Mathematical QM - Lecture 1

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Contents

1 "Postulates" of QM... and of classical mechanics

1.1 An analogy

	Quantum	Classical
State	Vector Ψ from $\mathcal{H}$	Prob. distr. $\rho(x,p)$ on phase space
Property	"Observable", lin. op. $\mathbf{A}$ on $\mathcal{H}$	Function $A(x,p)$ on phase space
Measur.val. a	Eigenvalues of $\mathbf{A}$	Function values of $A(x,p)$
Probab. for a	$\sum_{\|a\rangle\in\mathrm{Ker}(A-a)} \|\langle a\|\Psi\rangle\|^2$	$\int_{\mathrm{Ker}(A-a)} dx\,dp\,\rho(x,p)$
After result a	Vector $\Phi_a \in \mathrm{Ker}(\mathbf{A}-a)$	Prob. distr. $\phi_a(x,p)$ on $\mathrm{Ker}(A-a)$
Dynamics(I)	$\frac{d}{dt}\Psi(t) = -iH\Psi(t)$	$\frac{d}{dt}\rho = L_H\rho$ (Lie-derivative)
Dynamics(II)	$\Psi(t) = U_t\Psi(0)$, $\langle\Psi(t)\|\Psi(t)\rangle \equiv 1$	$\rho(t) = \Phi_t[\rho(0)]$, $\int dxdp\,\rho(t) \equiv 1$
(I) and (II) are equivalent alternatives		

1.1.1 $|a\rangle \in \mathrm{Ker}(A-a)$

By that we denote, somewhat loosely, a set of orthonormal eigenvectors to a (possibly degenerate) eigenvalue a:

$$A|a,i\rangle = |a,i\rangle a, \tag{1}$$

where i labels the degeneracy. We recover a spectral projector for the eigenvalue a of A:

$$P_a = \sum_i |a,i\rangle\langle a,i| \tag{2}$$

Projector implies $P_a = P_a^2$ and we will see that is a property that we expect of any operation that represents a "measurement". The probability for finding a is therefor the expectation value of

$$\langle\Psi|P_a|\Psi\rangle = \sum_i \langle\Psi|a,i\rangle\langle a,i|\Psi\rangle \tag{3}$$

After a measurement with result a we know the system is in a state from the subspace onto which P_a projects, namely

$$\Phi_a = P_a\Psi. \tag{4}$$

1.1.2 Lie-derivative and probability preserving flow

For the sake of making the analogy more striking, we have used the Lie-derivative, which, however, for the present purpose can be re-expressed by the Poisson-bracket of Hamiltonian mechanics:

$$L_{X_H}\rho = \{H,\rho\} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial \rho}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial \rho}{\partial q_i} \tag{5}$$

We see that, with the help of the Poisson-bracket, we can consider ordinary functions on phase-space as "operators", acting on other functions on phase-space. Intreaguing also the analogy:

$$\{q_i, p_j\} = \delta_{ij} \tag{6}$$

The Lie-derivative L_X is a more general object that can be defined for any given vector field X on a manifold. This is a subject of differential geometry. The differential equation defines a differential "flow" Φ_t that transports ρ across the manifold. The specific form of the Poisson brackets ensures that the integral over ρ — total probability — remains constant.

1.1.3 Classical uncertainty

The necessarily finite precision of measurement implies that $\rho(x, p)$ always extends over a finite domain. This implies an "uncertainty principle" also for classical mechanics. However, this is qualitatively different from quantum mechanics. For a more detailed discussion, see Asher Peres: Quantum Theory — Concepts and Methods (highly recommended!)

The differential geometric analogy to for quantum mechanics can be carried further by actually introducing quantization. See, for example, lecture notes by Bates and Weinstein
(http://math.berkeley.edu/~alanw/GofQ.pdf).

1.1.4 Differences between quantum and classical theory

Essential: two operators $\mathbf{A}, \mathbf{B}$ do not commute in general, but $A(x, p), B(x, p)$ *do*
Unessential: Ψ are complex valued from a Hilbert space, ρ are real-valued and normalized.

2 Do we really need quantum mechanics?

[see Asher Peres: Quantum Theory — Concepts and Methods]

There is a persistent uneasiness in our perception of quantum mechanics. It appears to be at odds with very fundamental elements of our concept of the world. This was brought to a point by the famous paper by Einstein, Podolsky, and Rosen, Phys. Rev. 47, 777 (1934).

Based on very plausible elements of our world concept, Bell [J.S. Bell, Physics 1, 195 (1964)] has derived his famous inequalities that put limits on the correlations of completely separated systems. The inequalities are at odds with quantum mechanics. The inequalities are also at odds with experiments — at least one of the "plausible elements of our world concept" is wrong.

2.1 Strange observations

The strangeness of observations on small systems can be abstracted into the following simple description.

Suppose there is a particle that occurs in two different colors, red and green (r/g) and it can be hard or soft (h/s). We can distinguish these two properties by two aparatuses, a "colorizer" $\widehat{C}$ and a "statesplitter" $\mathcal{S}$. On each aparatus sends particles of a given property to one of its two exits, spatially separted. We have some source for these particles, but do not know their status or color.

1. $\widehat{C}^0$: We determine the color distribution by sending the particles through $\widehat{C}$ and we find a distribution, say, 50%:50%. We conclude that half of the particles are red, half of them green.

2. We take the particles leaving one of the exits, say the r-exit, and send it through another colorizer $\widehat{C}^1$: now we find all that indeed all these particles leave through the r-exit of $\widehat{C}^1$: we have made a **measurement**, we know the particles "are" red. Same works for g and, analogously, for h/s.

3. We take the 50% r-particles leaving $\widehat{C}^0$ and send them through $\widehat{S}$ and, say, we find half of these, i.e. 25% of the original sample, to be h.

4. We tentatively conclude that our ensemble of particles is uniformily distributed over the four possible states rh,rs,gh,gs and a sorting maching consisting of a sequence $\widehat{C}\widehat{S}$ should have at its 4 respective exits particles with fully determined properties.

5. If we now check the colors, just to make sure, by sending each cohort through $\widehat{C}$, we are dispointed: we find that the population splits again into two different colors, obtaining a total of 8 separate cohorts. Denote this sequence of measurements as $\widehat{C}\widehat{S}\widehat{C}$.

6. We have already seen that $\widehat{C}\widehat{C}\widehat{S} = \widehat{C}\widehat{S}$ separates into only 4 different cohorts, i.e. $\widehat{C}\widehat{S}\widehat{C} \neq \widehat{C}\widehat{C}\widehat{S}$

7. Similarly, one finds $\widehat{S}\widehat{C}\widehat{S} \neq \widehat{S}\widehat{S}\widehat{C}$

8. Considering that $\widehat{S}$ and $\widehat{C}$ measure all properties of the particles, it appears that $\widehat{S}\widehat{C} \neq \widehat{C}\widehat{S}$ for any of the particles.

The above, of course, is describes an experiment of the Stern-Gerlach kind, with $\widehat{C}$ and $\widehat{S}$ being two spin-measurements at non-parallel axes.

We will see later that the non-commutativity (8) allows to (almost) “bootstrap” all that you know as standard quantum mechanics. The fact that there are non-commuting “properties” in quantum systems is the only fundamental distinction from classical mechanics.

A possible explanations for the observation is that our aparatuses are no good, as each changes the property that it does *not* measure. If this were true, we should try to improve the aparatuses. If we start to think that this is fundamentally impossible, as QM claims, we need to reconsider what is the **reality** of a property that cannot, in principle, be observed independently. Occam's razor (English, died in Munich, 14'th century!) would advice us to do away with this specific idea of independent properties.

2.2 The EPR paradox

Quantum mechanics claims that a particle does not “have” simultaneously a momentum and a position, the complete information is in the wave function.

EPR construct a quantum mechanical state that supposedly shows that a system *must* “have” position and momentum simultaneously, even if they may not be accessible to direct measurement.

There are two essential ingredients for setting up this paradox:

(1) *Locality*: the idea that large spatial separation can ensure independence of two systems.

(2) *Realism*: an operational concept of “physical reality” which should allow us to talk about which properties a system “has”.

Although (2) appears much more fuzzy, it seems that physicists' suspicion is also directed against (1).

We take QM at face value in the sense that two subsystems are represented by any state in the tensor product space of the two spaces characterizing each of the subsystems:

$$\Psi^{(a,b)} \in \mathcal{H}_a \otimes \mathcal{H}_b = L^2(dx_a\, dx_b, \mathbb{R}^3 \times \mathbb{R}^3) \tag{7}$$

while

$$\Psi^{(a)} \in \mathcal{H}_a, \quad \text{and} \quad \Psi^{(b)} \in \mathcal{H}_b. \tag{8}$$

Suppose at some instant in time, you have a system of two particles in a peculiar wave packet state:

$$\Psi = d(x_1 - x_2 - L)d(p_1 + p_2) \tag{9}$$

where d is a function very well localized near 0 (approximating the δ function). You might ask whether this is a state in the two particle Hilbert space.

It is: we can just change coordinates $(x_1, x_2) \to (x_1 - x_2, x_1 + x_2)$ and Fourier transform with respect to $x_1 + x_2 \to p_1 + p_2$ and then safely set up our wavepacket as above. Sure, this is an "entangled" state $\Psi(x_1, x_2)$, not just the product of two single-particle states $\Psi(x_1, x_2) \neq \Psi^{(a)}(x_1)\Psi^{(b)}(x_2)$, this is essential for the argument.

This is a very formal argument with the purpose to show that the wave function is a legitimate one within the formal framework of quantum mechanics. It describes two particles about which we only know (1) they are separated by L and (2) they move at equal momenta in opposite directions. The functions d can be as close to a δ-function as we like, i.e. the error in each of these two pieces of informations can be arbitrarily small. Quantum mechanics claims that this contains the complete information about the system. We cannot determine the system better, because through its wave-function we already know everything about it. EPR introduce the idea of "physical reality" to reason that even if this may be all that is accessible to us, there is more "reality" in such a system.

Element of physical reality: "If, without in any way disturbing a system, we can predict with certainty ... the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" (Quote from EPR).

A theory can only be legitimately called "complete", if it includes all "elements of physical reality".

The position of particle 2 is an "element of physical reality": we can determine it by measuring the position of particle 1. As the particles are arbitrarily far separated, by assumption (1) we do this measurement "without in any way disturbing" particle 2. By the same argument, we could just as well measure p_1, and therefore also momentum p_2 is an element of physical reality. Therefore, EPR reason, quantum mechanics is not "complete": The word "physical reality" implies that somehow particle 2 "has" a momentum p_2 and position x_2, which by quantum theory would be a meaningless statement.

The construction of the EPR paradox was criticized by Bohr on the basis of the notion of "elements of physical reality", as there is no measurement that would provide us with both, x_1 and p_1. This would deprive us of any predictions for x_2 and p_2 simultaneously: so in which sense could both quantities independently considered "real"? This may or may not be a justified argument. If it were justified, the EPR paradox would be reduced to the problem of particle-wave dualism, which I personally consider rather disquieting in the first place. I am inclined to this point of view.

2.3 Bell's inequalities

The assumption that somehow a particle *does* carry all properties that determine an experimental outcome, has immediate consequences, even if we cannot find any way to reveal the hidden properties directly: the simple assumption that each particle carries with it everything that determines a measurement, leads to a prediction about the statistical correlations of measurements on *independent* particles: these are *Bell's inequalities.*

The fact that the particle in that sense "has" all properties that determine a measurements means that these properties, in some sense, are **real**: these properties are not generated by a given observation, they do not depend on our intentions, or any other thing outside the particle. They will invariably cause a well-defined response of the aparatus to the particle.

"Independent" is important here. In physics we believe that sufficient separation in space ensures independence, so if we require the hidden properties to be **local**, independence is ensured. As you know, Bell's inequalities are violated in experiment. The assumption that a particle has properties that determine its behavior fully, is at variance with observation.

Note that, giving up locality is to open yet another Pandora's box: we start having a problems to split the universe into reasonably independent parts, or speak of something like a "particle" (local by definition), or any useful form of causality.

Bell, instead of discussing the specific form of quantum theory, he set up his famous inequalities based on pretty much the two assumptions underlying the EPR criticism of quantum theory. He then shows that for theories based on these assumptions some inequalities hold that are violated by quantum mechanics. It appears that they also are violated by experiments.

Except for locality, which seems to be a rather clear cut concept, the essence of Bell's realism is that it is meaningful to speak of a system to "have" a set of properties irrespective of whether we can measure them simultaneously or not, similar to Einstein's "element of physical reality".

Let us assume that we have two particles that are well separated such that manipulations (or measurement) on one particle cannot influence measurements on the other ("locality", requires space-like separation of the two measurement events in the sense of special relativity). Let us further assume that each particle "has" an internal state that completely determines the outcome of any measurement made on that particle ("realism" or "determinism"). A "particle" here is local *by definition* (different from a "wave", that is *defined* by its variation over space). It does not matter whether we can in principle measure the complete information of that internal state or not.

For an example we imagine the two particles to be photons originating from a common source. We measure passage of the photo through a polarizer with two possible outcomes: 1 for pass, -1 for do not pass. We do a series of measurements $j = 1, 2, 3, \ldots$, on these photons. We assume in the jth measurment the particles have the internal states λ_j and μ_j which uniquely determine the outcome of any possible measurement. Each λ_j is a sufficiently large set of numbers to fully characterize the internal state of the first particle, likewise μ_j for the second particle. The internal states are also called "hidden variables". As by assumption each particle has its own λ_j and μ_j we call them *local* hidden variables. In particular, the internal states would determine which result, $+1$ or -1, we would find if we measure polarization in arbitrary directions $\vec{\alpha}$, $\vec{\beta}$ or $\vec{\gamma}$. Denote the outcome of measurements in the corresponding directions on the first particle by the functions $a(\lambda_j)$, $b(\lambda_j)$, $c(\lambda_j)$, and for the second particle $a(\mu_j)$, $b(\mu_j)$, $c(\mu_j)$. With the restriction to values ± 1 this looks like we are extracting a very small part of the internal information, but maybe that is just what what typically happens in the lab.

Now assume that we generated the two particles in a correlated fashion, such that we know that for any measurment $c(\lambda_j) = c(\mu_j)$. This can be achieved e.g. with photons that for symmetry reasons must have parallel polarization.

For the first particle, we use two directions of the polarizer $\vec{\alpha}$ and $\vec{\gamma}$, for the second particle we use $\vec{\beta}$ and $\vec{\gamma}$. We call the measurement results $a(\lambda_j), b(\mu_j)$,$c(\lambda_j) = c(\mu_j)$ for polarizer angles α, β, γ, respectively. The possible outcomes for all measurements are ± 1 in quantum mechanics; in our general model think of a digital switch that can only show these to results. We could measure γ on either particle, and we *know for sure*, because of symmetry, that the outcome would be the same for either particle. (If you do not like this reasoning, there is a slightly more complex inequality by the name CHSH that does not use this, but rather relies on 4 different measurement angles.) The *potential* measurement results for each photon pair with internal states λ_j and μ_j fulfill

$$1 - b(\mu_j)c(\lambda_j) = +a(\lambda_j)[b(\mu_j) - c(\mu_j)] \text{ or } = -a(\lambda_j)[b(\mu_j) - c(\mu_j)] \tag{10}$$

which can be easily verified by inserting the values ± 1 for b and c.

Note that we cannot actually measure the $b(\mu_j) - c(\mu_j)$ *unless* we assume that we do not disturb μ_j by our measurement of $b(\mu_j)$. However, with a polarizer we *do disturb* the measured system. This is sometimes called "counterfactual reasoning". It assumes that a property is somehow "there", even if we cannot give a prescription how to determine it. It reasons that if we *were* able to do that measurement, we would get the inequality for each j. This is similar to the EPR concept of "physical reality": it imagines something could be done, even if nobody can tell us how.

Now we take the average value of these functions over many measurements $j = 1, 2, 3, ...$, i.e. sum up all *potential* results and divide by the number of measurements. As the left hand side is ≥ 0, while the right hand side changes sign, we find

$$1 - \langle bc \rangle \geq \langle a[b - c] \rangle \text{ and } [1 - \langle bc \rangle] \geq -\langle a[b - c] \rangle \tag{11}$$

or

$$|\langle ab \rangle - \langle ac \rangle| \leq 1 - \langle bc \rangle. \tag{12}$$

This is **Bell's inequality**.

We cannot measure $\langle ab - ac \rangle$, but we may measure the **"correlation functions"** $\langle ab \rangle$ and $\langle ac \rangle$ separately. If the distribution of λ_j and μ_j is statistical, random, we can split our measurement indices $j = 1, 2, 3, \ldots$ into three subsets J and K and L, compute the average values for each subset $\langle ab \rangle_J$, $\langle ac \rangle_K$ and $\langle bc \rangle_L$ and assume $\langle ab \rangle_J \sim \langle ab \rangle$ and likewise for $\langle bc \rangle$ and $\langle ac \rangle$. Randomness of the "internal states" λ_j

and μ_j may be ensured by randomly selecting the subsets J, K and L. This hypothesis may be experimentally corroborated by just extending the measurement series to ever larger numbers. Bell's inequality puts a rigorous bound on the correlation functions of different correlation functions of the same observable.

In view of this statistical argument in is difficult to see what could be wrong with counterfactual reasoning; but of course, we are here at the very limits of our imagination and logics, far form the terrain that is secured by everyday experience. Therefore we need to proof statements, not ask, why they should be wrong. Thus, it remains a sore point in this whole chain of reasoning. It may limit the validity of the arguments to models, where *in principle* we can *measure* $b(\mu_j) - c(\mu_j)$. After all, what is the point of talking of a property, if there is no effect which can be identify as an unambiguous consequence of this property? In the end, connecting an effect to a property is what we call a measurement. A property that does not lead to any effect *ever*, isn't a property. Permission for counterfactual reasoning after all is somehow subsumed in the "reality" of the "hidden" variables.

2.4 The correlation of polarization measurements

The crucial tests of the assumptions of "physical reality" and locality of nature to date were all performed with light, i.e. with photons. We therefore briefly discuss the quantum mechanical polarization measurements of photons.

Polarization state of a photon: we can measure the polarization of a photon by inserting a polarizer into its path of propagation, say, along direction z. Then the photon can have polarization directions in the xy-plane. A polarizer lets the photon pass, if the polarizer's direction is parallel to the polarization direction of the photon, it does not let it pass, if the polarization direction is perpendicular to the direction of the polarizer. After the polarizer we know the polarization direction of the photon to be the same a the polarizer's: measuring a photon behind a polarizer means to project the wave function on the polarization state in direction of the projector.

If $|x\rangle$ and $|y\rangle$ designate polarization states in the respective directions, a polarization measurement with a polarizer in direction $\vec{\alpha} = (\cos\alpha, \sin\alpha, 0)$ is represented by the operator

$$P_\alpha = (|x\rangle\cos\alpha + |y\rangle\sin\alpha)(\cos\alpha\langle x| + \sin\alpha\langle y|) \tag{13}$$

This is manifestly a projector with eigenvalues 0 and 1. For convenience, we will work with the derived operators

$$\sigma_\alpha = 2P_\alpha - 1 = (|x\rangle\langle x| - |y\rangle\langle y|)\cos 2\alpha + (|x\rangle\langle y| + |y\rangle\langle x|)\sin 2\alpha \tag{14}$$

with eigenvalues ± 1. This can also be written in matrix form

$$\sigma_\alpha = \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos 2\alpha + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin 2\alpha \right] \begin{pmatrix} \langle x| \\ \langle y| \end{pmatrix} \tag{15}$$

With respect to the basis $|x\rangle, |y\rangle$ these operators are represented by the Pauli matrices $\sigma_\alpha = \cos 2\alpha\sigma_z + \sin 2\alpha\sigma_x$ and we see that polarization measurement can be mathematically mapped onto measurements of spin directions in the xz-plane.

Problem 2.1: Verify the mathematical form of the observable for polarization measurement used for discussing the violation of Bell's inequalities:

$$P_\alpha = (|x\rangle \cos\alpha + |y\rangle \sin\alpha)(\cos\alpha\langle x| + \sin\alpha\langle y|) \tag{16}$$

$$\sigma_\alpha = 2P_\alpha - 1 = (|x\rangle\langle x| - |y\rangle\langle y|)\cos 2\alpha + (|x\rangle\langle y| + |y\rangle\langle x|)\sin 2\alpha \tag{17}$$

and finally

$$\sigma_\alpha = \begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos 2\alpha + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin 2\alpha\right] \begin{pmatrix} \langle x| \\ \langle y| \end{pmatrix} \tag{18}$$

Problem 2.2: Show that the expectation value for a simultaneous measurement of polarization for a state $|e\rangle := [|x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle]/\sqrt{2}$ is

$$\langle e|\sigma_\alpha \otimes \sigma_\beta|e\rangle = \cos 2(\alpha - \beta). \tag{19}$$

Assume you have a source of light that emits two photons at a time and that is rotationally invariant. Such a source could be an atom in an excited s-state ($L = 0$) that decays to its $L = 0$ ground state by emitting two photons. As the atomic initial state is rotationally invariant, the total system after decay must also be rotationally invariant. And as the final atomic state is $L = 0$ also the state of the photons must be $L = 0$. This is a requirement of symmetry only.

Assume we measure only photons emitted in a well-defined direction (call it the z-direction) from the atom at two far separated locations A and B. A complete basis for the polarization states of the two photons is $|x\rangle \otimes |x\rangle$, $|x\rangle \otimes |y\rangle$, $|y\rangle \otimes |x\rangle$, $|y\rangle \otimes |y\rangle$. As the total state is rotationally invariant, it is in particular invariant under rotations around the z-axis, which leave only the "entangled" two-photon polarization states

$$|x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle \text{ and } |x\rangle \otimes |y\rangle - |y\rangle \otimes |x\rangle \tag{20}$$

where the latter has odd particle exchange symmetry. That means that in two photons emitted from a rotationally invariant process have parallel polarizations. If we measure the polarization of one we can infer the polarization of the other. This discrete quantity now replaces what was momentum in the original formulation of the EPR paradox.

2.5 Experimental test of Bell's inequalities

It is easy to see that the expectation value of the photon-pair state for two polarizers at the angles α and β

$$[\langle x| \otimes \langle x| + \langle y| \otimes \langle y|]\sigma_\alpha \otimes \sigma_\beta[|x\rangle \otimes |x\rangle + |y\rangle \otimes |y\rangle] = \cos 2(\alpha - \beta) \tag{21}$$

Be that as it may, Bell's inequality relates expectation values of measurements to each other that can be computed by quantum mechanics. When we choose e.g. angles $\alpha = 0°$, $\beta = 30°$ and $\gamma = 60°$ we violate Bell's inequality

$$|\cos(-60°) - \cos(-120°)| + \cos(-60°) = |1/2 + 1/2| + 1/2 = 3/2 > 1 \tag{22}$$

This would be bad for quantum mechanics, if experiments had not found the same kind of violation of Bell's inequality. So, it is bad for our preferred, intuitive, and only known way of thinking about reality.

2.6 CHSH inequality

The actual experiment [Aspect et al., Phys. Rev. Lett. 49, 1804 (1982)] uses four different angles $\alpha, \beta, \gamma, \delta$ and the inequality

$$|\langle ab\rangle + \langle bc\rangle + \langle cd\rangle - \langle da\rangle| \leq 2 \tag{23}$$

which was found to be violated by 5 standard deviations, but in perfect agreement with the QM prediction.

The CHSH inequality is named after the authors Clauser, Horne, Shimony, Holt (Phys. Rev. Lett. 1969). It uses 4 different measurment arrangements (angles)

$$(a+c)b + (a-c)d \equiv \pm 2 \tag{24}$$

Here, the quantum prediction for angles differing pairwise α, β β, γ, and γ, δ by 22.5^o is

$$|\cos 45^o + \cos 45^o + \cos 45^o - \cos 135^o| = 2\sqrt{2}. \tag{25}$$

2.7 Experiment by Alain Aspect (1982)

The final breakthrough, as the experiment could be performed using non-blocking polarizers, where the photon polarization would be determined without destroying the photon. As an atomic photon source, Ca_{40} where a two-photon transition from the $4s^{21}S_0$ into the $4p^{21}S_0$ atomic state was driven using two lasers with 406 and 581 nm wave length. The two states are both spin singlet and rotationally invariant states. The spin singlet ensures the comples system is rotationally invariant and thus the above reasoning for the symmetry of emitted photons applies. The fluorescence de-excitation goes through emission of one photon at 551,3 nm into the $4s4p^1P_1$ state and then further into the initial state by emitting a 442,7 nm photon.

These optical photons can be efficientaly split into two polarization components $|x\rangle$ and $|y\rangle$ using a polarizing beam splitter consiting of two prism stuck together with a thin dielectric film between, causing reflection of the polarization component parallel to the surface and transimission of the other.

The find the expectation value

$$S = 2.697 \pm 0.015 \tag{26}$$

compared to a QM value

$$S = 2.7 \pm 0.05. \tag{27}$$

Note that the theoretical value includes corrections for detection efficiency and a small uncertainty due to the asymmtery of transimission and reflection in the polaritmeters. (The ideal theoretical value is $2\sqrt{2} \approx 2.82$.) A flagrant violation of Bell's inequalities.

2.8 "Unperformed experiments have no results"

It is esthetically unsatisfactory that we can measure only one pair of the observables in the CHSH arrangement at a time. What if we *imagine* an outcome for the unperformed measurements (even without doing them)?

Assume we have the two alternative polarimeter direcstion (a,c) on site one and (b,d) on site two. Assume further (as in CHSH) that (a,b) as (b,c) as (c,d) all differ by the same relative angle. With no direction in space distinguished, we must imagine the same probability for disagreement between measurements of the three pairs. For optimal angles $\pi/8$, probablity for disagreement is $\sin^2(\pi/8) \approx 1/7$ for each pairs. We can now fill in a table of ficticious measurements of c and d at will. Our only constraint is that only 1/7 of the cases disagree between measurements of (b,c), when you fill in c, and then (c,d), when you fill in d. Note that a measurement series (b,c), if performed, would indeed approximate this percentage of disagreement, the same holds for (c,d). This means, that in such a ficticious series of measurments, there would be a fraction of 1-3/7 *agreement* between the values of (a,d). However, if we actually *measure* (a,d) with its relative angle of $3\pi/8$ (or calculate by QM), we find agreement of only 1/7. Note that the only thing we have done wrong is to imagine doing something that we actually do not know how to do. So, simply by imagining measurements that actually cannot be performed, we we arrive at a conclusion for measurements that *can* be performed (a,d) that are incorrect. Premisses like "assume a quadratic cow", if quadratic cows do not exist, can lead to incorrect conclusions.

This reiterates the implication of Bell's theorem in yet another form: the idea that the photon "has" a property, even if not measured (and not measureable) is incorrect (realism) unless, possibly, there is some influence from the other photon (locality).

2.9 Conclusions

It appears that "local realism" is not a property of the world. The violation of Bell's inequalities has been confirmed many times since the first experiment, and with even more striking error margins. There is a struggle to close the remaining "detection efficiency" loophole and to get rid of the "fair sampling" assumption. Note that these appear rather contrived objections to the reasoning. Yet, of course, they need to be eliminated.

This is where we stand today. We do not know whether "realism", the idea that a system somehow "has" all properties that we can measure, and has them simultaneously, or the concept that far separated systems are independent of each other, or both are wrong. Certain explicit formulations of non-local theories have been ruled out by an experiment 2007, but this is not a universal statement for all non-local theories. Currently there seems to be popular inclination to think of the world as being inherently non-local. Thinking of the possible implications of this for our ability to understand and predict events makes me shudder. The alternative of it being not “real”, i.e. the properties of things not being only “their” properties but rather a joint product of “us” and “them”, is not much of a consolation. I dare say that all of mankind’s thinking is based on the concept of objects “out there” which we can perceive and about which we can think, but which have their “nature” or “reality” independently of us. Philosophers have always known that this may be an untenable position because the idea is very difficult to make precise. However, they have not offered useful alternatives. Now we have measurements that seem to tell us that the idea *is wrong*. Quantum mechanics *may be right*. But who understands it? So how shall we form a correct image of reality?