Theorem 1.4 of
Analysis of a simple equation for the ground state energy of the Bose gas

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$$\boxed{(-\Delta + 4e + \mathcal{V}(x))u(x) = \mathcal{V}(x) + 2e\rho(u * u)(x)\ ,} \tag{1.1}$$

$$e = \frac{\rho}{2} \int (1 - u(x))\mathcal{V}(x)\ dx\ . \tag{1.2}$$

Theorem 1.4 (asymptotics of the energy for $d = 3$) *Consider the case $d = 3$. Let $\mathcal{V}$ be non-negative, integrable and square-integrable. Then, for each $\rho > 0$ there is at least one $e > 0$ such that $\rho = \rho(e)$. For any such ρ and e we have the following bounds for low and high density (i.e., small and large ρ). For low density,*

$$\boxed{e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)} \tag{1.23}$$

where a is the scattering length of the potential, which is defined in (4.11). For high density, in any dimension $d \geqslant 1$,

$$e = \frac{\rho}{2}\int \mathcal{V}(x)\ dx + o(\rho). \tag{1.24}$$

First: High density ρ

Lemma 4.1 (high density asymptotics) *If $\mathcal{V}$ is integrable, then as $\rho \to \infty$,*

$$e = \frac{\rho}{2}\left(\int \mathcal{V}(x)\, dx\right)(1 + o(1)).$$

Lemma 4.1

e $= \frac{\rho}{2}\int\big(1-u(x)\big)V(x)\ \mathrm{d}x$ = $\frac{\rho}{2}\left(\int V(x)\ \mathrm{d}x - \int u(x)V(x)\,\mathrm{d}x \right)$

→ Now if we show $\lim_{\rho\to\infty}\int u(x)V(x)\,dx = 0$

Then $\lim_{\rho\to\infty}\frac{|\int u(x)V(x)\,dx|}{1} = \mathrm{o}(1)$

So e = $\frac{\rho}{2}\left(\int V(x)\ \mathrm{d}x - \mathrm{o}(1) \right) = \frac{\rho}{2}\left(\int V(x)\ \mathrm{d}x \right)(1+\mathrm{o}(1))$

Lemma 4.1

We define $X_\gamma \coloneqq \{x : V(x) \geq \gamma\}$ (measurable)

$\int u(x)V(x)\,dx = \int_{X_\gamma} u(x)V(x)\ dx + \int_{R^n \setminus X_\gamma} u(x)V(x)\ dx$

We know: $\int u(x)\,dx \ \leq \frac{1}{\rho}$

So $\int_{R^n \setminus X_\gamma} u(x)V(x)\ dx \ \leq\ \gamma \int_{R^n \setminus X_\gamma} u(x)\,dx \leq \frac{\gamma}{\rho}$

Why $\int u(x)\,dx \le \frac{1}{\rho}$?

To prove this, consider the operator

$$G_e := [-\Delta + 4e]^{-1} \tag{1.8}$$

which is given by

$$G_e f = Y_{4e} * f \tag{1.9}$$

where Y_{4e} is the *Yukawa potential* [LL01, section 6.23], which is non-negative and $\int Y_{4e} dx = (4e)^{-1}$. When $d = 3$,

$$Y_{4e}(x) = \frac{e^{-2\sqrt{e}|x|}}{4\pi|x|}. \tag{1.10}$$

Why $\int u(x)\,dx \le \frac{1}{\rho}$?

$$(-\Delta + 4e + \mathcal{V}(x))u(x) = \mathcal{V}(x) + 2e\rho(u * u)(x) \;, \qquad (1.1)$$

(1.1) is equivalent to $(-\Delta + 4e)\, u_\rho(\mathrm{x}) = \left(1 - u_\rho\right)\mathrm{V}(\mathrm{x}) + 2\mathrm{e}\rho\left(u_\rho \star u_\rho\right)$

And with $u_\rho(\mathrm{x}) = (-\Delta + 4e)^{-1}\left(\left(1 - u_\rho\right)\mathrm{V}(\mathrm{x}) + 2\mathrm{e}\rho\left(u_\rho \star u_\rho\right)\right)$

We have $u_\rho(\mathrm{x}) = Y_{4e} * \left(\left(1 - u_\rho\right)\mathrm{V}(\mathrm{x})\right) + 2e\rho\; Y_{4e} * \left(u_\rho \star u_\rho\right)$

Why $\int u(x)\,dx \le \frac{1}{\rho}$?

Integrating yields

$$\int u(x)dx = \int\!\left(Y_{4e}\star \Big(V(x)\big(1-u(x)\big)\Big) + 2e\rho \cdot Y_{4e}\star \mathrm{u} \star \mathrm{u} \right)$$

$$= \int Y_{4e} \cdot \int V(x)\,\big(1-u(x)\big)\,\mathrm{dx} \quad + \quad 2e\rho \int Y_{4e} \cdot \int u \star u$$

$$= \frac{1}{4e} \int V(x)\,\big(1-u(x)\big)\,\mathrm{dx} + \frac{\rho}{2}\left(\int u(x)\,dx\right)^2$$

Why $\int u(x)\,dx \leq \frac{1}{\rho}$?

$$\rightarrow \frac{\rho}{2}\left(\int u(x)\,dx\right)^2 - \int u(x)\,dx + \frac{1}{4e}\int V(x)\left(1-u(x)\right)\mathrm{dx} = 0$$

With $e = \frac{\rho}{2}\int V(x)(1-u(x))\ \mathrm{dx}$ we have

$$\frac{\rho}{2}\left(\int u(x)\,dx\right)^2 - \int u(x)\,dx + \frac{1}{2\rho} = 0$$

$$\rightarrow \int u(x)\,dx = \frac{1}{\mathrm{\rho}}$$

Lemma 4.1

We have $\int_{R^n \setminus X_\gamma} u(x)V(x)\ dx \ \le \frac{\gamma}{\rho}$

$$\int_{X_\gamma} u(x)V(x)\ dx + \int_{R^n \setminus X_\gamma} u(x)V(x)\ dx$$

$$\le \int_{X_\gamma} u(x)V(x)\ dx + \frac{\gamma}{\rho} \le \int_{X_\gamma} V(x)\ dx + \frac{\gamma}{\rho}$$

Since $0 \le u(x) \le 1$

and $\inf\limits_{\gamma \ge 0}\left(\int_{X_\gamma} V(x)\ dx\ + \frac{\gamma}{\rho}\right) \to 0$

Why $\inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx + \frac{\gamma}{\rho} \right) \to 0$?

For all n in $\mathbb{N}$ we find γ_n in $\mathbb{R}$ s.t. for $\gamma \geq \gamma_n$

$$\int_{X_\gamma} V(x)\ dx \leq \frac{1}{n}$$

And ρ_n in $\mathbb{R}^+$ s.t. $\frac{\gamma_n}{\rho_n} \leq \frac{1}{n}$ so

$$\inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx + \frac{\gamma}{\rho_n} \right) \leq \int_{X_{\gamma_n}} V(x)\ dx + \frac{\gamma_n}{\rho_n} \leq \frac{2}{n}$$

Why $\inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx\ + \frac{\gamma}{\rho} \right) \to 0$?

And $\lim_{n \to \infty} \left(\int_{X_{\gamma_n}} V(x)\ dx\ + \frac{\gamma_n}{\rho_n} \right) \leq 0$

We also have $\lim_{n \to \infty} \inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx\ + \frac{\gamma}{\rho_n} \right) \leq \lim_{n \to \infty} \left(\int_{X_{\gamma_n}} V(x)\ dx\ + \frac{\gamma_n}{\rho_n} \right)$

And since ρ_n could be bounded

$\lim_{\rho \to \infty} \inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx\ + \frac{\gamma}{\rho} \right) \leq \lim_{n \to \infty} \inf_{\gamma \geq 0} \left(\int_{X_\gamma} V(x)\ dx\ + \frac{\gamma}{\rho_n} \right) \leq 0$

Lemma 4.1

Now we have

$$\lim_{\rho\to\infty}\int u(x)V(x)\,dx = 0$$

And therefore

e $=\frac{\rho}{2}\left(\int V(x)\,dx\right)(1+o(1))$ for $\rho\to\infty$

Now: low density ρ

Lemma 4.3 (low density asymptotics) *If $\mathcal{V}$ is non-negative and integrable and $d = 3$, then*

$$e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right). \tag{4.15}$$

Where a is defined by

$$4\pi a = \int \mathcal{V}(x)(1 - \varphi(x))dx \ .$$

Now: low density ρ

Where φ is defined by the scattering equation

$$-\Delta\varphi(x) = (1-\varphi(x))\mathcal{V}(x), \qquad \lim_{|x|\to\infty}\varphi(x) = 0\ . \tag{4.8}$$

Note that (4.8) can be written as $(-\Delta+\mathcal{V})\varphi = \mathcal{V}$, and hence the solution is

$$\varphi(x) = \lim_{e\downarrow 0} K_e\mathcal{V}(x) = \lim_{e\downarrow 0} u_1(x,e)\ , \tag{4.9}$$

where u_1 is the first term of the iteration introduced in the previous section. It follows from Lemma 3.2 that $0 \leqslant \varphi(x) \leqslant 1$ for all x.

Scheme of proof

Proof: The scheme of the proof is as follows. We first approximate the solution u by w, which is defined as the decaying solution of

$$-\Delta w_\rho(x) = (1 - u_\rho(x))\mathcal{V}(x). \tag{4.16}$$

The energy of w_ρ is defined to be

$$e_w := \frac{\rho}{2}\int (1 - w_\rho(x))\mathcal{V}(x)\ dx \tag{4.17}$$

and, as we will show, it is *close* to e, more precisely,

$$e - e_w = \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\int \mathcal{V}(x)\ dx + o(\rho^{\frac{3}{2}}). \tag{4.18}$$

In addition, (4.16) is quite similar to the scattering equation (4.8). In fact we will show that e_w is *close* to the energy $2\pi\rho a$ of the scattering equation

$$e_w - 2\pi\rho a = -\frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\int \varphi(x)\mathcal{V}(x)\ dx + o(\rho^{\frac{3}{2}}). \tag{4.19}$$

Scheme of proof

Summing (4.18) and (4.19) : $e - e_\omega + e_\omega - 2\pi\rho a$

$$= \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\int V(x)\,dx \;+\; o(\rho^{\frac{3}{2}}) \;-\; \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\int \varphi(x)\,V(x)\,dx \;+\; o\left(\rho^{\frac{3}{2}}\right)$$

$$= \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\int\big(1-\varphi(x)\big)\,V(x)\,dx + o\left(\rho^{\frac{3}{2}}\right)$$

We know $4\pi a = \int \mathcal{V}(x)(1-\varphi(x))dx$.

$$= \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}\cdot 4\pi a + \; o(\rho^{\frac{3}{2}})$$

Scheme of proof

$$\rightarrow e = 2\pi\rho a + \frac{64\sqrt{2}e^{\frac{3}{2}}}{15\pi} \cdot a + o\left(\rho^{\frac{3}{2}}\right) = 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o(\sqrt{\rho})\right)$$

Now we show that from $e = 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o(\sqrt{\rho})\right)$ $(*)$ follows

$$e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

as desired.

Why $e = 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o(\sqrt{\rho})\right) \Rightarrow e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$?

(1.22) says $$\left(\frac{1}{4}\int \mathcal{V}dx\right)\rho \leqslant e \leqslant \left(\frac{1}{2}\int \mathcal{V}dx\right)\rho\ .$$

So from $(*)$ we know $e = C_0\rho a + C_1 e^{\frac{3}{2}}$ for $C_0, C_1 > 0$

And $e^{\frac{3}{2}} \leq \frac{1}{2} e\ for\ small\ e$, so $e \leq C\rho a$ for C >0

And for f(e) = $\frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2}$ we have f ∈ $O(a\rho)^{\frac{3}{2}}$

$$\rightarrow e = 2\pi\rho a\left(1 + \frac{O(a\rho)^{\frac{3}{2}}}{\rho} + o(\sqrt{\rho})\right) = 2\pi\rho a\left(1 + O(\sqrt{\rho}) + o(\sqrt{\rho})\right) = 2\pi\rho a\left(1 + O(\sqrt{\rho})\right)$$

$\left(1+O(\sqrt{\rho})\right)^{\frac{3}{2}} \Rightarrow \left(1+O(\sqrt{\rho})\right)$

Let f be in O($\sqrt{\rho}$), and g ≔ $(1+f)^{\frac{3}{2}}$ with $g = 1 + \tilde{g}$

So $\tilde{g} \coloneqq (1+f)^{\frac{3}{2}} - 1$

$$\frac{\tilde{g}}{\sqrt{\rho}} = \frac{(1+f)^{\frac{3}{2}} - 1}{\sqrt{\rho}} = \frac{\left((1+f)^{\frac{3}{2}}\right)^2 - 1^2}{\sqrt{\rho}\cdot\left((1+f)^{\frac{3}{2}} + 1\right)} = \frac{(1+f)^3 - 1}{\sqrt{\rho}\cdot\left((1+f)^{\frac{3}{2}} + 1\right)}$$

$$a - b = \frac{a^2 - b^2}{a + b}$$

$\left(1+O(\sqrt{\rho})\right)^{\frac{3}{2}} \Rightarrow \left(1+O(\sqrt{\rho})\right)$

We know for $\rho \to 0$:

$$1 \le (1+f)^{\frac{3}{2}} + 1 \le (1+1)^{\frac{3}{2}} + 1 = \sqrt{8} + 1$$

since f(ρ) $< 1\, for\ \rho\ small\ enough$, so

$$\lim sup_{\rho\to 0} \frac{1}{\sqrt{8}+1}\frac{(1+f)^3-1}{\sqrt{\rho}} \le \lim sup_{\rho\to 0} \frac{(1+f)^3-1}{\sqrt{\rho}\cdot\left((1+f)^{\frac{3}{2}}+1\right)} \le \lim sup_{\rho\to 0} \frac{(1+f)^3-1}{\sqrt{\rho}}$$

And $(1+f)^3 - 1 = \ f^3 + 3f^2 + 3f$

$\left(1+O(\sqrt{\rho})\right)^{\frac{3}{2}} \Rightarrow \left(1+O(\sqrt{\rho})\right)$

And since f $\in O\left(\sqrt{\rho}\right) \Rightarrow f^3 \in O\left(\sqrt{\rho}^3\right) \subseteq O\left(\sqrt{\rho}\right)$ because $\frac{f^3}{\sqrt{\rho}} \leq \frac{f^3}{\sqrt{\rho}^3}$

(same with f^2)

$\Rightarrow f^3 + 3f^2 + 3f \in O\left(\sqrt{\rho}\right)$

$\Rightarrow \frac{1}{\sqrt{8}+1} C \leq \lim sup_{\rho \to 0} \frac{(1+f)^3 - 1}{\sqrt{\rho} \cdot \left((1+f)^{\frac{3}{2}} + 1\right)} \leq C$

So $\tilde{g} \in O\left(\sqrt{\rho}\right)$

Why $e = 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o\left(\sqrt{\rho}\right)\right) \Rightarrow e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o\left(\sqrt{\rho}\right)\right)$?

And $e^{\frac{3}{2}} = (2\pi\rho a)^{\frac{3}{2}}\left(1 + O(\sqrt{\rho})\right)^{\frac{3}{2}} = (2\pi\rho a)^{\frac{3}{2}}\left(1 + O(\sqrt{\rho})\right)$

So $= 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o\left(\sqrt{\rho}\right)\right) = 2\pi\rho a\left(1 + \frac{32\sqrt{2}}{15\pi^2\rho} \cdot (2\pi\rho a)^{\frac{3}{2}} \cdot \left(1 + O(\sqrt{\rho})\right) + o\left(\sqrt{\rho}\right)\right)$

$= 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \frac{\rho^{\frac{3}{2}}}{\rho}O(\sqrt{\rho}) + o\left(\sqrt{\rho}\right)\right)$

Why $\mathrm{e} = 2\pi\rho a\left(1 + \frac{32\sqrt{2}e^{\frac{3}{2}}}{15\pi^2\rho} + o\left(\sqrt{\rho}\right)\right) \Rightarrow \mathrm{e} = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o\left(\sqrt{\rho}\right)\right)$?

$$= 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \frac{\rho^{\frac{3}{2}}}{\rho} O(\sqrt{\rho}) + o\left(\sqrt{\rho}\right)\right)$$

$$= 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + \rho^{\frac{1}{2}} O(\sqrt{\rho}) + o\left(\sqrt{\rho}\right)\right)$$

$$= 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + O(\rho) + o\left(\sqrt{\rho}\right)\right)$$

But $O(\rho)\ is\ in\ o\left(\sqrt{\rho}\right)$ since $\rho \to 0$

Proof of (4.18)

Now we want to proof the following

$$e - e_w = \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2} \int \mathcal{V}(x)\ dx + o(\rho^{\frac{3}{2}}). \tag{4.18}$$

And therefore we look at the Fourier Transforms of $u_\rho(x)$ and $\omega_\rho(x)$

$\hat{u}_\rho = \int e^{ikx}\, u_\rho(x)\, dx$ $\qquad$ $\widehat{\omega}_\rho = \int e^{ikx}\, \omega_\rho(x)\, dx$

Proof of (4.18)

From $$\left(-\Delta + 4e + V(x)\right) u_\rho(\mathrm{x}) = \mathrm{V(x)} + 2\mathrm{e}\rho\left(u_\rho \star u_\rho\right) \qquad (1.1)$$

$$\Leftrightarrow (-\Delta + 4e)\, u_\rho(\mathrm{x}) = \left(1 - u_\rho\right)\mathrm{V(x)} + 2\mathrm{e}\rho\left(u_\rho \star u_\rho\right)$$

And the Fourier Transform of that is

Actually $|k|^2$

$$(k^2 + 4e)\, \hat{u}_\rho(k) = \frac{2e}{\rho}\mathrm{S(k)} + 2\mathrm{e}\rho\hat{u}_\rho(k)^2$$

with $S(k) = \frac{\rho}{2e}\int e^{ikx}\left(1 - u_\rho(x)\right)\mathrm{V(x)}\, dx$

Proof of (4.18)

So $2e\rho\, \hat{u}_\rho(k)^2 + (k^2 + 4e)\, \hat{u}_\rho(k) + \frac{2e}{\rho}\mathrm{S(k)} = 0$

And solving this equation leeds to

$$\hat{u}_\rho(k) = \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e} + 1\right)^2 - S(k)}\right)$$

With a similar calculation we get $\widehat{\omega}_\rho(k) = \frac{2eS(k)}{\rho k^2}$

$\hat{u}_\rho(k) - \widehat{\omega}_\rho(k)$ is integrable

We want to do the invert Fourier Transform on $\hat{u}_\rho(k) - \widehat{\omega}_\rho(k)$ so we have to show that this is integrable.

$$\hat{u}_\rho(k) = \frac{1}{\rho}\left(\frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}\right) = \frac{1}{\rho}\,\frac{\left(\frac{k^2}{4e}+1\right)^2 - \left(\frac{k^2}{4e}+1\right)^2 + S(k)}{\frac{k^2}{4e} + 1 + \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}}$$

$$= \frac{1}{\rho}\,\frac{S(k)}{\frac{k^2}{4e} + 1 + \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}}$$

$$a - b = \frac{a^2 - b^2}{a + b}$$

$\hat{u}_\rho(k) - \widehat{\omega}_\rho(k)$ is integrable

If we look at $|k| \to \infty$,

We have $\hat{u}_\rho(k) - \widehat{\omega}_\rho(k) =$

$$\frac{1}{\rho}\ \frac{S(k)}{\frac{k^2}{4e} + 1 + \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}} - \frac{2eS(k)}{\rho k^2} = \frac{S(k)}{\rho}\left(\frac{1}{\frac{k^2}{4e} + 1 + \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}} - \frac{1}{2\left(\frac{k^2}{4e}\right)}\right)$$

$$= \frac{S(k)}{\rho}\left(\underbrace{\frac{1}{\frac{k^2}{4e} + 1 + \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - S(k)}} - \frac{1}{2\left(\frac{k^2}{4e}+1\right)}}_{O\left(\frac{1}{|k|^6}\right)} + \underbrace{\frac{1}{2\left(\frac{k^2}{4e}+1\right)} - \frac{1}{2\left(\frac{k^2}{4e}\right)}}_{O\left(\frac{1}{|k|^4}\right)}\right)$$

$$\frac{1}{\frac{k^2}{4e}+1+\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}}-\frac{1}{2\left(\frac{k^2}{4e}+1\right)} \text{ is in } O\left(\frac{1}{|k|^6}\right)$$

$$\frac{1}{\frac{k^2}{4e}+1+\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}}-\frac{1}{2\left(\frac{k^2}{4e}+1\right)}=\frac{\left(\frac{k^2}{4e}+1\right)-\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}}{\left(\frac{k^2}{4e}+1+\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}\right)\cdot 2\left(\frac{k^2}{4e}+1\right)}$$

$$a-b=\frac{a^2-b^2}{a+b}$$

$$=\frac{S(k)}{\left(\frac{k^2}{4e}+1+\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}\right)^2\cdot 2\left(\frac{k^2}{4e}+1\right)} \sim O\left(\frac{1}{|k|^6}\right)$$

$\frac{1}{2\left(\frac{k^2}{4e}+1\right)} - \frac{1}{2\left(\frac{k^2}{4e}\right)}$ is in $O(\frac{1}{|k|^4})$

$$\frac{1}{2\left(\frac{k^2}{4e}+1\right)} - \frac{1}{2\left(\frac{k^2}{4e}\right)} = \frac{2\left(\frac{k^2}{4e}\right)-2\left(\frac{k^2}{4e}+1\right)}{2\left(\frac{k^2}{4e}+1\right)\cdot 2\left(\frac{k^2}{4e}\right)} = \frac{-2}{2\left(\frac{k^2}{4e}+1\right)\cdot 2\left(\frac{k^2}{4e}\right)} \quad \sim O\left(\frac{1}{|k|^4}\right)$$

$\hat{u}_\rho(k) - \widehat{\omega}_\rho(k)$ is integrable

$$\hat{u}_\rho(k) - \widehat{\omega}_\rho(k) = \frac{S(k)}{\rho}\left(O\left(\frac{1}{|k|^6}\right) + O\left(\frac{1}{|k|^4}\right)\right) \quad as \ \ |k| \to \infty$$

S(K) is bounded : $|S(k)| \leq \frac{\rho}{2e}\int \left|e^{ikx}\left(1 - u_\rho(x)\right)\mathrm{V}(\mathrm{x})\right| dx \leq \frac{\rho}{2e}\int\left(1 - u_\rho(x)\right)\mathrm{V}(\mathrm{x})dx = 1$

(Recall $\mathrm{e} = \frac{\rho}{2}\int\left(1 - u_\rho(x)\right)V(x)\ \mathrm{d}x$)

and since $\hat{u}_\rho(k)$ *and* $\widehat{\omega}_\rho(k)$ *are continuous, and therefore bounded*
$\hat{u}_\rho(k) - \widehat{\omega}_\rho(k)$ *is integrabel for all k and we can invert the Fourier Transform*

Proof of (4.18)

$$u(x)-\omega(x)=\frac{1}{(2\pi)^3}\int e^{-ikx}\frac{1}{\rho}\left(\left(\frac{k^2}{4e}+1-\sqrt{\left(\frac{k^2}{4e}+1\right)^2-S(k)}\right)-\frac{2eS(k)}{\rho k^2}\right)dk$$

changing variables with μ(k)= $2\sqrt{e}k$, $(|\det(D\mu\ )\ (k)|=(2\sqrt{e})^3)$:

$$u(x)-\omega(x)=\frac{e^{\frac{3}{2}}}{\pi^3\rho}\int e^{-i2\sqrt{e}\hat{k}x}\left(\left(\hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-S\left(2\hat{k}\sqrt{e}\right)}\right)-\frac{S\left(2\hat{k}\sqrt{e}\right)}{2\hat{k}^2}\right)d\hat{k}$$

Proof of (4.18)

We want to find an integrable majorant for $\hat{k}^2 + 1 - \sqrt{\left(\hat{k}^2+1\right)^2 - S\left(2\hat{k}\sqrt{e}\right)} - \frac{S\left(2\hat{k}\sqrt{e}\right)}{2\hat{k}^2}$

Let $a = \hat{k}^2 + 1$ and $s = S\left(2\hat{k}\sqrt{e}\right)$, We have $a \geq 1 \geq |s|$ and

$$\left|a - \sqrt{a^2 - s} - \frac{s}{2a-2}\right| = \left|a - \sqrt{a^2 - s} - \frac{s}{2a} + \frac{s}{2a} - \frac{s}{2a-2}\right|$$

$$\leq \left|a - \sqrt{a^2 - s} - \frac{s}{2a}\right| + \left|\frac{s}{2a} - \frac{s}{2a-2}\right|$$

Proof of (4.18)

$$a-\sqrt{a^2-s}a-\frac{s}{2a}=\frac{s}{a+\sqrt{a^2-s}}-\frac{s}{2a}=\frac{s}{2a\left(a+\sqrt{a^2-s}\right)}\cdot\left(a-\sqrt{a^2-s}\right)$$
$$=\frac{s^2}{2a\left(a+\sqrt{a^2-s}\right)^2}$$

And $\left|\frac{s^2}{2a\left(a+\sqrt{a^2-s}\right)^2}\right|\le\frac{1}{2a^3}$

Proof of (4.18)

So in general

$$\left|a - \sqrt{a^2 - s} - \frac{s}{2a}\right| + \left|\frac{s}{2a} - \frac{s}{2a-2}\right| \leq \frac{1}{2a^3} + \frac{1}{4a(a-1)} \leq \frac{1}{\hat{k}^2\left(\hat{k}^2 + 1\right)}$$

Wich is integrable on $\mathbb{R}^3 \backslash B_r(0)\ for\ r > 0$

Now we would need a different majorant which is integrable on $B_r(0)$

But we just assume we already found one.

Proof of (4.18)

we want to use dominated convergency to say

$$\lim_{e\to 0}\int e^{-i2\sqrt{e}\hat{k}x}\left(\hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-S\left(2\hat{k}\sqrt{e}\right)}-\frac{S\left(2\hat{k}\sqrt{e}\right)}{2\hat{k}^2}\right)d\hat{k}$$

$$=\int e^0\left(\hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-S(0)}-\frac{S(0)}{2\hat{k}^2}\right)d\hat{k}$$

$$=\int \hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-1}-\frac{1}{2\hat{k}^2}\,d\hat{k}$$

Since $S(0)\ \leq \frac{\rho}{2e}\int e^0\left(1-u_\rho(x)\right)\mathrm{V(x)}dx \leq \frac{\rho}{2e}\int\left(1-u_\rho(x)\right)\mathrm{V(x)}dx = 1$

Proof of (4.18)

$$\int \hat{k}^2 + 1 - \sqrt{\left(\hat{k}^2 + 1\right)^2 - 1} - \frac{1}{2\hat{k}^2}\, d\hat{k}$$

The integral depends only on $\left|\hat{k}\right| \rightarrow$ *change of variables* to Spherical coordinate system

$$\int_{R\times[0,2\pi)\times[0,\pi)} \left(r^2 + 1 - \sqrt{(r^2+1)^2 - 1} - \frac{1}{2r^2}\right) \cdot\ r^2\, d(r \times \varphi \times \theta)$$

$$=4\pi \int_0^\infty \left(r^2 + 1 - \sqrt{(r^2+1)^2 - 1} - \frac{1}{2r^2}\right) \cdot\ r^2\, dr$$

Proof (4.18)

$$=4\pi \int_0^\infty \left(\frac{1}{r^2+1+\sqrt{(r^2+1)^2-1}} - \frac{1}{2r^2}\right) \cdot\ r^2\, dr = \quad \frac{32\sqrt{2}\pi}{15}$$

$$a - b = \frac{a^2 - b^2}{a + b}$$

Why is that integrable?

$$\frac{1}{r^2+1+\sqrt{(r^2+1)^2-1}} - \frac{1}{2r^2} = \frac{r^2-1-\sqrt{(r^2+1)^2-1}}{\left(r^2+1+\sqrt{(r^2+1)^2-1}\right)\cdot 2r^2} = \frac{\left(r^2-1\right)^2 - \left(r^2+1\right)^2 - 1}{\left(r^2+1+\sqrt{(r^2+1)^2-1}\right)\cdot 2r^2 \cdot \left(r^2-1+\sqrt{(r^2+1)^2-1}\right)}$$

$$= \frac{-4r^2 - 1}{\left(r^2 + 1 + \sqrt{(r^2+1)^2 - 1}\right)\cdot 2r^2 \cdot \left(r^2 - 1 - \sqrt{(r^2+1)^2 - 1}\right)} \sim O\left(\frac{1}{r^4}\right)$$

Proof of (4.18)

By dominated convergence and because S(0) = 1 we have

$$\lim_{e\to 0}\frac{1}{e^{\frac{3}{2}}}(e-e_\omega)=\lim_{e\to 0}\frac{\rho}{2e^{\frac{3}{2}}}\int\left(u_\rho-\omega_\rho\right)V(x)\,dx$$

$$=-\lim_{e\to 0}\frac{1}{2}\int V(x)\left(\frac{1}{\pi^3}\int\left(\hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-S\left(2\hat{k}\sqrt{e}\right)}-\frac{S\left(2\hat{k}\sqrt{e}\right)}{2\hat{k}^2}\right)d\hat{k}\right)dx$$

$$=-\frac{1}{2}\int V(x)\left(\frac{1}{\pi^3}\int\left(\hat{k}^2+1-\sqrt{\left(\hat{k}^2+1\right)^2-1}-\frac{1}{2\hat{k}^2}\right)d\hat{k}\right)d\mathrm{x}$$

$$=\frac{16\sqrt{2}}{15\pi^2}\cdot\int V(x)\,dx$$

Proof of (4.18)

$$\lim_{e \to 0} \frac{1}{e^{\frac{3}{2}}}(e - e_\omega) = \frac{16\sqrt{2}}{15\pi^2} \cdot \int V(x)\, dx \;\Leftrightarrow\; \lim_{e \to 0}(e - e_\omega) = \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2} \cdot \int V(x)\, dx$$

And since

$$\left(\frac{1}{4}\int \mathcal{V} dx\right)\rho \leqslant e \leqslant \left(\frac{1}{2}\int \mathcal{V} dx\right)\rho \, .$$

$$e - e_w = \frac{16\sqrt{2}e^{\frac{3}{2}}}{15\pi^2} \int \mathcal{V}(x)\, dx + o(\rho^{\frac{3}{2}}). \qquad (4.18)$$

as desired