, \forall \omega$, and positivity, i.e. that a state will give non-negative expectation value for any function that has all non-negative values:

Definition: State of a system
A state is a normalized positive linear functional on the the C^* algebra of bounded functions.

1.2 Justification of the C^* algebraic nature of observables

Above we have seen that classical mechanics is "embedded" in a C^* algebra, i.e. any observable that corresponds to realizable experiments can be considered as an element in a larger structure, which forms a C^* algebra. The assumptions underlying this are very "natural", in the sense that this is how we perceive our experimental procedures. However, one must warn already here, the C^* structure cannot be fully motivated by such considerations: it may well be that there is a meaningful (maybe even the ultimately correct?) description of nature that cannot be embedded into a C^* algebra but still fulfills all procedural requirements (linearity, normalizability, etc.) that we put *a priori* into a scientific theory. The Hilbert space formulation follows from the C^*-structure and the definition of states. It is a convenient technical representation of the algebra and it is guaranteed to have the essential properties of meaningful theory. There is further structure, though, not required by reason *a priori*. Therefor this specific form may not be the only choice. However, we find to this point no statements in the theory that would contradict experiment.

We will now justify a few aspects of the algebraic form of the physical theory and than make the leap of faith that it is best to realize these properties as part of a C^* algebra. Much of the justification was already anticipated in the discussion of the classical mechanics case.

1.2.1 Observables

An observable $\mathbf{A}$ is an experimental apparatus that produces real numbers $a \in \mathbb{R}$ as its results. Clearly, we can, e.g. just by changing measurement units, multiply an observable by a real number λ. We can form powers of the measurement results by forming the power of the result a. We can define an observable $\mathbf{A}$ as *positive* if any measurement results in a non-negative number a.

1.2.2 State

A state ω of a physical system is an experimental procedure that can be repeated an arbitrary number of times. The expectation value for the results is a (real) number $\omega(\mathbf{A})$ for any observable. In the sense of empirical science, a state is *completely characterized*, if we know the expectation values of all possible observables. Two states that give the same result for anything we measure are plausibly called equal: there is simply no experiment that can distinguish them. By its definition as the average over repeated measurements, we find the properties for a given observable $\mathbf{A}$:

$$\omega(\lambda \mathbf{A}) = \lambda \omega(\mathbf{A}), \quad \omega(\mathbf{A}^m + \mathbf{A}^n) = \omega(\mathbf{A}^m) + \omega(\mathbf{A}^n), \tag{4}$$

where the power is simply formed by replacing every measurement value in the a_i by a_i^m.

1.2.3 Normalization and positivity of a state

Normalization and positivity (for $A := B^2$)

$$\omega(1) = 1 \text{ and } \omega(A) = \omega(B^2) \geq 0 \tag{5}$$

follow easily.

1.2.4 Equality of observables

We can reason in a similar way for the observables: any two observables, that give the same result for all possible states defined are equal:

$$\omega(A) = \omega(B) \quad \forall \omega \leftrightarrow A = B. \tag{6}$$

Here we assume that a state can be applied to all observables. This is not completely obvious if we adhere to a strict operational description as a certain way of preparing a systems may be incompatible with certain ways of measuring things. Of course, as soon as we abstract the procedure into some internal property of the "system" we should be able to perform any possible measurement on it.

1.2.5 C^*-structure

Linear structure and norm can be plausibly deduced along the same lines (see Strocchi): for the linear structure, we use that we call two observables equal, when their expectation value for all states ω is equal. This, in particular, means that we *know* an observable, if we know its expectation value for all states

$$A + B = C : \omega(C) = \omega(A + B) \quad \forall \omega \tag{7}$$

We can define the norm of an observable as "the largest possible expectation value in any experiment". As it is not *a priori* obvious whether this largest value will ever be actually assumed by for a single state, we use the supremum sup rather than the maximum:

$$||A|| = \sup_{\omega} \omega(A). \tag{8}$$

Problem 1.1: Show that this has the properties of a norm in a linear space.

One may ask, however whether the observable $\mathbf{C} = \mathbf{A} + \mathbf{B}$ can be mapped onto a single realizable experiment. If not, it may define an object that does not comply with the original definition of observables. Our forming of powers of C relies on this operational prescription: we need to take powers of the individual measurement results $a_i + b_i$. If we cannot obtain a_i and b_i for the same individual measurement i, we have no procedure to form a power of $\mathbf{C}$ by averaging over powers of $c_i = a_i + b_i$.

At this point we withdraw from the real world of experimental arrangements into mathematics by assuming that, for whichever are the mathematical objects that represent an observable $\mathbf{C}$, the powers of $\mathbf{C}$ can be formed in a meaningful way. This is certainly the case in classical and quantum mechanics, but we see no reasons for considering it an *a priori* necessity of a theory describing experimental practice.

If we accept the assumptions so far, we can make a few more steps to prove the triangular inequality for the norm, but this is where our beautiful story ends. We terminate our motivation here and *assume* that the observables are embedded in a C^*-algebra. This is the case for classical and quantum mechanics, but it is not a requirement for mapping measured numbers into a mathematical structure.

Our goal is not to "deduce" quantum mechanics, but a different one: we will show that the structure of a C^*-algebra together with a state generates a Hilbert space and that, actually, any Hilbert space with a set of (bounded) observables on it is equivalent to a sum of such Hilbert space representations for states ω_i. The states may be infinitely many. This structure comprises both, quantum and classical mechanics. Classical mechanics is the special case, where the algebra is commutative.

1.2.6 Basic structure of a physical theory

1. A physical system is defined by a C^*-algebra that contains all its observables
2. A state is a positive normalized linear functional on the C^* algebra
3. An observable is fully defined by its expectation values on all possible states. Conversely, a state is fully defined by its expectation values for all elements of the algebra.

Note:

- There is a duality between states and observables, as we know it from linear functionals on vector spaces.
- In reality, we know exactly two theories that comply with this definition: classical and quantum mechanics. In case of classical mechanics, it appears to be an overblown definition, but it precisely fits quantum mechanics as we know it.

2 C^* algebra

Definition: Algebra
An algebra over the complex numbers $\mathbb{C}$:

1. $\mathcal{A}$ is a vector space
2. there is an associative multiplication: $(\mathbf{P}, \mathbf{Q}) \to \mathbf{O} =: \mathbf{PQ} \in \mathcal{A}$ with the properties

$$\begin{aligned}
&\mathbf{P}(\mathbf{Q}_1 + \mathbf{Q}_2) = \mathbf{PQ}_1 + \mathbf{PQ}_2 \text{ for } \mathbf{P}, \mathbf{Q}_1, \mathbf{Q}_2 \in \mathcal{A} \\
&\mathbf{O}(\mathbf{PQ}) = (\mathbf{OP})\mathbf{Q} \\
&\mathbf{P}(\alpha\mathbf{Q}) = \alpha(\mathbf{PQ}) \text{ for } \alpha \in \mathbb{C} \\
&\exists \mathbf{1} \in \mathcal{A} : \mathbf{1Q} = \mathbf{Q1} = \mathbf{Q}
\end{aligned}$$

Definition: $*$-Algebra
An algebra is a called a $*$-algebra if there is a map $* : \mathcal{A} \to \mathcal{A}$ with the properties $(\mathbf{PQ})^* = \mathbf{Q}^*\mathbf{P}^*$, $(\mathbf{P} + \mathbf{Q})^* = \mathbf{P}^* + \mathbf{Q}^*$, $(\alpha\mathbf{Q}) = \alpha^*\mathbf{Q}^*$, $\mathbf{Q}^{**} = \mathbf{Q}$. The element $\mathbf{Q}^*$ is called the **adjoint** of $\mathbf{Q}$.

Obviously this abstracts what you have encountered as “hermitian conjugate” of a matrix or an operator.

Definition: C^*-Algebra
A *-algebra $\mathcal{A}$ is called C^* if it has the following properties:

1. There is a norm $0 \le ||\mathbf{Q}|| \in \mathbb{R}$ with the usual properties (positivity, compatible with scalar multiplication, and triangular inequality) of a norm on a vector space and in addition

$$\begin{aligned}
&||\mathbf{PQ}|| \le ||\mathbf{P}||\,||\mathbf{Q}|| \\
&||\mathbf{Q}^*|| = ||\mathbf{Q}|| \\
&||\mathbf{QQ}^*|| = ||\mathbf{Q}||\,||\mathbf{Q}^*|| \\
&||\mathbf{1}|| = 1.
\end{aligned}$$

2. The algebra is **complete** w.r.t. to that norm (Banach space).

2.1 Classes of elements of a C^* algebra

1. Hermitian: $Q^* = Q$
2. Unitary: $Q^*Q = QQ^* = 1$
3. Normal: $Q^*Q = QQ^*$
4. Projector: $Q^2 = Q = Q^*$ (more strictly: orthogonal projector)
5. Positive: $\exists C \in \mathcal{A} : Q = C^*C$

2.2 The spectrum of $A \in \mathcal{A}$

Definition: Resolvent set and spectrum
The set

$$z \in \mathbb{C} : \exists (A-z)^{-1} \tag{9}$$

is called **resolvent set** of A. Its complement $\sigma(A)$ is called the **Spectrum** of A.

2.2.1 Series expansion of the resolvent

$$(z-A)^{-1} = z^{-1}\sum_{n=0}^{\infty}(z^{-1}A)^n. \tag{10}$$

That series, depending on the size of $|z|$, may or may not be convergent.

2.2.2 The resolvent set is an open set

Let z_0 be in the resolvent set and let $|z-z_0| < \epsilon < 1/||(z_0-A)^{-1}||$. Then the formal series

$$(z-A)^{-1} = (z-z_0+z_0-A)^{-1} = (z_0-A)^{-1}\sum_{n=0}^{\infty}\left\{(z-z_0)(z_0-A)]^{-1}\right\}^n \tag{11}$$

is norm-convergent and therefore has its limit in the C^* algebra. To see norm-convergence, we need to the estimate

$$||\sum_{n=0}^{N}[(z-z_0)(z_0-A)]^{-1}|| \quad \underbrace{\leq}_{\text{triangular}} \quad \sum_{n=0}^{N}|||[(z-z_0)(z_0-A)]^{-n}|| \tag{12}$$

$$\underbrace{\leq}_{||\mathbf{PQ}||\leq||\mathbf{P}||||\mathbf{Q}||} \quad \sum_{n=0}^{N}|(z-z_0)|^n\,||(z_0-A)]^{-1}||^n \tag{13}$$

and choose $|z-z_0|||(z_0-A)^{-1}|| < 1$ or $z \in B_\epsilon(z_0)$ with $\epsilon = 1/||(z_0-A)^{-1}||$

2.2.3 Algebra element properties and spectrum

There is a close relation between the class of an element and its spectrum. In fact, the spectral properties fully characterize hermitian, unitary, positive and projector elements, if we assume that Q is *normal*. You can study this as an exercise (with instructions).

2.3 Problems

Continuity is an essential property of maps, as it allows us to take limits, i.e. to approximate, e.g. an idealized measurement by a sequence of increasingly accurate measurements and assume that the perfect measurement would be close to our "increasingly accurate" measurements. In a sense, it is the idea of "continuity" makes the concept of "accuracy" meaningful.

For functions f between sets with a norm, continuity can be best understood as "the limit of the function values is the function value of the limit", i.e.

$$f \text{ is continuous } \Leftrightarrow f(x_n) \to f(x) \text{ if } x_n \to x. \tag{14}$$

This is not the most general definition of continuity, but it serves all our purposes and matches intuition.

A **bounded operator** is an operator with finite operator norm (problem above). In linear spaces, boundedness of a linear map and continuity are intimately related.

Problem 2.2: Boundedness and continuity Let $\mathbf{E}$ and $\mathbf{F}$ be two normed linear spaces and let $\widehat{B} : \mathbf{E} \to \mathbf{F}$ be a linear map. Show that the following statements are equivalent:

1. $\widehat{B}$ is continuous at the point $0 = x \in \mathbf{E}$
2. $\widehat{B}$ is bounded
3. $\widehat{B}$ is continuous everywhere

w.r.t. to the supremum-norm.

Hint: This is a standard result for linear maps. For those who have not been exposed to it, here a few hints: show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. In the first step use the fact that in a linear space, any vector $||x|| = 1$ can be scaled to length ϵ. Apply this to $||\widehat{B}x||/||x||$ that appear in the definition of the norm. For the step bounded $\Rightarrow$ continuous, observe that $||B(x_n - x)|| < ||B||||x_n - n||$ by construction of the operator norm.

Problem 2.3: Positivity Show that for hermitian A the element $||A||\mathbf{1} \pm A$ is positive .
Hint: Write $||A|| \pm A = C^2$ for hermitian C and explicitly construct C using the fact that there is a convergent Taylor series for $\sqrt{1 \pm x}$. Use the properties of the algebra to show the series converges with a limit in the algebra.

Problem 2.4: Positivity and boundedness Show that any positive linear functional on a C^* algebra is bounded.
Hint: Use Cauchy-Schwartz and positivity of $||A|| \pm A$ for hermitian A.

Problem 2.5: Resolvent Let $A \in \mathcal{A}$ be an element of a C^* algebra. Then also $A - z \in \mathcal{A}$, where we use the notation $A - z$ for $A - z\mathbf{1}, z \in \mathbb{C}$. We call $(A-z)^{-1} : (A-z)^{-1}(A-z) = (A-z)(A-z)^{-1} = \mathbf{1}$ the **resolvent** $R_A(z)$ of A at z if it exists and $(A - z)^{-1} \in \mathcal{A}$.

(a) Show that the formal series expansion

$$(z - A)^{-1} = z^{-1} \sum_n^\infty (A\, z^{-1})^n \tag{15}$$

converges for $|z| > ||A||$.
Hint: Use the properties of the norm on a C^*-algebra to show the the series of finite sums is Cauchy.

(b) Conclude from the above that $(A - z)^{-1} \in \mathcal{A}$ for $||A|| < |z|$.

Problem 2.6: Classes of algebra elements and spectrum Normality is an important property that is not at all guaranteed.

(a) Using the C^* algebra of $n \times n$ matrices, give an example (*the* example ;-)) of a non-normal matrix.

Properties like hermiticity, projector, unitarity, positivity of elements of a C^* algebra are reflected in the spectra of the respective elements. This is trivial once we can use the spectral theorem, but it can be proven directly as well. For the following cases this can be done without any particular tricks:

(b) $\sigma(A^*) = \sigma(A)^*$
Hint: Of course, you should prove this for the complement of the spectrum, the resolvent set, first.

(c) $\sigma(U) \subset$ (unit circle) for U unitary.
Hint: Determine $||U||$ and $||U^{-1}||$ and Taylor-expand $(U - z)^{-1}$ and $(U^* - z*)^{-1} = (U^{-1} - z^*)$.

The next to cases need some tricks, that are hard to guess. For the enthusiasts, try taking advantage of the hints.

(d) $\sigma(P) \subset \{0, 1\}$ for P (orthogonal) projector.
Hint: Study $R_a = (1 + P/(a - 1))/a$, which exist for all $a \ni 0, 1$.

(e) $\sigma(A) \subset \mathbb{R}$ for A hermitian.

2.4 State on a C^* algebra

A linear functional ω on a C^* algebra $\mathcal{A}$ is called **positive** if $\omega(A^*A) \geq 0 \forall A \in \mathcal{A}$. If $\omega(\mathbf{1}) = 1$, we call it a **state**.

Examples

1. In quantum mechanics: $\langle x|Ax\rangle$
2. In classical mechanics: $\int_\Gamma dx dp \rho(x,p) A(x,p)$ for $\rho \geq 0$.
3. On matrices, $\mathrm{Tr}\,\rho A$ for a "density matrix" ρ, i.e. a positive matrix (equivalent hermitian with non-negative eigenvalues) with $\mathrm{Tr}\,\rho = 1$.
4. Positive measures on function spaces

2.4.1 Notes

1. There holds a Cauchy-Schwartz inequality, i.e.
$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B) \tag{16}$$
This will be left to you as an exercise.
2. Positivity of ω implies its boundedness.
3. As always with linear maps, boundedness implies continuity. Intuitively, if two vectors do not differ too much, $||P - Q|| < \epsilon$, the also the images do not differ too much: $\omega(P - Q) \leq C_\omega ||P - Q||$, which is the motivation for general definition of continuity: the pre-image of any open set is an open set. (See exercises).

Problem 2.7: Cauchy-Schwartz Prove the generalized Cauchy-Schwartz inequality for states ω
$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B) \tag{17}$$

Hint: This can be modeled after one standard form of proving Cauchy-Schwartz. Follow these steps: (a) study $0 \leq \omega((A^* + B^*)(A + B))$, (b) show that $\omega(A^*B)$ can be considered as real without loss of generality; (c) use positivity of ω to show that $\omega(A^*B) = \omega(B^*A)$, (d) look at the square of (a).

2.4.2 The states form a convex set

$$\omega = \alpha\omega_1 + (1-\alpha)\omega_2, \quad \alpha \in [0,1], \quad \omega_1, \omega_2 \text{ states} \tag{18}$$

is a state. They do *not* form a linear space, e.g. 2ω is not a state, if ω is one.

2.4.3 Pure and mixed states

There are special states, the **pure** states, which cannot be written as a linear combination of states. Conversely, one can prove that any state can be written as a convex combination of pure states:

$$\omega = \sum_i \alpha_i \vec{u}_i, \quad \sum_i \alpha_i = 1, \quad \nu_i \text{ pure} \tag{19}$$

States that are not pure are called **mixed**.

We know and we will provide the prove later that a quantum mechanical wave function ψ defines a pure state ω_ψ

$$\omega_\psi(\widehat{A}) = \langle\psi|\widehat{A}|\psi\rangle.$$

We see that this, according to our general principles, is not the only possibility for a state. The more general mixed states come by the name of **density matrix** that is

$$\omega_\rho = \sum_i \rho_i \omega_{\psi_i}, \quad \sum_i \rho_i = 1, \tag{20}$$

with the action

$$\omega_\rho(\widehat{A}) = \sum_i \rho_i \omega_{\psi_i}(\widehat{A}) = \sum_i \langle\psi_i|\widehat{A}|\psi_i\rangle \tag{21}$$

3 Gelfand isomorphism

The essence of this isomorphism is that Abelian C^* algebras are isomorphic to algebras of functions and further that the algebra spanned by $\{\mathbf{1}, A, A^*\}$ for normal A is equivalent to the functions over $\sigma(A)$. This contains the essence of the spectral theorem for normal bounded operators.

Definition: Abelian algebra
is an algebra where the product is commutative.

In general, in mathematics, the term "homomorphism" designates a map from algebra into another that preserves the algebraic operations in the sense that the image of a sum is the sum of the images, the image of a product is the product of the images. For *-algebras, one defines the

Definition: Algebraic *-Homomorphism
A map $\pi : \mathcal{A} \to \mathcal{B}$ between *-algebras $\mathcal{A}, \mathcal{B}$ is called *-homomorphism if

1. $\pi(AB) = \pi(A)\pi(B)$
2. $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$
3. $\pi(A^*) = \pi(A)^*$.

Definition: Character
An algebraic *-homomorphism χ of an Abelian C^* algebra into the complex numbers $\mathbb{C}$ is called **character**. We denote the set of all characters of an algebra $\mathcal{A}$ as $X(\mathcal{A})$.

Illustration

1. Let $\vec{e}$ be a joint eigenvector of all elements the algebra formed by commutating matrices. Then $\chi_{\vec{e}}(\widehat{A}) = \vec{e} \cdot \widehat{A}\vec{e}$ is a character of the algebra.
2. Bounded functions on a compact set: values at a given element from the compact set.

Comments

1. $(\chi(A) - z)^{-1}$ exists if $z \notin \sigma(A)$, therefore $\sigma(\chi(A)) \subset \sigma(A)$.
2. A character is trivially positive $\chi(A^*A) = \chi(A)^*\chi(A) \geq 0$. Therefore it is also bounded.
3. Any character is a state: it is a positive, linear map $\mathcal{A} \to \mathbb{C}$ and $\chi(A\mathbf{1}) = \chi(\mathbf{1})\chi(A)$ shows normalization.
4. But not all states are characters: characters do not form a convex space.
5. As states, also characters are bounded by $||A||$: $|\chi(A)| \leq ||A||$.
6. The Cauchy-Schwartz inequality that holds for states and therefore for characters $|\chi(A^*B)|^2 \leq \chi(A^*A)\chi(B^*B)$ implies $|\chi(A)| \leq ||A||$: choose $B = \mathbf{1}$ and use that $0 \leq \chi(||A||^2 - A^*A) = ||A||^2 - \chi(A)\chi(A)^*$.
7. Characters are *pure* states: let χ be a character and ω_1 and ω_2 two general states, then $(\alpha_1\omega_1 + \alpha_2\omega_2)$ $(\alpha_1 + \alpha_2 = 1)$ will not be a homomorphism

8. There exists character that corresponds to a pure state ω_χ $\omega_\chi(A) = ||A||$: this conforms with our analogy that characters are correspond to eigenvectors and norm of a *matrix* $||\widehat{A}||$ is the modulus of its largest eigenvalue.

   One can explicitly construct that state ω_χ as follows: Given any $A \in \mathcal{A}$, we construct the subspace of all vectors $\alpha\mathbf{1} + \beta A^*A$. On that subspace, $\chi_s(\alpha\mathbf{1} + \beta A^*A) = \alpha + \beta A^*A$ is a positive functional with $\chi_s(\mathbf{1}) = 1$ and $\chi_s(A^*A) = ||A||^2$. This χ_s is only defined on the subspace. However (theorems of Hahn-Banach and Krein) there are states on *all* of $\mathcal{A}$ that behave as χ_s on the subspace. Now that we know that such states exist, let us denote the states with $\omega(A^*A) = ||A||^2$ by Z: Z is a clearly a convex set. The extremal states $\chi_e \in Z$ are pure states: because of $\omega(A^*A) \leq ||A||^2$, $\chi_s = \alpha\omega_1 + (1-\alpha)\omega_2$ implies $\omega_i(A^*A) = ||A||^2$, i.e. $\omega_i \in Z$ and χ_e would not be extremal in Z. Note that we have not shown that ω_χ is a *character*.

We had not proven that states are bounded. For characters that prove is really easy and all that we need for now: For hermitian A we know (from one of the problems) $0 \leq ||A|| \pm A$, therefore $0 < ||A|| \pm \chi(A)$, i.e. $|\chi(A)| \leq ||A||$. For general A, we note that A^*A is hermitian and therefore (using properties of $||\cdot||$

$$||A||^2 = ||A^*A|| \geq \chi(A^*A) = \chi(A)^*\chi(A) = |\chi(A)|^2. \tag{22}$$

3.1 Weak *-topology

Strictly a topology on any set is given, if we define all open subsets of the set.

Topology is required to define continuity: a function is continuous at a point, if the pre-image ("Urbild") of any open neighborhood of a image of the point is an open neighborhood. This is an abstraction and generalization of our intuition for continuous functions on $\mathbb{C}^n$.

We will want to define *continuous* functions from the character set $X(\mathcal{A})$ into the real numbers. We need a topology, i.e. we need to define the "open sets" $\subset X(\mathcal{A})$. Now the characters are subsets of the *linear functionals* on the algebra: they are subsets of the *dual space* of the algebra.

The open sets M in the dual space are defined at those where for any point $W^* \in M$ and any v there is a $B_{v,\epsilon}(W^*) = \{U^* \in \mathrm{V} | |U^*(v) - W^*(v)| < \epsilon\} \subset M$.

We define when we call a sequence in the dual space as convergent in the sense of this topology: a sequence of elements χ_n from the dual space of some vector space $\mathcal{V}$ is said to converge in the sense of *weak convergence*, if $\chi_n(v)$ converges as a sequence in $\mathbb{C}$ for each $v \in \mathcal{V}$.

Illustration and comments on weak convergence In finite dimensions most topologies are equivalent and also the distinction between the a linear space and the linear functionals on that space is not essential. In infinite dimensions the distinctions become crucial.

1. Continuous functions on a compact set S: a character χ_x maps a function $f \in C_0 : \chi(f) = f(x)$ into its value at one point. A sequence of characters χ_{x_n} converges, if the function values $f(x_n)$ converge at that point, which, as f is continuous, is the case if x_n converge in the sense of the topology of S. (You may recognize the δ-functions as characters of that algebra.)

2. n-component complex vectors $\in \mathbb{C}^n$: we do not have an algebraic structure and therefor no characters. We have a dual, of course. Choosing a basis $e_i \in \mathbb{C}^n$, any linear functional W^* can be written as $W^*(v) = \sum_i W^*(e_i)v_i = \sum_i w_i v_i$, where the w_i are components in the dual space. A sequence W^*_n converges, if each of its components $w_i^{(n)}$ converges. This dual is the isomorphic to ("for all practical purposes the same as") the original space.

3. On a *finite dimensional* vector space we can use an alternative concept with equal result: As we have a norm for the $v \in \mathbb{C}^n$ and as we can identify every $W^* \in \mathrm{dual}(\mathbb{C}^n)$ with some $w \in \mathbb{C}^2$, we can use that norm also for the W^*: $||W^*|| := ||w||$. Sequences that are convergent w.r.t. the norm ("strongly convergent") are also weakly convergent. In finite dimensions, also the converse is true.

4. In infinity dimensions weak convergence does no longer imply strong convergence. Hilbert space: weak convergence does not imply strong convergence: trivial, but also typical example is $\vec{v}_n : (\vec{v}_n)_i = \delta_{in}$

5. Example from physics: a (purely continuous) spreading wave packet converges $\to 0$ in the weak topology!

$A \in \mathcal{A}$ is a continuous function on $X(\mathcal{A})$: We are seemingly going in circles: we just introduced the characters χ as functionals on $\mathcal{A}$. We can turn the roles around and consider the elements of the algebra as functions on the set of characters

$$A \ni \mathcal{A} : X \to \mathbb{C} : \chi \to \chi(A). \tag{23}$$

As a direct consequence of the construction of the weak-* topology, A is a *continuous* function w.r.t. the *-topology on $X(\mathcal{A})$.

Problem 3.8: Continuity of $A : X(\mathcal{A}) \to \mathbb{C}$ Show that the map $A : \chi \to \chi(A)$ is continuous w.r.t. the weak *-topology on X.

3.2 Gelfand isomorphism

An Abelian C^* algebra is isomorphic to the *continuous* functions $C(X)$ on the character set $X = X(\mathcal{A})$ if we use the weak *-topology for X and the norm $C(X) \ni f : ||f|| = \sup_\chi |f(\chi)|$.

Isomorphic By isomorphic we denote maps that are bijections and that conserve the essential structural characteristics such as algebraic properties, *-map, and norm. In colloquial language, things that are isomorphic "are the same", just represented in different ways.

Outline of the proof Even if this remains very incomplete, the reasoning with approximations by algebras, dense sets, and compact sets is of general value and therefore useful to look at:

1. We have seen that any element A of the algebra defines a continuous function $f_A \in C(X)$. Let $f_A, A \in \mathcal{A}$ denote the set of these functions. Clearly, as characters are *-homomorphisms, any algebraic combination of these f_A's matches the algebraic combination of the corresponding A's e.g.
$$f_{AB}(\chi) = \chi(AB) = \chi(A)\chi(B) = f_A(\chi) f_B(\chi), \tag{24}$$
and similarly for all other algebraic operations. That means that each element of the algebra re-appears as a function on $X(\mathcal{A})$ and that these functions form a *-algebra. The map is injective.

2. Next we convince ourselves that the C^*-algebra norm maps onto the (supremum) norm of $f_A \in C(X)$:
$$||A|| \overset{!}{=} \sup_\chi |f_A(\chi)| = \sup_\chi |\chi(A)|. \tag{25}$$
As also all χ are states, $|\chi(A)| \leq ||A||$. Here we use the fact that all characters are states and that there is a character with $|\chi(A)| = ||A||$, *a fortiori* the supremum of $f_A(\chi)$ over all χ is indeed the norm.

3. The main mathematical task is to show that $A \to f_A$ is also *surjective*, i.e. that indeed *all* continuous functions $f \in C(X)$ have a correspondence in the algebra. Rather then proving anything here, we appeal to your knowledge of a fact from the theory of ordinary functions (Weierstraß): *polynomials* approximate any continuous function on a compact set arbitrarily well in the sense of the supremum norm: they are *dense* in the continuous functions. Compactness must be understood for the *-topology on X.

In fact, this holds not only for polynomials but for any *algebra* of complex-valued functions on a compact set. Now our f_A definitely form an algebra of complex valued functions on $X(A)$, which is compact. Therefore the f_A are dense in $C(X)$. This means that any function $f \in C(X)$ can be approximated by a sequence f_{A_n} with $A_n \in \mathcal{A}$. As $||A_n|| = ||f_{A_n}||$ we know that A_n is a Cauchy sequence and $A_n \to A \in \mathcal{A}$ with $f_A = f$.

Comment on this sketch: There are a few very important holes in this; but as it is not our main purpose to follow up on this, we only list these holes here clearly:

1. The proof that the pure state with $\omega(A) = ||A||$ is a character is missing. This proof will be delivered later: the GNS construction shows that pure states for abelian algebras indeed produce *-homomorphisms into the complex numbers, i.e. characters.

2. We have not discussed compactness, which for the special case of weak *-topologies can get tricky. Nonetheless we assert that the character set is compact as needed as input for Weierstraß.

3. Even if we take Weierstraß for granted, its generalization for algebras needs to be looked up in math literature.

3.2.1 Terms: Closure and dense sets

Closure of a subset Let D be a subset of a space X where a distance is defined between all elements (metric space). Then the **closure** $\overline{D}$ of D is

$$\overline{D} = D \cup \{x \in X | \exists d_n \to x\} \tag{26}$$

Dense subset A subset D of a metric space X (space with a norm) is called **dense** in X if

$$\overline{D} = X. \tag{27}$$

3.2.2 The spectral theorem *in nuce*

Let A be a normal element of any (also non-commutative) C^*-algebra $\mathcal{A}$ and consider the sub-algebra generated by $\{\mathbf{1}, A, A^*\}$. Normality implies that the algebra of all polynomials of A and A^* is Abelian. If we consider the *closure* of the algebra of the polynomials, we obtain again a now Abelian C^* algebra. (Strictly, one needs to show that commutativity survives the limit). By the Gel'fand theorem, this algebra is *isomorphic* to the functions on the character set. This implies, in particular, that

$$\exists (A - z)^{-1} \Leftrightarrow \exists (f_A(\chi) - z)^{-1} \forall \chi \tag{28}$$

or

$$\sigma(A) = \mathrm{ran}(f_A) \tag{29}$$

i.e. the “range” of the function f_A (=image $f_A(X)$) is just the spectrum of A. Considering now that

$$f_A(\chi) = \chi(A) \quad \in \sigma(A) \tag{30}$$

we see that a character maps A into the spectrum. We can also consider polynomials P

$$f_{P(A)}(\chi) = \chi(P(A)) = P(\chi(A)) = P(\chi(A)) = P(a) \text{ for } a := \chi(A) \in \sigma(A) \tag{31}$$

to see that all f's are just functions on the spectrum of A.

To make the spectral theorem as you know it complete, we need the Hilbert space. This rabbit we will magically pull out using any innocent *state* as our top-hat.

Problem 3.9: Spectra and Gelfand isomorphism Use Gelfand isomorphism to prove the spectral characterization of hermitian, unitary, positive and projector elements of a C^* algebra.

4 Representations in Hilbert space

4.1 Representation of an algebra

We have dealt with algebras reduced to their abstract properties, i.e. by defining the results of addition, multiplication. In practice, such an algebra is realized by objects like matrices, differential operators etc. One and the same algebra can be realized in different ways. Also, the realization may only represent part of the complete properties (it may not be an isomorphism).

Definition: Representation
A representation π of a C^*-algebra $\mathcal{A}$ is a *-homomorphism from $\mathcal{A}$ into the bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. If $\ker(\pi) = 0 \in \mathcal{A}$ the representation is **faithful**. Two representations π_1 on $\mathcal{H}_1$ and π_2 on $\mathcal{H})2$ are **equivalent**, if there is an isomorphism $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_2(A) = U\pi_1(A)U^{-1} \quad \forall A \in \mathcal{A}. \tag{32}$$

4.1.1 Note

- Continuity of π follows from positivity.
- As representations may not be faithful, we can have $\ker(\pi) \neq 0$. A simple example is to have the C^*-algebra of $N \times N$ diagonal matrices and consider a representation on $\mathbb{C}$, considered as a rather trivial one-dimensional Hilbert space. Specifically, $\pi_i(A) = a_i$ with a_i the i'th diagonal element would not be faithful.
- This example also hints upon the relation of characters to the spectral values of operators.

4.2 Irreducible representation

Definition: Invariant subspace
Let $\mathcal{V} \subseteq \mathcal{H}$ be a *subspace* of $\mathcal{H}$, i.e. $\mathcal{V}$ is a linear space. $\mathcal{V}$ is called an **invariant subspace** of $\mathcal{H}$ for a representation π, if

$$\vec{v} \in \mathcal{V} \Rightarrow \pi(A)\vec{v} \in \mathcal{V}, \quad \forall A \in \mathcal{A} \tag{33}$$

Definition: Cyclic vector
A vector $\vec{c} \in \mathcal{H}$ is called *cyclic vector* for a representation π, if

$$\mathcal{C} := \{\pi(A)\vec{c} | A \in \mathcal{A}\} \tag{34}$$

is *dense* in $\mathcal{H}$, i.e.

$$\overline{\mathcal{C}} = \mathcal{H}. \tag{35}$$

Definition: Irreducible representation
The following two properties are equivalent

1. The only closed invariant subspaces $\mathcal{V} \subseteq \mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.
2. Any vector $\phi \in \mathcal{H}$ is cyclic.

The first version can also be formulated as "there are no non-trivial projectors $\in \mathcal{H}$ that commute with $\pi(A) \forall A \in \mathcal{A}$.

4.3 Sum and tensor product of representations

The concepts of direct sum and tensor product can be introduced more abstractly than what we do here. We, in both cases, assume that our Hilbert spaces are equipped with a denumerable bases $\{|i\rangle\}$ (it is a *separable* Hilbert space) and use these for the following definition.

Definition: Direct sum of Hilbert spaces
Let $\mathcal{J}$ and $\mathcal{K}$ be Hilbert spaces with the bases $\{|j\rangle\}$ and $\{|k\rangle\}$. Then the space spanned by the basis

$$|i\rangle_\oplus = |j_i\rangle \oplus |k_i\rangle \tag{36}$$

with the addition rules

$$|m\rangle_\oplus + |n\rangle_\oplus = |j_m\rangle \oplus |k_m\rangle + |j_n\rangle \oplus |k_n\rangle = |j_m + j_n\rangle \oplus |k_m + k_n\rangle \tag{37}$$

and the scalar product

$$\langle m|n\rangle_\oplus = \langle j_m|j_n\rangle + \langle k_m|k_n\rangle \tag{38}$$

and analogously for the multiplication by a scalar $\alpha \in \mathbb{C}$ is a Hilbert space. **Show completeness**

Definition: Tensor product of Hilbert spaces
Let $\mathcal{J}$ and $\mathcal{K}$ be Hilbert spaces with the bases $\{|j\rangle\}$ and $\{|k\rangle\}$. Then the space spanned by the basis

$$|i\rangle_\otimes = |j_i\rangle \otimes |k_i\rangle \tag{39}$$

and the scalar product

$$\langle m|n\rangle_\oplus = \langle j_m|j_n\rangle\langle k_m|k_n\rangle \tag{40}$$

and analogously for the multiplication by a scalar $\alpha \in \mathbb{C}$ is a pre-Hilbert space. Its completion

$$\overline{\mathrm{span}(|j\rangle \otimes |k\rangle)} = \mathcal{J} \otimes \mathcal{K} = \mathcal{H} \tag{41}$$

and is called the **tensor product** of $\mathcal{J}$ with $\mathcal{K}$.

4.3.1 Illustration

Definition: Direct sum of operators
Let $\mathbf{B} \in \mathcal{B}(\mathcal{J})$ and $\mathbf{C} \in \mathcal{B}(\mathcal{K})$ be operators on two Hilbert spaces. Their **direct sum** is defined by its action on the basis functions:

$$\mathcal{B}(\mathcal{J} \oplus \mathcal{K}) \ni \mathbf{C}|i\rangle =: (\mathbf{B} \oplus \mathbf{C})(|j_i\rangle \oplus |j_i\rangle) = (\mathbf{B}|j_i\rangle) \oplus (\mathbf{C}|k_i\rangle). \tag{42}$$

As the operators are bounded (=continuous), this can be extended to a definition on the complete space.

Definition: Tensor product of operators
Let $\mathbf{B} \in \mathcal{B}(\mathcal{J})$ and $\mathbf{C} \in \mathcal{B}(\mathcal{K})$ be operators on two Hilbert spaces. Their **tensor product** is defined by its action on the basis functions:

$$\mathcal{B}(\mathcal{J} \otimes \mathcal{K}) \ni \mathbf{C}|i\rangle =: (\mathbf{B} \otimes \mathbf{C})(|j_i\rangle \otimes |j_i\rangle) = (\mathbf{B}|j_i\rangle) \otimes (\mathbf{C}|k_i\rangle). \tag{43}$$

As the operators are bounded (=continuous), this can be extended to a definition on the complete space.

Definition: Direct sum of representations
Let π_1 and π_2 be two representations of the same algebra on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The direct sum of the representations is defined by

$$\pi_1 \oplus \pi_2 = \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) : \pi(A) = \pi_1(A) \oplus \pi_2(A) \tag{44}$$

Direct sums of representations are the prototypical form a reducible representations.

4.4 Illustration

- Draw vectors and matrices
- Point out the important differences!
- Remind of tensor product; as to direct sum: archetypical form of reducible representation

4.5 Problems

Problem 4.10: Irreducibility Let π be a representation of an algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Show that the two definitions of irreducibility are equivalent:

1. The only closed invariant subspaces under π are $\mathcal{H}$ and $\{0\}$.
2. Any non-zero vector of $\mathcal{H}$ is cyclic.

Problem 4.11: Positivity and boundedness

(a) Use the Gel'fand isomorphism to argue that $A^*A \leq ||A^*A||\mathbf{1}$

(b) Convince yourself, that for operators in Hilbert space the definition of positivity translates into $\langle\psi|B\psi\rangle \geq 0\,\forall\psi \in \mathcal{H}$.

(c) Show that positivity is conserved in representation, i.e. A positive $\Rightarrow \pi(A)$ positive

(d) Use the above to show that any representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a continuous map. (Remember the relation between continuous and bounded for linear maps.)

5 The GNS construction

It is easy to see that, given a C^* algebra (which in particular is a linear space) and a state, one can construct a linear space with a hermitian sesquilinear form

$$\langle A|B\rangle := \omega(A^*B). \tag{45}$$

One can easily show hermiticity of the scalar product using positivity. Unfortunately, positive definiteness is missing: $\langle A|A\rangle = 0 \not\Rightarrow A = 0$. We never excluded that a given ω has 0 expectation value for some observable A^*A. We resolve this by removing these vectors with zero "ω-length" $\sqrt{\omega(A^*A)} = 0$.

To make this precise, we need to introduce a few concepts. First, we note that elements of the algebra with ω-length 0 transmit this property by multiplication:

The left-sided ideal $\mathcal{N}_\omega$ Denote the set of elements with zero length by

$$\mathcal{N}_\omega = \{A \in \mathcal{A} | \omega(A^*A) = 0\}. \tag{46}$$

Then for any $N \in \mathcal{N}_\omega$ we have $\omega(N^*A^*AN) \leq ||A^*A||\omega(N^*N) = 0$, i.e

$$N \in \mathcal{N} \Rightarrow AN \in \mathcal{N}_\omega. \tag{47}$$

(One says: $\mathcal{N}_\omega$ is a *left-sided ideal* of $\mathcal{A}$).

Proof.
- $A^*A < ||A^*A||\mathbf{1}$ - through Gel'fand isomorphism, function is positive, $\sqrt{||A||^2 - A^*A} =: C \in \mathcal{A}$ exists.
- then $0 < (CB)^*CB = B^*C^2B = B^*(||A||^2 - A^*A)B$ or $B^*A^*AB < ||A^*A||B^*B$.
- by positivity and linearity of states: $0 < \omega(||A^*A||B^*B - B^*A^*AB) = ||A^*A||\omega(B^*B) - \omega(B^*A^*AB)$
- $N \in \mathcal{N}$, $A \in \mathcal{A}$: $\omega(N^*A^*AN) \leq ||A^*A||\omega(N^*N) = 0$

□

We can now define an algebra, from which $\mathcal{N}$ is "divided out" (the *quotient algebra*): the idea is that if any two elements differ only by an element of ω-length 0, they are considered as equivalent, both being the valid representatives of an equivalence class.

Definition: Quotient space
We denote by b_B the sets

$$b_B := \{A \in \mathcal{A} | \exists N \in \mathcal{N}_\omega : A = B + N\} \tag{48}$$

The set of the equivalence classes b is denoted by $\mathcal{A}/\mathcal{N}_\omega$ (the *quotient space*).

Equivalence class The relation

$$A \sim A' : A = A' + N \in \mathcal{N} \tag{49}$$

defines an equivalence relation

$$\begin{aligned} A &\sim A \\ A &\sim B \Leftrightarrow B \sim A \\ A &\sim B, \quad B \sim C \Rightarrow A \sim C \end{aligned}$$

The sets b_B form **equivalence classes**, meaning

$$A \sim B \Leftrightarrow b_B = a_A \tag{50}$$

Any representative B from the class is equally good. Note in particular that the $0 \in \mathcal{A}/\mathcal{N}_\omega$ corresponds to $\mathcal{N}_\omega \subset \mathcal{A}$ (it is a *true subset*, why?).

With this we can construct a pre-Hilbert space

Lemma Let ω be a state, then the quotient space $\mathcal{A}/\mathcal{N}_\omega$ with the scalar product

$$\langle b_B | b_{B'} \rangle = \omega(B^*B') \tag{51}$$

is a pre-Hilbert space.

Proof. The only non-trivial thing to show is $b_B = 0 \Rightarrow \langle b_B | b_B \rangle = 0$. Assume $b_B \neq 0$, i.e. $B = A + N$ with $A \notin \mathcal{N}_\omega$:

$$\omega(B^*B) = \omega(A^*A) + \underbrace{\omega(N^*A)}_{|\omega(N^*A)|^2 \leq \omega(N^*N)\omega(A^*A)} + \omega(A^*N) + \underbrace{\omega(N^*N)}_{=0} > 0 = \omega(A^*A).$$

We know $|\omega(A^*N)| \leq \omega(A^*A)\omega(N^*N) = 0$ and we get $\mathcal{A}/\mathcal{N}_\omega \ni b \neq 0 \Rightarrow \langle b|b \rangle \neq 0$. □

From this we can obtain a Hilbert space by including all limits w.r.t. to the ω-length into our space, by operating in the closure $\overline{\mathcal{A}/\mathcal{N}_\omega}$.

We can immediately define an action of our algebra on the vectors of this space: let $b \in \mathcal{A}/\mathcal{N}_\omega$ and $B \in \mathcal{A}$ a representative of b, i.e. $b = b_B$. Then

$$\pi(A)|b\rangle = |c\rangle, \quad c_{AB} = \{C \in \mathcal{A} | C = AB + N, N \in \mathcal{N}_\omega\}. \tag{52}$$

Strictly speaking this is only defined for the quotient algebra $\mathcal{A}/\mathcal{N}_\omega$, but by *continuity* we can extend it to being defined also on its completion $\overline{\mathcal{A}/\mathcal{N}_\omega}$.

5.1 Reducible and irreducible GNS representations

5.1.1 $1 \in \mathcal{A}/\mathcal{N}$ is cyclic

Trivially so, as any $\mathcal{A}/\mathcal{N} \ni b = b_B = \pi(B)1$.

5.1.2 Irrep ⇔ pure state

In the exercises you have shown that the GNS representation corresponding to a pure state is irreducible. We do not prove that also the converse holds: if a GNS representation is irreducible, the corresponding state is pure.

5.1.3 Irreps for abelian C^* algebras

For an abelian algebra, any irreducible representation must be one-dimensional. This may be intuitive (if somewhat circular), if you have bought into the the analogy to diagonal matrices for an abelian C^* algebra, similarly using the Gel'fand isomorphism.

For any finite-dimensional representation of an abelian C^* on can quite naïvely show that it cannot be irrep: irreducibility implies that any two vectors in the N-dimensional Hilbert space are connected by at least one element of the algebra. By that one can construct more than N linearly indepmenent matrices within the algebra. As the spaces of commuting matrices is N-dimensional, the space spanned by these then includes non-commuting matrices, e.g. the typical non-normal matrix with one non-zero elemente above the diagonal.

For those familiar with the terms: The more general proof of this goes through the equivalent characterization of irreducibility by the the commutant being proportional to the unit matrix and the fact that the bi-commutant is the closure of the algebra under strong convergence. The closure is still abelian and then includes also projectors onto true subspaces. These projectors define non-trivial invariant subspaces.

The GNS representation derived from a pure state is irrep. Therefore for an abelian it is one-dimensional algebra i.e. it is equivalent to a *-homomorphism into $\mathbb{C}$. That implies that a pure state on an abelian algebra is a character. We had used this for Gel'fand theorem. Note that in the present construction we have not made any reference to Gel'fand, ie. the argument is not circular.

Problem 5.12: Pure states and characters Accepting the fact that any irrep of an Abelian C^* algebra is one-dimensional, show that pure states on Abelian C^* algebras are characters.
Hint: Evaluate for GNS $\langle 1|(\pi_\omega(A)|1\rangle)$ and note that in 1-d $\pi_\omega(A)|b\rangle \propto |b\rangle$.

5.1.4 Any representation is the (possibly infinite) sum of irreps

Suppose we have a representation in a separable Hilbert space $\mathcal{H}$. Remember that any vector $|i\rangle \in \mathcal{H}$ defines a pure state. According to the previous statement, the selected state defines an irrep GNS, and the GNS space is isomorphic to an invariant subspace of $\mathcal{H}_i \subset \mathcal{H}$. We can now pick a state $|j\rangle \notin \mathcal{H}_i$ and repeat the procedure to obtain an new irrep on $\mathcal{H}_j \cap \mathcal{H}_i = \{0\}$ (being irreps, if they share one vector they need to be identical). As the Hilbert space is separable, the iteration of the procedure exhausts the complete space.

Each of the irreps is isomorphic to the corresponding irrep GNS.

Problem 5.13: GNS representation

(a) Find at least one cyclic vector for the GNS representation

Show that a pure state ω produces a irreducible GNS representation π_ω. We denote by $\mathcal{G}$ the GNS space, by $\mathcal{B}(\mathcal{G})$ the set of *all* bounded operators on it, and by $|\Omega\rangle \in \mathcal{G}$ the vector that corresponds to $\mathbf{1} \in \mathcal{A}$. Follow the steps:

(b) Show that ω pure $\Leftrightarrow$ any positive linear functional μ with $\mu \leq \omega$ is $\mu = \lambda\omega$.
Hint: $\Rightarrow$: convince yourself that $\omega = \mu + (\omega - \mu)$ is a convex linear combination of two states
Hint: $\Leftarrow$: construct $\mu \neq \lambda\omega$, $0 < \mu < \omega$ from $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$. E.g., imagine a density matrix ρ with two entries on the diagonal and construct a matrix that is not proportional to that one, but still smaller.

(c) Let $P^2 = P, P = P^* \in \mathcal{B}(\mathcal{G})$ be a projector onto an invariant subspace. Convince yourself that $P\pi(A) = \pi(A)P \quad \forall A$.

(d) Show that $\mu(A) : A \to \langle P\Omega|\pi(A)P\Omega\rangle$ is a positive linear functional on $\mathcal{A}$ with the property $\mu(C^*C) \leq \omega(C^*C), \quad \forall C \in \mathcal{A}$, i.e. $\mu \leq \omega$.

(e) Consider the matrix elements $\langle\pi(A)\Omega|P\pi(B)\Omega\rangle$ and determine the possible values of λ of $\mu = \lambda\omega$. Conclude that $P = \mathbf{1}$ or $= 0$.

Interestingly, also the converse is true, i.e. a GNS representation for a state ω is irreducible *if and only if* ω is pure.

Problem 5.14: Convergence of operators

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded operators on a finite-dimensional Hilbert space $\mathcal{H}$ and let $\{B_n\}$ denote a sequence of operators. Show that the three following definitions of convergence are equivalent

1. Norm-convergence: $B_n \to B \Leftrightarrow ||B_n - B|| \to 0$ (Remember: $||B|| = \sup_{||\psi||=1} ||B\psi||$)
2. "Strong" convergence: $B_n \xrightarrow{s} B \Leftrightarrow ||B_n\psi - B\psi|| \to 0 \forall\psi \in \mathcal{H}$
3. "Weak" convergence: $B_n \rightharpoonup B \Leftrightarrow \langle\phi|(B_n - B)\psi\rangle \to 0 \forall\phi, \psi \in \mathcal{H}$
   (Note the notation by $\rightharpoonup$).

On infinite-dimensional Hilbert spaces this is not the case, there is only the implication norm-convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.

(a) Using $\mathcal{H} = l^2$, i.e. the space of infinite length vectors, construct $B_n \xrightarrow{s} B$, but $B_n \not\to B$

(b) Construct $B_n \rightharpoonup B$, but $B_n \overset{s}{\not\to} B$.

Hint: In both cases the trick is to let escape the maps B_n to ever new directions, e.g. by a sequence of operators that connect ever new pairs of entries etc.