Reading seminar summer 2021 - An effective theory for interacting Bose gases

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Part I

Theorem 1.5 in [2]

1 Introduction

$$\boxed{(-\Delta + 4e + \mathcal{V}(x))u(x) = \mathcal{V}(x) + 2e\rho(u * u)(x)\ ,} \tag{1.1}$$

$$e = \frac{\rho}{2}\int (1 - u(x))\mathcal{V}(x)\ dx\ . \tag{1.2}$$

Theorem 1.1(Positivity) Suppose that $\mathcal{V}$ is non-negative and integrable and that u is an integrable solution of (1.1)-(1.2) such that $u(x) \leqslant 1$ for all x. Then $u(x) \geqslant 0$ for all x, and all such solutions have fairly slow decay at infinity in that they satisfy

$$\int |x|u(x)dx = \infty\ . \tag{1.5}$$

Thus, any physical solutions of (1.1)-(1.2) must necessarily satisfy the *pair* of inequalities

$$0 \leqslant u(x) \leqslant 1 \quad \text{for all } x\ . \tag{1.6}$$

Theorem 1.3 (existence and uniqueness) Let $\mathcal{V} \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, $p > \max\{\frac{d}{2}, 1\}$, be non-negative. Then there is a constructively defined continuous function $\rho(e)$ on $(0,\infty)$ such that $\lim_{e\to 0}\rho(e) = 0$ and $\lim_{e\to\infty}\rho(e) = \infty$ and such that for any $e \geqslant 0$ and $\rho = \rho(e)$, the system (1.1) and (1.2) has a unique integrable solution $u(x)$ satisfying $u(x) \leqslant 1$. Moreover, if $\rho \neq \rho(e)$, the system (1.1) and (1.2) has *no* integrable solution $u(x)$ satisfying (1.6).

Remark:

- We do not assume here that the potential is radially symmetric. However, the uniqueness statement implies that u is radially symmetric whenever $\mathcal{V}$ is radially symmetric.

- The function $\rho(e)$ is the *density function*, which specifies the density as a function of the energy. Thus, our system together with (1.6) constrains the parameters e and ρ to be related by a strict functional relation $\rho = \rho(e)$. In most of the early literature on the Bose gas, ρ is taken as the independent parameter, as suggested by (**??**): One puts N particles in a box of volume N/ρ, and seeks to find the ground state energy per particle, e, as a function of ρ. Our theorem goes in the other direction, with ρ specified as a function of e. We prove that $e \mapsto \rho(e)$ is continuous, and we conjecture that $\rho(e)$ is a strictly monotone increasing function. In that case, the functional relation could be inverted, and we would have a well-defined function $e(\rho)$.
- Since $\lim_{e\to 0}\rho(e) = 0$ and $\lim_{e\to\infty}\rho(e) = \infty$, the continuity of $e \to \rho(e)$ implies that for each $\rho \in (0,\infty)$ there is *at least one* e such that $\rho(e) = \rho$.

Theorem 1.5 (decay of u at infinity) In all dimensions, provided $\mathcal{V}$ is spherically symmetric with $\int |x|^2\mathcal{V}dx < \infty$ in addition to satisfying the hypotheses imposed in Theorem 1.3, all integrable solutions of (1.1)-(1.2) with $u(x) \leqslant 1$ for all x satisfy

$$\int |x|u(x)dx = \infty \quad \text{and} \quad \int |x|^r u(x)dx < \infty \quad \text{for all } 0 < r < 1\ . \tag{1.25}$$

Thus, if $u(x) \sim |x|^{-m}$ for some m, the only possibility is $m = d+1$. Under stronger assumptions on the potential, this is actually the case. For $d = 3$, if $\mathcal{V}$ is non-negative, square-integrable, spherically symmetric (that is, $\mathcal{V}(x) = \mathcal{V}(|x|)$), and, for $|x| > R$,

$$\mathcal{V}(|x|) \leqslant Ae^{-B|x|} \tag{1.26}$$

for some $A, B > 0$ then there exists $\alpha > 0$ such that

$$u(x) \underset{|x|\to\infty}{\sim} \frac{\alpha}{|x|^4}. \tag{1.27}$$

5 Decay of u

In this section, we prove Theorem 1.5. Our proof assumes that $\mathcal{V}$ decays exponentially, because we will use analyticity properties of the Fourier transform of the potential $\mathcal{V}$. In particular, the theorem holds if $\mathcal{V}$ has compact support. We expect the result to hold for any potential that decays faster than $|x|^{-4}$. Algebraic decay for u seems natural: by (1.1), $u * u$ must decay at infinity in the same way as u. This is the case if u decays algebraically, but would not be so if, say, it decayed exponentially.
Take the Fourier transform to (1.1), then we have

$$(k^2 + 4e)\widehat{u}(k) + (\widehat{\mathcal{V}} * \widehat{u})(k) = \widehat{\mathcal{V}}(k) + 2e\rho\widehat{u}^2(k).$$

If $u(x) = \exp(-|x|)$, the $\widehat{u}(k) \sim (1+k^2)^{-2}$ in 3D.

Proof of theorem 1.5: We begin by proving (1.25) in arbitrary dimension. Recall that the first part has already been proved in Theorem 1.1 without the additional assumption on the potential. For the second part, recall that by the first remark after Theorem 1.3, u is also radial, and hence $\mathcal{V}(1-u)$ is non-negative and radial. It then follows from the hypotheses on $\mathcal{V}$ that $g := 2\rho e Y_{4e} * Y_{4e} * [\mathcal{V}(1-u)]$ satisfies

$$\int |x|^2 g(x)dx < \infty \quad \text{and} \quad \int xg(x)dx = 0 \ . \tag{5.1}$$

Then, as explained in Section 2, if $f := 2e\rho Y_{4e} * u$, $f - f * f = g \geqslant 0$, and then by [CJLL20, Theorem 4], the second part of (1.25) follows. Note that if

$$u(|x|) \underset{|x|\to\infty}{\sim} \frac{\alpha}{|x|^m} \tag{5.2}$$

for some $\alpha > 0$, then the only choice of m that is consistent with (1.25) is $m = d + 1$.
It can be seen by the following:

$$\int_{|x|>R} |x|^r \frac{1}{|x|^m} \mathrm{d}x \sim \int_R^\infty \rho^{r-m+d-1} \mathrm{d}\rho < \infty \iff r - m + d - 1 < -1.$$

Then, we have $r < 1$ only when $m = d + 1$.

We now specialize to $d = 3$, and impose the additional assumption on the potential.

Recall that the Fourier transform of u (4.22) satisfies (4.25):

$$\hat{u}(|k|) = \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(|k|)}\right) \tag{5.3}$$

where S was defined in (4.24):

$$S(|k|) := \frac{\rho}{2e}\int e^{ikx}(1-u(|x|))\mathcal{V}(|x|)\ dx. \tag{5.4}$$

We split

$$\hat{u}(|k|) = \widehat{\mathcal{U}}_1(|k|) + \widehat{\mathcal{U}}_2(|k|) \tag{5.5}$$

with

$$\widehat{\mathcal{U}}_1(|k|) := \frac{2eS(|k|)}{\rho(4e + k^2)} \tag{5.6}$$

so that, taking the large $|k|$ limit in (4.25),

$$\widehat{\mathcal{U}}_2(|k|) = O(|k|^{-6} S^2(|k|)) \tag{5.7}$$

so $\widehat{\mathcal{U}}_2$ is integrable.

1 - Decay of $\mathcal{U}_1$. We first show that

$$\mathcal{U}_1(|x|) := \frac{1}{(2\pi)^3}\int e^{-ikx}\widehat{\mathcal{U}}_1(|k|)\ dk \tag{5.8}$$

decays exponentially in $|x|$. We have

$$\mathcal{U}_1(|x|) = (-\Delta+1)^{-1}(1-u(|x|))\mathcal{V}(|x|) = Y_1 * ((1-u)\mathcal{V})(|x|) \tag{5.9}$$

with

$$Y_1(|x|) := \frac{e^{-|x|}}{4\pi|x|}. \tag{5.10}$$

Therefore, by (1.26),

$$\mathcal{U}_1(|x|) \leqslant \frac{A}{4\pi}\int_{|y|>R} \frac{e^{-|x-y|-B|y|}}{|x-y|}\;dy + \frac{1}{4\pi}\int_{|y|<R} \frac{e^{-|x-y|}}{|x-y|}\mathcal{V}(|y|)\;dy \tag{5.11}$$

so, denoting $b := \min(B,1)$,

$$\mathcal{U}_1(|x|) \leqslant \frac{A}{4\pi}\int \frac{e^{-b(|x-y|+|y|)}}{|x-y|}\;dy + \frac{e^{-(|x|-R)}}{4\pi(|x|-R)}\int \mathcal{V}(|y|)\;dy \tag{5.12}$$

and since

$$\begin{aligned}\frac{A}{4\pi}\int \frac{e^{-b(|x-y|+|y|)}}{|x-y|}\;dy &= \frac{A}{4\pi}\int \frac{e^{-b(|y|+|y+x|)}}{|y|}\;dy\\ \leq \frac{A}{4\pi}\int_{y\leq|x|} \frac{e^{-b|x|}}{|y|}\;dy &+ \frac{A}{4\pi}\int_{y>|x|} \frac{e^{-b|y|}}{|y|}\;dy \leq C(b)e^{-b|x|}(|x|^2+|x|+1)\end{aligned} \tag{5.13}$$

we have

$$\mathcal{U}_1(|x|) \leqslant C(b)e^{-b|x|}(|x|^2+|x|+1) + \frac{e^{-(|x|-R)}}{4\pi(|x|-R)}\int \mathcal{V}(|y|)\;dy. \tag{5.14}$$

2 - Analyticity of $\mathcal{U}_2$. We now turn to

$$\mathcal{U}_2(|x|) := \frac{1}{(2\pi)^3}\int e^{-ikx}\widehat{\mathcal{U}}_2(|k|)\;dk = \frac{1}{4i\pi^2|x|}\sum_{\eta=\pm}\eta\int_0^\infty e^{i\eta\kappa|x|}\kappa\widehat{\mathcal{U}}_2(\kappa)\;d\kappa. \tag{5.15}$$

We start by proving some analytic properties of $\widehat{\mathcal{U}}_2$, which, we recall from (4.25) and (5.5), is

$$\widehat{\mathcal{U}}_2(|k|) = \frac{1}{\rho}\left(\frac{k^2}{4e}+1-\sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(|k|)} - \frac{2eS(|k|)}{4e+k^2}\right). \tag{5.16}$$

2-1 - First of all, S is analytic in a strip about the real axis:

$$S(\kappa) = 4\pi\int_0^\infty \mathrm{sinc}(\kappa r)r^2\mathcal{V}(r)(1-u(r))\;dr, \quad \mathrm{sinc}(\xi) := \frac{\sin(\xi)}{\xi} \tag{5.17}$$

so

$$\partial^n S(\kappa) = 4\pi\int_0^\infty \partial^n\mathrm{sinc}(\kappa r)r^{n+2}\mathcal{V}(r)(1-u(r))\;dr. \tag{5.18}$$

We will show that if $\mathcal{I}m(\kappa) \leqslant \frac{B}{2}$ (the factor $\frac{1}{2}$ can be improved to any factor that is < 1, but this does not matter here), then there exists $C > 0$ which only depends on A and B such that

$$|\partial^n S(\kappa)| \leqslant n! C^n. \tag{5.19}$$

As a consequence, S is analytic in a strip around the real line of height $\frac{B}{2}$. Because the Taylor series of S at κ converges. In particular, if we define the strip

$$H_\tau := \{z : \ |\mathcal{I}m(z)| \leqslant r^{-\tau}, \ \mathcal{R}e(z) > 0\} \tag{5.20}$$

with $0 < \tau < 1$, and take

$$r > \left(\frac{B}{2}\right)^{-\frac{1}{\tau}} \tag{5.21}$$

then S is analytic in H_τ.

2-1-1 - We now prove (5.19). We first treat the case $|\kappa| \leqslant \frac{B}{2}$. We have

$$\mathrm{sinc}(\xi) = \sum_{p=0}^{\infty} \frac{(-1)^p \xi^{2p}}{(2p+1)!} \tag{5.22}$$

so

$$\partial^n \mathrm{sinc}(\xi) = \sum_{p=\lceil \frac{n}{2} \rceil}^{\infty} \frac{(-1)^p \xi^{2p-n}}{(2p+1)(2p-n)!}. \tag{5.23}$$

Therefore

$$|\partial^n \mathrm{sinc}(\xi)| \leqslant \sum_{p=\lceil \frac{n}{2} \rceil}^{\infty} \frac{|\xi|^{2p-n}}{(2p-n)!} \leqslant \cosh(|\xi|). \tag{5.24}$$

Thus,

$$|\partial^n S(\kappa)| \leqslant 4\pi \int_0^\infty \cosh(|\kappa| r) r^{n+2} \mathcal{V}(r)(1-u(r)) \ dr \tag{5.25}$$

so, by (1.26),

$$|\partial^n S(\kappa)| \leqslant 4A\pi \int_R^\infty \cosh(|\kappa| r) r^{n+2} e^{-Br} \ dr + 4\pi \int_0^R \cosh(|\kappa| r) r^{n+2} \mathcal{V}(r) \ dr \tag{5.26}$$

and

$$|\partial^n S(\kappa)| \leqslant 8A\pi \int_0^\infty r^{n+2} e^{-(B-|\kappa|)r} \ dr + 8\pi e^{|\kappa| R} R^n \int r^2 \mathcal{V}(r) \ dr \tag{5.27}$$

which, if $|\kappa| \leqslant \frac{B}{2}$, implies that

$$8A\pi \int_0^\infty r^{n+2} e^{-(B-|\kappa|)r} \ dr \leqslant 8A\pi \int_0^\infty r^{n+2} e^{-\frac{B}{2} r} \ dr = \frac{2^{n+6} A\pi}{B^{n+3}} (n+2)! \tag{5.28}$$

and

$$8\pi e^{|\kappa| R} R^{n+2} \int \mathcal{V}(r) \ dr \leqslant 8\pi e^{\frac{B}{2} R} R^n \int r^2 \mathcal{V}(r) \ dr \tag{5.29}$$

which implies (5.19) in this case.

2-1-2 - We now turn to $|\kappa| \geqslant \frac{B}{2}$:

$$\partial^n \mathrm{sinc}(\xi) = \sum_{p=0}^{n} \binom{n}{p} \partial^p \sin(\xi) \frac{(n-p)!(-1)^{n-p}}{\xi^{n-p+1}} \tag{5.30}$$

so

$$|\partial^n \mathrm{sinc}(\xi)| \leqslant 2e^{\mathcal{I}m(\xi)} \sum_{p=0}^{n} \frac{n!}{p!} |\xi|^{-(n-p+1)}. \tag{5.31}$$

Therefore,

$$|\partial^n S(\kappa)| \leqslant 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_0^\infty e^{\mathcal{I}m(\kappa)r} r^{p+1} \mathcal{V}(r)(1-u(r))\ dr \tag{5.32}$$

so, by (1.26),

$$|\partial^n S(\kappa)| \leqslant \sigma_1 + \sigma_2 \tag{5.33}$$

with

$$\sigma_1 := 8A\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_R^\infty r^{p+1} e^{-(B-\mathcal{I}m(\kappa))r}\ dr \tag{5.34}$$

and

$$\sigma_2 := 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} \int_0^R r^{p+1} e^{\mathcal{I}m(\kappa)r} \mathcal{V}(r)\ dr. \tag{5.35}$$

Furthermore,

$$\sigma_1 = 8A\pi n! \sum_{p=0}^{n} \frac{p+1}{(B-\mathcal{I}m(\kappa))^{p+2}|\kappa|^{n-p+1}} \tag{5.36}$$

so, as long as $|\kappa| \geqslant \frac{1}{2}B$ and $\mathcal{I}m(\kappa) \leqslant \frac{1}{2}B$,

$$\sigma_1 \leqslant \frac{2^{n+6}A\pi}{B^{n+3}} n! \sum_{p=0}^{n} (p+1) = \frac{2^{n+5}A\pi}{B^{n+3}} (n+2)!. \tag{5.37}$$

In addition,

$$\sigma_2 \leqslant 8\pi \sum_{p=0}^{n} \frac{n!}{p!|\kappa|^{n-p+1}} R^{p-1} e^{\mathcal{I}m(\kappa)R} \int_0^R r^2 \mathcal{V}(r)\ dr \tag{5.38}$$

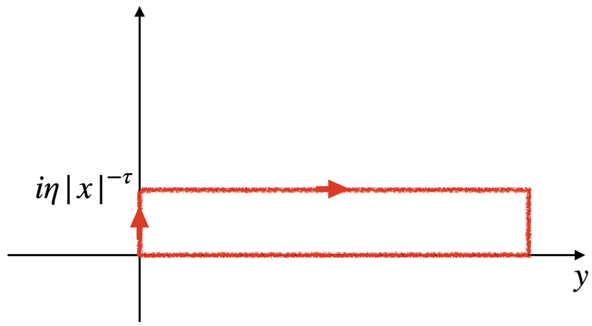
so

$$\sigma_2 \leqslant 8\pi \sum_{p=0}^{n} \frac{n!2^{n-p+1}}{p!B^{n-p+1}} R^{p-1} e^{\mathcal{I}m(\kappa)R} \int_0^R r^2 \mathcal{V}(r)\ dr \leqslant \frac{2^{n+4}\pi}{RB^{n+1}} n! e^{RB} \int_0^R r^2 \mathcal{V}(r)\ dr \tag{5.39}$$

which implies (5.19) in this case.

2-2 - We have thus proved that S is analytic in H_τ, which implies that the singularities of $\widehat{\mathcal{U}}_2$ in H_τ all come from the branch points of $\sqrt{F(|k|)}$ with $F(|k|) := (\frac{k^2}{4e}+1)^2 - S(|k|)$. For $\kappa \in \mathbb{R}$,

$$|S(\kappa)| \leqslant 1 \tag{5.40}$$



so, for $\kappa \in \mathbb{R}$,

$$F(\kappa) \geqslant \frac{\kappa^2}{2e}. \tag{5.41}$$

Therefore, since F is analytic in a strip around the real axis, there exists an open set containing the real axis in which F has one and only one root, at 0. Thus the only branch point of $\sqrt{F}$ on the real axis is 0. Thus, $\widehat{\mathcal{U}}_2$ is analytic in H_τ.

3 - Decay of $\mathcal{U}_2$. We deform the integral to the path

$$\{i\eta y,\ 0 < y < |x|^{-\tau}\} \cup \{i\eta|x|^{-\tau} + y,\ y > 0\} \tag{5.42}$$

and find

$$\int_0^\infty e^{i\eta\kappa|x|}\kappa\widehat{\mathcal{U}}_2(\kappa)\ d\kappa = I_1 + I_2 \tag{5.43}$$

with

$$I_1 := -\int_0^{|x|^{-\tau}} e^{-y|x|}y\widehat{\mathcal{U}}_2(i\eta y)\ dy \tag{5.44}$$

and

$$I_2 := e^{-|x|^{1-\tau}}\int_0^\infty e^{i\eta y|x|}(i\eta|x|^{-\tau} + y)\widehat{\mathcal{U}}_2(i\eta|x|^{-\tau} + y)\ dy. \tag{5.45}$$

3-1 - We first estimate I_1. We expand S:

$$S(\kappa) = 1 - \beta\kappa^2 + O(|\kappa|^4) \tag{5.46}$$

with $\beta > 0$ (since S is analytic and symmetric, and $|S(|k|)| \leqslant 1$). Therefore, $y \mapsto \widehat{\mathcal{U}}_2(iy)$ is

$\mathcal{C}^2$ for $y \neq 0$, and

$$\widehat{\mathcal{U}}_2(|k|) = \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(|k|)} - \frac{2eS(|k|)}{4e+k^2}\right)$$
$$= \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\frac{k^4}{16e^2} + \frac{k^2}{2e} + 1 - 1 + \beta k^2 + O(|k|^4)} - \frac{2e}{4e+k^2}(1+\beta k^2 + O(|k|^4))\right)$$
$$= \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\frac{k^2}{2e} + \beta k^2 + O(|k|^4)} - \frac{2e}{4e+k^2}(1-\beta k^2 + O(|k|^4))\right)$$
$$= \frac{1}{\rho}\left(1 - k\sqrt{\frac{1}{2e}+\beta} - \frac{1}{2} + O(k^2)\right)$$

Thus,

$$\widehat{\mathcal{U}}_2(i\eta y) = \frac{1}{2\rho} - \frac{i\eta y}{\rho}\sqrt{\frac{1}{2e}+\beta} + O(y^2) \tag{5.47}$$

Furthermore,

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y\ dy = -\frac{1}{|x|^2} + \frac{1+|x|^{1-\tau}}{|x|^2} e^{-|x|^{1-\tau}} \tag{5.48}$$

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y^2\ dy = -\frac{2}{|x|^3} + \frac{1+|x|^{1-\tau}(2+x^{1-\tau})}{|x|^3} e^{-|x|^{1-\tau}} \tag{5.49}$$

and

$$-\int_0^{|x|^{-\tau}} e^{-y|x|} y^3\ dy = O(|x|^{-4}) \tag{5.50}$$

$$I_1 = -\frac{1}{2\rho|x|^2} + \frac{2i\eta}{\rho|x|^3}\sqrt{\frac{1}{2e}+\beta} + O(|x|^{-4}) \tag{5.51}$$

so

$$\frac{1}{4i\pi^2|x|}\sum_{\eta=\pm}\eta I_1 = \frac{1}{\pi^2\rho|x|^4}\sqrt{\frac{1}{2e}+\beta} + O(|x|^{-5}). \tag{5.52}$$

3-2 - We now bound I_2. Recall that, for $\kappa\in\mathbb{R}$, $|S(\kappa)| \leqslant 1$. Recalling (5.19),

$$|S(\kappa + i\eta|x|^{-\tau})| \leqslant \sum_{n=0}^{\infty} \frac{1}{n!}|\partial^n S(\kappa)|^n |x|^{-n\tau} \leqslant \frac{1}{1-C|x|^{-\tau}} \leqslant 2 \tag{5.53}$$

provided $|x|^\tau > 2C$. Therefore, for large κ, by (5.7),

$$|\widehat{\mathcal{U}}_2(\kappa + i\eta)| = O(\kappa^{-4}) \tag{5.54}$$

so

$$I_2 \leqslant C' e^{-|x|^{1-\tau}} \tag{5.55}$$

for some constant $C' > 0$.

3-3 - Inserting (5.52) and (5.55) into (5.43) and (5.15), we find that

$$\mathcal{U}_2(|x|) = \frac{1}{\pi^2\rho|x|^4}\sqrt{\frac{1}{2e}+\beta} + O(|x|^{-5}) \tag{5.56}$$

which, using (2.10), concludes the proof of the theorem. □

Part II

Theorem 1.2 in [4]

1 Introduction

Theorem 1.2 (Large $|x|$ asymptotics of u) If $(1+|x|^4)v(x) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, then

$$\rho u(x) = \frac{\sqrt{2+\beta}}{2\pi^2\sqrt{e}}\frac{1}{|x|^4} + R(x) \tag{1.26}$$

where

$$\beta = \rho\int |x|^2 v(1-u)dx \leqslant \rho\|x^2 v\|_1, \tag{1.27}$$

and where $|x|^4R(x)$ is in $L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$, uniformly in e on all compact sets. Moreover, for every $\rho_0 > 0$, there is a constant C that only depends on ρ_0 such that for all x, for all $\rho < \rho_0$,

$$u(x) \leqslant \min\left\{1, \frac{C}{\rho e^{\frac{1}{2}}|x|^4}\right\} . \tag{1.28}$$

2 Pointwise bounds on $u(x)$ – Proof of Theorem 1.2

Let

$$\kappa := \frac{|k|}{2\sqrt{e}} \tag{2.1}$$

in terms of which (**??**) becomes

$$\rho\widehat{u} = (\kappa^2+1)\left(1-\sqrt{1-\frac{\frac{\rho}{2e}\widehat{S}}{(\kappa^2+1)^2}}\right). \tag{2.2}$$

For small κ, since x^4v is integrable, $\widehat{S}$ is $\mathcal{C}^4$

$$\frac{\rho}{2e}\widehat{S} = 1-\beta\kappa^2+O(e^2\kappa^4) \tag{2.3}$$

and β is defined in (1.27):

$$\beta = -\frac{\rho}{4e}\partial_\kappa^2\widehat{S} \leqslant \rho\|x^2v\|_1. \tag{2.4}$$

Therefore, defining

$$\widehat{U}_1 := (\kappa^2+1)^{-2}\left(1-\sqrt{1-\frac{(1-\beta\kappa^2)}{(\kappa^2+1)^2}}\right) \tag{2.5}$$

$\widehat{U}_1$ coincides with $\widehat{u}$ asymptotically as $\kappa \to 0$ and we chose the prefactor $(\kappa^2+1)^{-2}$ in such a way that $\widehat{U}_1$ is integrable. Define the remainder term

$$\widehat{U}_2 := \rho\widehat{u} - \widehat{U}_1 = (\kappa^2+1)\left(1-\sqrt{1-2\zeta_1}\right) - (\kappa^2+1)^{-2}\left(1-\sqrt{1-2\zeta_2}\right) \tag{2.6}$$

with

$$\zeta_1 := \frac{\frac{\rho}{4e}\widehat{S}}{(\kappa^2+1)^2}, \quad \zeta_2 := \frac{1-\beta\kappa^2}{2(\kappa^2+1)^2}. \tag{2.7}$$

The rest of the proof proceeds as follows: we show that the Fourier transform of $\widehat{U}_1$ decays like $|x|^{-4}$ by direct analysis, then we show that $\Delta^2\widehat{U}_2$ is integrable and square integrable, which implies that it is subdominant as $|x|\to\infty$.

1 - We compute $U_1(x) := \int \frac{dk}{(2\pi)^3} e^{-ikx}\widehat{U}_1(k)$. We write

$$\sqrt{1-\frac{1-\beta\kappa^2}{(1+\kappa^2)^2}} = \frac{\kappa}{1+\kappa^2}\sqrt{2+\beta+\kappa^2} = \frac{1}{\pi}\frac{|\kappa|(2+\beta+\kappa^2)}{1+\kappa^2}\int_0^\infty \frac{1}{2+\beta+t+\kappa^2}t^{-1/2}dt \ . \tag{2.8}$$

Therefore,

$$\widehat{U}_1 := (\kappa^2+1)^{-2} - \frac{\kappa}{\pi}(\kappa^2+1)^{-2}\left(1+(\beta+1)\frac{1}{1+\kappa^2}\right)\int_0^\infty \frac{1}{2+\beta+t+\kappa^2}t^{-1/2}dt. \tag{2.9}$$

We take the inverse Fourier transform of $\widehat{U}_1$, recalling the definition of κ (2.1)

$$U_1(x) = \frac{e^{\frac{3}{2}}}{\pi}e^{-2\sqrt{e}|x|} - \frac{1}{\pi}\left(\delta(x) + \frac{(\beta+1)e}{\pi}\frac{e^{-2\sqrt{e}|x|}}{|x|}\right) * f_1 * f_2 \tag{2.10}$$

where

$$f_1(x) := \frac{e^{\frac{3}{2}}}{\pi^3}\int dk\ e^{-ik(2\sqrt{e}x)}\frac{|k|}{(k^2+1)^2} \tag{2.11}$$

and

$$f_2(x) := \frac{e^{\frac{3}{2}}}{\pi^3}\int dk\ e^{-ik(2\sqrt{e}x)}\int_0^\infty \frac{dt}{\sqrt{t}}\frac{1}{2+\beta+t+k^2} = \frac{e}{\pi|x|}\int_0^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)}t^{-1/2}dt \ , \tag{2.12}$$

now, for all $T>0$,

$$\begin{aligned}
\int_0^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)}t^{-1/2}\ dt &= \int_0^T e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)}t^{-1/2}dt + \int_T^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)}t^{-1/2}\ dt \\
&\leqslant \int_0^T e^{-\sqrt{2+\beta}}e^{-\sqrt{t}}(2\sqrt{e}|x|)t^{-1/2}dt + \int_T^\infty e^{-\sqrt{2+\beta+t}(2\sqrt{e}|x|)}t^{-1/2}\ dt \\
&= 2(1-e^{-\sqrt{T}})e^{-\sqrt{2+\beta}(2\sqrt{e}|x|)} + \frac{1}{\sqrt{e}|x|}e^{-\sqrt{T}(2\sqrt{e}|x|)} \\
&\leqslant 2T^{1/2}e^{-\sqrt{2+\beta}(2\sqrt{e}|x|)} + \frac{1}{\sqrt{e}|x|}e^{-\sqrt{T}(2\sqrt{e}|x|)}\ .
\end{aligned} \tag{2.13}$$

Where we have use that

$$\exp(-x) = 1 - x + \frac{1}{2}x^2 + O(x^3)$$

for the last inequality.

Choosing $T = 2+\beta$, we see that for large $(2\sqrt{e}|x|)$, $0 \leqslant f_2(x) \leqslant Ce^{-\sqrt{2+\beta}(2\sqrt{e}|x|)}$. Furthermore,

$$f_1(x) = \frac{e^{\frac{3}{2}}}{\pi^3}\int dk\ e^{-ik(2\sqrt{e}x)}\frac{1}{|k|}\frac{k^2}{(k^2+1)^2} = \frac{e^{\frac{3}{2}}}{\pi^3}\frac{1}{|x|^2} * g, \quad g(x) = \frac{(1-\sqrt{e}|x|)e^{-(2\sqrt{e})|x|}}{|x|} \tag{2.14}$$

Using

$$\frac{1}{|x-y|^2} = \frac{1}{|x|^2} + \frac{-|y|^2+2x\cdot y}{|x|^2|x-y|^2} \tag{2.15}$$

twice and the fact that $g(y)$ is even, integrates to zero, and $\int yg(y)\ dy = 0$,

$$f_1(x) = \frac{1}{|x|^4}\frac{e^{\frac{3}{2}}}{\pi^3}\left(-\int_{\mathbb{R}^3}|y|^2g(y)\mathrm{d}y + \int_{\mathbb{R}^3}\frac{(-|y|^2+2x\cdot y)^2}{|x-y|^2}g(y)\mathrm{d}y\right) \tag{2.16}$$

We compute $\int_{\mathbb{R}^3}|y|^2g(y)\mathrm{d}y = -\frac{3\pi}{2e^2}$, and then using the symmetry of g once more,

$$\lim_{|x|\to\infty}\int_{\mathbb{R}^3}\frac{(x\cdot y)^2}{|x-y|^2}g(y)\mathrm{d}y = \frac{1}{3}\int_{\mathbb{R}^3}|y|^2g(y)\mathrm{d}y = -\frac{\pi}{2e^2}\ , \tag{2.17}$$

Therefore,

$$\lim_{|x|\to\infty}|x|^4f_1(x) = -\frac{1}{2\pi^2\sqrt{e}} \qquad \text{and} \qquad \lim_{|x|\to\infty}|x|^4U_1(x) = \frac{1}{2\pi^2\sqrt{e}}\sqrt{2+\beta}\ . \tag{2.18}$$

We now turn to an upper bound of U_1. First of all, if $|x| \leqslant \frac{1}{\sqrt{e}}$, then by (2.14) and (2.16),

$$f_1(x) \geqslant 0 \tag{2.19}$$

and if $|x| > \frac{1}{\sqrt{e}}$, then

$$f_1(x) \geqslant -\frac{1}{|x|^4}\frac{e^2}{\pi^3}\int_{\mathbb{R}^3}\frac{(-|y|^2+2x\cdot y)^2}{|x-y|^2}e^{-(2\sqrt{e})|y|}\mathrm{d}y. \tag{2.20}$$

We split the integral into two parts: $|y-x| > |x|$ and $|y-x| < |x|$. We have, (recalling $|x| > \frac{1}{\sqrt{e}}$),

$$\int_{|y-x|>|x|}\frac{(-|y|^2+2x\cdot y)^2}{|x-y|^2}e^{-(2\sqrt{e})|y|}\mathrm{d}y \leqslant e^{-\frac{5}{2}}C \tag{2.21}$$

for some constant C (we use a notation where the constant C may change from one line to the next). Now,

$$\int_{|y-x|<|x|}\frac{(-|y|^2+2x\cdot y)^2}{|x-y|^2}e^{-(2\sqrt{e})|y|}\mathrm{d}y \leqslant e^{-\sqrt{e}|x|}\int_{|y-x|<|x|}\frac{(|y|^2+2|x||y|)^2}{|x-y|^2}\mathrm{d}y \leqslant |x|^5e^{-\sqrt{e}|x|}C. \tag{2.22}$$

Therefore, for all x,

$$f_1(x) \geqslant -\frac{1}{|x|^4}C(e^{-\frac{1}{2}} + e^2|x|^4e^{-\sqrt{e}|x|}). \tag{2.23}$$

Finally, by use (2.13),

$$|x|^4 \left(\delta(x) + \frac{(\beta+1)e}{\pi} \frac{e^{-2\sqrt{e}|x|}}{|x|} \right) * f_1 * f_2(x) \geqslant -Ce^{-\frac{1}{2}}. \tag{2.24}$$

All in all, by (2.10), (since $|x|^4 e^{\frac{3}{2}} e^{-2\sqrt{e}|x|} < Ce^{-\frac{1}{2}}$)

$$|x|^4 U_1(x) \leqslant Ce^{-\frac{1}{2}}. \tag{2.25}$$

2 - We now show that $\Delta^2 \widehat{U}_2$ is integrable and square-integrable. We use the fact that

$$16e^2 \Delta^2 \equiv \partial_\kappa^4 + \frac{4}{\kappa} \partial_\kappa^3. \tag{2.26}$$

One can derive this by using radial Laplacian.

$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial f}{\partial r}) + \text{non-radial derivatives.}$$

We have, by the Leibniz rule,

$$\partial_\kappa^n \widehat{U}_2 = \sum_{i=0}^n \binom{n}{i} \left(\partial_\kappa^{n-i} (\kappa^2+1) \partial_\kappa^i (1 - \sqrt{1-2\zeta_1}) - \partial_\kappa^{n-i} (\kappa^2+1)^{-2} \partial_\kappa^i (1 - \sqrt{1-2\zeta_2}) \right). \tag{2.27}$$

Furthermore,

$$\partial_\kappa^n (1 - \sqrt{1-2\zeta_j}) = \sum_{p=1}^n \partial_{\zeta_j}^p (1 - \sqrt{1-2\zeta_j}) \sum_{\substack{l_1,\cdots,l_p \in \{1,\cdots,n\} \\ l_1+\cdots+l_p=n}} c_{l_1,\cdots,l_p}^{(p,n)} \prod_{i=1}^n \partial_\kappa^{l_i} \zeta_j \tag{2.28}$$

for some family of constants $c_{l_1,\cdots,l_p}^{(p,n)}$ which can easily be computed explicitly, but this is not needed. Now, since $S \geqslant 0$, $\frac{\rho}{1e}|\widehat{S}| \leqslant 1$, so $|\zeta_1| \leqslant \frac{1}{2}$ and $\zeta_1 = \frac{1}{2}$ if and only if $\kappa = 0$. Therefore, $\widehat{U}_2$ is bounded when κ is away from 0, so it suffices to show that $\Delta^2 \widehat{U}_2$ is integrable and square integrable at infinity and at 0.

2-1 - We first consider the behavior at infinity, and assume that κ is sufficiently large. The fact that $\partial_\kappa^{n-i}(\kappa^2+1)^{-2}\partial_\kappa^i(1-\sqrt{1-2\zeta_2})$ is integrable and square integrable at infinity follows immediately from (2.7). To prove the corresponding claim for ζ_1, we use the fact that $|x|^4 v$ square integrable, which implies that $\widehat{S}$ is as well. Therefore, by (2.7) for $0 \leqslant n \leqslant 4$, $\kappa^2 \partial_\kappa^n \zeta_1$ is integrable at infinity, and, therefore, square-integrable at infinity. Furthermore, by (2.7), $\zeta_1 < \frac{1}{2} - \varepsilon$ for large κ, and $\partial^n \zeta_1$ is bounded, so $\partial_\kappa^{n-i}(\kappa^2+1)\partial_\kappa^i(1-\sqrt{1-2\zeta_1})$ is integrable and square integrable.

2-2 - As $\kappa \to 0$

$$\zeta_i = \frac{1}{2}(1 - (\beta+2)\kappa^2) + O(\kappa^4) \tag{2.29}$$

and since $\beta \geqslant 0$,

$$1 - 2\zeta_i \geqslant \kappa^2 + O(\kappa^4). \tag{2.30}$$

therefore, for $p \geqslant 1$

$$\partial_{\zeta_j}^p (1 - \sqrt{1 - 2\zeta_j}) = O(\kappa^{1-2p}) \tag{2.31}$$

and, since ζ_i is $\mathcal{C}^4$, for $3 \leqslant n \leqslant 4$,

$$\partial\zeta_i = -(\beta + 2)\kappa + O(\kappa^3), \quad \partial^2\zeta_i = -(\beta + 2) + O(\kappa^2), \quad \partial^n\zeta_i = O(\kappa^{4-n}). \tag{2.32}$$

Therefore, for $1 \leqslant i \leqslant 4$, by (2.28)

$$\partial_\kappa^i (1 - \sqrt{1 - 2\zeta_1}) - \partial_\kappa^i (1 - \sqrt{1 - 2\zeta_2}) = O(\kappa^{3-i}) \tag{2.33}$$

and

$$\partial_\kappa^i (1 - \sqrt{1 - 2\zeta_1}) = O(\kappa^{1-i}), \quad \partial_\kappa^i (1 - \sqrt{1 - 2\zeta_2}) = O(\kappa^{1-i}). \tag{2.34}$$

Thus, by (2.27), as $\kappa \to 0$,

$$|\partial_\kappa^4 \widehat{U}_2| = O(\kappa^{-1}), \quad \frac{4}{\kappa} |\partial_\kappa^3 \widehat{U}_2| = O(\kappa^{-1}). \tag{2.35}$$

Thus, $\Delta^2 \widehat{U}_2$ is integrable and square integrable. And since the $O(\cdot)$ hold uniformly in e on all compact sets, by (2.26),

$$|x|^4 U_2(x) \leqslant \frac{8e^{\frac{3}{2}}}{16e^2} \int \left(\partial_{|k|}^4 + \frac{4}{|k|} \partial_{|k|}^3 \right) \hat{U}_2(|k|) \, dk \leqslant \frac{C}{\sqrt{e}}. \tag{2.36}$$

This along with (2.18) and (2.25) implies (1.26) and (1.28). □