Homework Sheet 12

(Released 5.2.2021 - Discussed 11.2.2021)

12.1. Given $\{R_k\}_{k=1}^M \in \mathbb{R}^3$ and $Z_k > 0$. Consider the Thomas–Fermi functional

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^3} (\rho(x)^{5/3} - V(x)\rho(x))\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y, \quad V(x) = \sum_{k=1}^M \frac{Z_k}{|x-R_k|}.$$

(a) Prove that for every $m > 0$ the following minimization problem

$$E^{\mathrm{TF}} = \inf_{\substack{0\le\rho\in L^1(\mathbb{R}^3)\cap L^{5/3}(\mathbb{R}^3)\\ \int\rho\le m}} \mathcal{E}(\rho)$$

has a unique minimizer ρ_0.

(b) Prove that the minimizer ρ_0 solves the Thomas–Fermi equation

$$\frac{5}{3}\rho_0(x)^{2/3} = [V(x) - \rho_0 * |x|^{-1} - \mu]_+ \quad \text{for a.e.} x \in \mathbb{R}^3$$

with a constant $\mu \in \mathbb{R}$.

(c) Prove that

$$\int_{\mathbb{R}^3} \rho_0 = \min\left\{m, \sum_{k=1}^M Z_k\right\}.$$

12.2. For any open bounded set $\Omega \subset \mathbb{R}^d$, we denote the energy $E(\Omega) \in (-\infty, 0]$. Assume that we have the following properties

- (Translation-invariant) $E(\Omega + z) = E(\Omega)$ for all $z \in \mathbb{R}^3$.
- (Sub-additivity) $E(\Omega_1 \cup \Omega_2) \le E(\Omega_1) + E(\Omega_2)$ if $\Omega_1 \cap \Omega_2 = \emptyset$.
- (Stability) $E(\Omega) \ge -C|\Omega|$.

Prove that the following thermodynamic limit exists

$$\lim_{\substack{\Omega=[-L,L]^3\\ L\to\infty}} \frac{E(\Omega)}{|\Omega|}.$$

Homework Sheet 11

(Released 29.1.2021 - Discussed 4.2.2021)

11.1. On the fermionic Fock space $\mathcal{F}(L^2(\mathbb{T}^3))$ denote the annihilation operators $a_p = a(u_p)$ with $u_p(x) = (2\pi)^{-3/2}e^{-ip\cdot x}$, $\forall p \in \mathbb{Z}^3$. Let A be a non-negative trace class operator on $L^2(\mathbb{T}^3)$. Let $0 \neq k \in \mathbb{Z}$ and define

$$\mathbb{A} = \sum_{p,q\in\mathbb{Z}^3} \langle u_p, Au_q\rangle b_p^*(k)b_q(k), \quad b_p^*(k) = a_p^* a_{p-k}^*.$$

Prove that

$$0 \leq \mathbb{A} \leq \|A\|_{\mathrm{op}}\mathcal{N}, \quad \mathcal{N} = \sum_{p\in\mathbb{Z}^3} a_p^* a_p.$$

11.2. Let $f : \mathbb{N} \to \mathbb{R}$ be a decreasing function satisfying that

$$f(m+n) \leq f(m) + f(n), \quad \forall m, n \in \mathbb{N}$$

and that

$$f(n) \geq -Cn, \quad \forall n \in \mathbb{N}$$

for some constant $C > 0$. Prove that the limit $\lim_{n\to\infty} f(n)/n$ exists.

11.3. Let $\{Z_k\}_{k=1}^K$ be positive numbers and let $\{R_k\}_{k=1}^K$ be distinct points in $\mathbb{R}^3$. Let $0 \leq f \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$. Consider the minimization problem

$$E = \inf_{0\leq g\leq f} \left\{ \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{g(x)g(y)}{|x-y|} - \int_{\mathbb{R}^3} g(x)V(x) \right\} \quad \text{with } V(x) := \sum_{k=1}^K \frac{Z_k}{|x-R_k|}.$$

(a) Prove that E has a unique minimizer g_0.
(b) Prove that $g_0 * |x|^{-1} \leq V(x)$ for a.e. $x \in \mathbb{R}^3$.
(c) Prove that $g_0 * |x|^{-1} = V(x)$ on $\{x : g_0(x) < f(x)\}$.

Homework Sheet 10

(Released 22.1.2021 - Discussed 28.1.2021)

10.1. Consider the atomic Hartree–Fock energy with $Z > 0$, $N \in \mathbb{N}$,

$$E^{\rm HF} = \inf_{\substack{0\le\gamma=\gamma^2\le 1\\ \operatorname{Tr}\gamma=N}} \Big(\operatorname{Tr}((-\Delta - Z|x|^{-1})\gamma) + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2}{|x-y|} {\rm d}x{\rm d}y. \Big)$$

Assume that $E^{\rm HF}$ has a minimizer $\gamma^{\rm HF}$. Define the mean–field operator

$$h = -\Delta - Z|x|^{-1} + \rho_{\gamma^{\rm HF}} * |x|^{-1} - G$$

where G is the operator on $L^2(\mathbb{R}^3)$ with kernel $G(x,y) = \gamma^{\rm HF}(x,y)|x-y|^{-1}$.

(a) Prove that $\gamma^{\rm HF} = \sum_{i=1}^N |u_i\rangle\langle u_i|$ where all u_i's are eigenfunctions of h with N lowest eigenvalues.

(b) Assume further that there exists a constant $\mu < 0$ such that $\operatorname{Tr} \mathbb{1}(h \le \mu) = N$. Prove that if Z is large enough, then $\gamma^{\rm HF}$ is the unique minnimizer for $E^{\rm HF}$.

10.2. Prove that for every $f \in L^2(\Omega)$, we have $(a^*(f))^2 = 0$ on the fermionic Fock space $\mathcal{F}(L^2(\Omega))$.

10.3. Prove that for all $f \in L^2(\Omega)$, we have

$$\langle a^*(f)\Psi, \Phi\rangle_{\mathcal{F}} = \langle \Psi, a(f)\Phi\rangle_{\mathcal{F}}, \quad \forall \Psi, \Phi \in \mathcal{F}(L^2(\Omega)).$$

10.4. Let W be a self-adjoint operator on $L^2(\Omega^2)$ such that $W_{12} = W_{21}$. Let $\{u_n\}_{n\ge1}$ be an orthonormal basis for $L^2(\Omega)$. Prove that in the fermionic Fock space $\mathcal{F}(L^2(\Omega))$, we have

$$\bigoplus_{n=0}^{\infty} \Big(\sum_{1\le i<j\le n} W_{ij} \Big) = \frac{1}{2} \sum_{m,n,p,q\ge 1} \langle u_m \otimes u_n, W u_p \otimes u_q\rangle \quad a^*(u_m)a^*(u_n)a(u_q)a(u_p).$$

10.5. Let Ψ be a normalized vector in the fermionic Fock space $\mathcal{F}(L^2(\Omega))$ with $\langle \Psi, \mathcal{N}\Psi\rangle < \infty$. Prove that its one-body density matrix satisfies

$$0 \le \gamma_\Psi^{(1)} \le 1, \quad \operatorname{Tr} \gamma_\Psi^{(1)} = \langle \Psi, \mathcal{N}\Psi\rangle.$$

Homework Sheet 9

(Released 15.1.2021 - Discussed 21.1.2021)

9.1. Prove Vitali covering lemma: for any family $\{B_j\}_J$ of balls in $\mathbb{R}^d$ such that $\sup_{j\in J}\operatorname{diam}(B_j)<\infty$, there exists a subfamily of disjoint balls $\{B_j\}_{J'}$ such that

$$\bigcup_{j\in J} B_j \subset \bigcup_{j\in J'} 5B_j.$$

Here if $B_j = B(x_j, r_j)$, then $5B_j = B(x_j, 5r_j)$.

9.2. Let $d \geq 1$ and $\lambda \in (0,d)$. Prove that for every normalized wave function $\Psi \in L^2_a(\mathbb{R}^{dN})$ we have the Lieb–Oxford inequality

$$\left\langle \Psi, \sum_{1\le i<j\le N} \frac{1}{|x_i-x_j|^\lambda}\Psi \right\rangle \geq \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\rho_\Psi(x)\rho_\Psi(y)}{|x-y|^\lambda}\mathrm{d}x\mathrm{d}y - C\int_{\mathbb{R}^d}\rho_\Psi(x)^{1+\lambda/d}\mathrm{d}x.$$

Here the constant $C = C(d,\lambda)$ is independent of N and Ψ.

9.3. Prove that for every constant $\mu > 0$, when $N\to\infty$ we have

$$\operatorname{Tr}(-N^{-2/3}\Delta - |x|^{-1} + \mu)_- + NL^{\mathrm{cl}}_{1,3}\int_{\mathbb{R}^3}(|x|^{-1}-\mu)_+^{5/2}\mathrm{d}x = \frac{1}{8}N^{2/3} + O(N^{1/3}).$$

(You can use the spectral property of $-h^2\Delta - |x|^{-1}$ on $L^2(\mathbb{R}^3)$.)

9.4. Let A be a nonnegative trace class operator on $L^2(\mathbb{R}^d)$. Prove that

$$\sum_{i=1}^N A_i \leq \operatorname{Tr}(A) \quad \text{ on } L^2_a(\mathbb{R}^{dN}).$$

9.5. Let $0 \le V \in C^\infty_c(\mathbb{R}^3)$. Prove that

$$\operatorname{Tr}(-\Delta - \kappa V)_- = -L^{\mathrm{cl}}_{1,3}\int_{\mathbb{R}^3}|\kappa V|^{5/2} + O(\kappa^{5/2-\varepsilon})_{\kappa\to\infty}$$

for some constant $\varepsilon > 0$. Try to get ε as large as possible.

Homework Sheet 8

(Released 8.1.2021 - Discussed 14.1.2021)

8.1. In this exercise we discuss the Lewin–Lieb–Seiringer construction of a trial density matrix for the kinetic energy functional. For $\rho \geq 0$, $\sqrt{\rho} \in H^1(\mathbb{R}^d)$, define

$$\gamma = \int_0^\infty \frac{\mathrm{d}t}{t} \varphi\Big(\frac{t}{\rho(x)}\Big) \mathbb{1}\Big(-\Delta \leq \frac{d+2}{d} K_d^{\mathrm{cl}} t^{2/d}\Big) \varphi\Big(\frac{t}{\rho(x)}\Big) \quad \text{on } L^2(\mathbb{R}^d)$$

where $\varphi\Big(\frac{t}{\rho(x)}\Big)$ is the multiplication operator on $L^2(\mathbb{R}^d)$ with a given function

$$0 \leq \varphi \in C_c^\infty(0,\infty), \quad \int_0^\infty \varphi(t)^2 \mathrm{d}t = 1, \quad \int_0^\infty \frac{\varphi(t)^2}{t} \mathrm{d}t \leq 1.$$

(a) Prove that $0 \leq \gamma \leq 1$, $\rho_\gamma = \rho$ and

$$\mathrm{Tr}(-\Delta\gamma) = K_d^{\mathrm{cl}} \int_{\mathbb{R}^d} \rho^{1+2/d} \int_0^\infty \varphi(t)^2 t^{2/d} \mathrm{d}t + 4 \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 \int_0^\infty \varphi'(t)^2 t^2 \mathrm{d}t$$

(b) Prove that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ independent of ρ such that

$$\inf_\varphi \mathrm{Tr}(-\Delta\gamma) \leq K_d^{\mathrm{cl}}(1+\varepsilon) \int_{\mathbb{R}^d} \rho^{1+2/d} + C_\varepsilon \int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2.$$

8.2. (a) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} f(x)\varphi(x)\mathrm{d}x \geq 0, \quad \forall 0 \leq \varphi \in C_c^\infty(\mathbb{R}^d)$. Prove that $f(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$.

(b) Prove that if $f_n(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$ for all $n \in \mathbb{N}$ and $f_n \rightharpoonup f$ weakly in $L^p(\mathbb{R}^d)$ as $n \to \infty$ for some $p \in (1,\infty)$, then $f(x) \geq 0$ for a.e. $x \in \mathbb{R}^d$.

8.3. Given real–valued functions $V, w \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$ with $p, q \in [1 + d/2, \infty)$ and $\widehat{w} \geq 0$. Recall the Thomas–Fermi functional

$$\mathcal{E}^{\mathrm{TF}}(f) := K_d^{\mathrm{cl}} \int_{\mathbb{R}^d} f^{1+2/d} + \int_{\mathbb{R}^d} Vf + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x)f(y)w(x-y)\mathrm{d}x\mathrm{d}y.$$

Prove that the variational problem

$$E^{\mathrm{TF}} := \inf\left\{ \mathcal{E}^{\mathrm{TF}}(f) : 0 \leq f \in L^1(\mathbb{R}^d) \cap L^{1+2/d}(\mathbb{R}^d), \int_{\mathbb{R}^d} f \leq 1 \right\}$$

has a minimizer f^{TF}. Moreover,

$$E^{\mathrm{TF}} = \inf\left\{ \mathcal{E}^{\mathrm{TF}}(f) : 0 \leq f \in L^1(\mathbb{R}^d) \cap L^{1+2/d}(\mathbb{R}^d), \int_{\mathbb{R}^d} f = 1 \right\}.$$

Homework Sheet 7

(Released 21.12.2020 - Discussed 7.1.2021)

7.1. Let $\{u_i\}_{i=1}^N$ be orthonormal functions in $L^2(\mathbb{R}^d)$ and consider the Slater determinant $\Psi_N = u_1 \wedge u_2 \wedge ... \wedge u_N$.

(i) Prove that the one–body density matrix of Ψ_N is

$$\gamma^{(1)}_{\Psi_N} = \sum_{i=1}^{N} |u_i\rangle\langle u_i|.$$

(ii) Prove that for every interaction potential $w : \mathbb{R}^d \to \mathbb{R}$, $w(x) = w(-x)$,

$$\Big\langle \Psi_N, \sum_{1\le i<j\le N} w(x_i - x_j)\Psi_N \Big\rangle = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\rho_{\Psi_N}(x)\rho_{\Psi_N}(y) - |\gamma^{(1)}_{\Psi_N}(x,y)|^2\Big) w(x-y)\mathrm{d}x\mathrm{d}y.$$

7.2. Let γ be a trace class operator on $L^2(\mathbb{R}^d)$ such that

$$0 \le \gamma \le 1 \quad \text{on } L^2(\mathbb{R}^d), \quad \operatorname{Tr}\gamma = N \in \mathbb{N}.$$

Assume further that γ has $N-1$ eigenvalues equal to 1, but γ is not a projection. Prove that there exists no normalized function $\Psi_N \in L^2_a(\mathbb{R}^{dN})$ such that $\gamma^{(1)}_{\Psi_N} = \gamma$.

7.3. Consider the Thomas–Fermi functional

$$\mathcal{E}^{\mathrm{TF}}(f) = \frac{3}{5}(6\pi^2)^{2/3}\int_{\mathbb{R}^3} f^{5/3} - \int_{\mathbb{R}^3} \frac{f(x)}{|x|}\mathrm{d}x.$$

Prove that the variational problem

$$E = \inf\Big\{\mathcal{E}^{\mathrm{TF}}(f) \mid 0 \le f \in L^1(\mathbb{R}^3)\cap L^{5/3}(\mathbb{R}^3), \int_{\mathbb{R}^3} f \le 1\Big\}$$

has a unique minimizer f_0. Moreover, $\int_{\mathbb{R}^3} f_0 = 1$ and $E = -(3)^{1/3}/4$.

7.4. Let $d \ge 1$ and let $\mathbb{1}_{B_r}$ be the characteristic function of the ball $B(0,r)$ in $\mathbb{R}^d$. Prove that for every $0 < \lambda < d$, there exists a constant $C_{\lambda,d} > 0$ such that

$$\frac{1}{|x|^\lambda} = C_{\lambda,d}\int_0^\infty \frac{1}{r^{d+\lambda+1}}(\mathbb{1}_{B_r} * \mathbb{1}_{B_r})(x)dr, \quad \forall x \in \mathbb{R}^d\backslash\{0\}.$$

Homework Sheet 6

(Released 11.12.2020 - Discussed 17.12.2020)

6.1. Consider a non-negative trace class operator on $L^2(\mathbb{R}^d)$

$$\gamma = \sum_{n=1}^{\infty} \lambda_n |u_n\rangle\langle u_n|$$

where $\{u_n\}_{n=1}^{\infty}$ are orthonormal functions.

(i) Prove that

$$\mathrm{Tr}(-\Delta\gamma) := \mathrm{Tr}(\sqrt{-\Delta}\gamma\sqrt{-\Delta}) = \int_0^\infty \mathrm{d}\tau \sum_{n=1}^{\infty} \mathbb{1}(\lambda_n > \tau) \int_{\mathbb{R}^d} \mathrm{d}x |\nabla u_n(x)|^2.$$

(ii) Deduce that if $0 \leq \gamma \leq 1$, then

$$\mathrm{Tr}(-\Delta\gamma) \geq K_d \int_{\mathbb{R}^d} \rho_\gamma^{1+2/d}, \quad \rho_\gamma(x) = \sum_{n=1}^{\infty} \lambda_n |u_n(x)|^2$$

with the constant K_d in the Lieb–Thirring kinetic energy for orthonormal functions.

6.2. Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $\{v_n\} \subset H_0^1(\Omega)$ such that

$$0 \leq \gamma := \sum_{n=1}^{N} |v_n\rangle\langle v_n| \leq 1 \quad \text{on } L^2(\Omega).$$

Prove the following extension of the Berezin–Li-Yau inequality

$$\sum_{n=1}^{N} \int_\Omega |\nabla v_n|^2 \geq \frac{K_d^{\mathrm{cl}}}{|\Omega|^{2/d}} \Big(\mathrm{Tr}\,\gamma \Big)^{1+\frac{2}{d}}.$$

6.3. Let $\{\Omega_j\}_{j=1}^J$ be disjoint open sets in $\mathbb{R}^d$. Let $N_{\mathrm{D}}(\lambda, \Omega)$ be the number of eigenvalues $< \lambda$ of the Dirichlet Laplacian on $L^2(\Omega)$. Prove that

$$N_{\mathrm{D}}(\lambda, \Omega) \geq \sum_{j=1}^{J} N(\lambda, \Omega_j), \quad \Omega = \text{interior of } \Big(\overline{\bigcup \Omega_j}\Big).$$

(We have the reversed inequality for Neumann eigenvalues.)

6.4. Consider the operator $A = -\Delta$ on $L^2(0,1)$ with domain

$$D(A) = \{u \in H^2(0,1) \mid u(0) = 0, u'(1) = 0\}.$$

(i) Prove that A is a self-adjoint operator.

(ii) Prove that $A > 0$ and it has compact resolvent.

Homework Sheet 5

(Released 4.12.2020 - Discussed 10.12.2020)

5.1. Consider Bessel function $J_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$J_1(t) = \frac{1}{i\pi} \int_0^\pi e^{it\cos\theta} \cos\theta \mathrm{d}\theta.$$

Prove that $J_1(t) \leq Ct^{-1/2}$ for all $t > 0$.

5.2. Let $\Omega = \mathbb{R}^d \backslash \{0\}$. Prove that $H_0^1(\Omega) = H^1(\Omega)$ if and only if $d \geq 2$.

5.3. Let $d \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Define the extension $\widetilde{u} : \mathbb{R}^d \to \mathbb{C}$ by

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

Prove that if $u \in H_0^1(\Omega)$, then $\widetilde{u} \in H^1(\mathbb{R}^d)$ and

$$\nabla\widetilde{u}(x) = \begin{cases} \nabla u(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

5.4. Prove that the domain of the Neumann Laplacian on $L^2(0,1)$ is

$$D(-\Delta_{\mathrm{N}}) = \{u = H^2(0,1) \mid u'(0) = u'(1) = 0\}.$$

Homework Sheet 4

(Released 27.11.2020 - Discussed 3.12.2020)

4.1. Here we discuss a simplified proof of the upper bound for Weyl's law. Let $d \geq 1$. Assume that $V_- \in L^{1+\frac{d}{2}}(\mathbb{R}^d)$ and $V_+ \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ with $p \geq \max(1, d/2)$ if $d \neq 2$ and $p > 1$ if $d = 2$. Let $F_{k,y}(x) = e^{2\pi i k\cdot x} G(x-y)$ with a radial function $0 \leq G \in C_c^\infty(\mathbb{R}^d)$ satisfying $\|G\|_{L^2(\mathbb{R}^d)} = 1$ and define the operator on $L^2(\mathbb{R}^d)$

$$\widetilde{\gamma} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |F_{k,y}\rangle\langle F_{k,y}| \mathbb{1}\Big(|2\pi k|^2 + \lambda(G^2 * V)(y) + \|\nabla G\|_{L^2}^2 < 0\Big) \mathrm{d}k\mathrm{d}y.$$

(i) Prove that

$$\mathrm{Tr}((-\Delta + \lambda V)\widetilde{\gamma}) = -L^{\mathrm{cl}}_{1,d} \int_{\mathbb{R}^d} \Big|\big(\lambda G^2 * V + \|\nabla G\|_{L^2}^2\big)_-\Big|^{1+\frac{d}{2}}.$$

(ii) Using an appropriate choice of G to deduce that

$$\limsup_{\lambda\to\infty} \lambda^{-(1+d/2)} \,\mathrm{Tr}((-\Delta+\lambda V)_-) \leq -L^{\mathrm{cl}}_{1,d}\int_{\mathbb{R}^d} |V_-|^{1+\frac{d}{2}}.$$

4.2. Let $d \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $\mu_1 \leq \mu_2 \leq ...$ be the min-max values of the Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ on $L^2(\mathbb{R}^d)$.

(i) Use the Berezin-Li-Yau inequality to prove that $-\Delta_{\mathrm{D}}$ has compact resolvent (hence all $\{\mu_n\}$ are eigenvalues).

(ii) Prove that $0 < \mu_1 < \mu_2$.

4.3. Let $d \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Let $\mu_1 \leq \mu_2 \leq ...$ be the eigenvalues of the Dirichlet Laplacian $-\Delta_{\mathrm{D}}$ on $L^2(\Omega)$. Use the asymptotic formula for $\sum_{i=1}^N \mu_i$ to prove that

$$\sum_{i=1}^\infty [\mu_i - \lambda]_- = -L^{\mathrm{cl}}_{1,d}|\Omega|\lambda^{1+\frac{d}{2}} + o(\lambda^{1+\frac{d}{2}})_{\lambda\to\infty}$$

4.4. Given an increasing sequence $0 \leq \mu_1 \leq \mu_2 \leq ...$ satisfying

$$\lim_{N\to\infty} N^{-a}\mu_N = A(1+a)$$

for two constants $A > 0$, $a > 0$. Prove that

$$\lim_{N\to\infty} N^{-1-a}\sum_{n=1}^N \mu_n = A.$$

Homework Sheet 3

(Released 20.11.2020 - Discussed 26.11.2020)

3.1. Let $A \geq 0$, $B \geq 0$ be self-adjoint operators on a Hilbert space such that $\sqrt{B}(A+1)^{-\frac{1}{2}}$ is a compact operator. Prove that $A+B$ can be defined by Friedrichs method as a self-adjoint operator with quadratic form domain $Q(A+B) = Q(A)$ and

$$\sigma_{\text{ess}}(A+B) = \sigma_{\text{ess}}(A).$$

3.2. Let $d \geq 1$. Let $0 \leq U \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$ with $\infty > p, q \geq \max(1, d/2)$ if $d \neq 2$ and $\infty > p, q > 1$ if $d = 2$. Prove that $\sqrt{U(x)}(-\Delta + 1)^{-1/2}$ is a compact operator on $L^2(\mathbb{R}^d)$.

3.3. Let $3 \geq d \geq 1$ and $V \in L^2(\mathbb{R}^d)$. Prove that for every $E > 0$

$$\mathcal{N}(-\Delta + V + E) \leq C_d E^{\frac{d-4}{2}} \int_{\mathbb{R}^d} |V|^2.$$

Here $\mathcal{N}(-\Delta + V + E)$ is the number of negative eigenvalue of $-\Delta + V + E$.

3.4. Let $A \geq 0$ be a self-adjoint operator on a Hilbert space. Let $\infty > q > 1$. Assume that for every $\varepsilon > 0$, we have the operator inequality

$$A \leq \varepsilon + B_\varepsilon \quad \text{with an operator } B_\varepsilon \geq 0, \quad \operatorname{Tr}(B_\varepsilon) \leq \varepsilon^{1-q}.$$

Prove that A is a compact operator and its eigenvalues $\lambda_1 \geq \lambda_2 \geq ...$ satisfy

$$\lambda_n \leq C n^{-1/q}, \quad \forall n \geq 1.$$

3.5. Let A be a self-adjoint operator on a Hilbert space such that $A_- = A\mathbb{1}(A < 0)$ is a trace class operator. Prove that

$$\operatorname{Tr}(A_-) = \inf_{0 \leq \gamma \leq 1} \operatorname{Tr}(A\gamma).$$

Here we use the convention $\operatorname{Tr}(A\gamma) = \operatorname{Tr}(\sqrt{\gamma} A \sqrt{\gamma}) = \operatorname{Tr}(\sqrt{\gamma} A_- \sqrt{\gamma}) + \operatorname{Tr}(\sqrt{\gamma} A_+ \sqrt{\gamma})$.

Homework Sheet 2

(Released 13.11.2020 - Discussed 19.11.2020)

2.1. Let $F, G : \mathbb{R}^d \to \mathbb{R}$ be locally bounded functions satisfying $F(x), G(x) \to \infty$ as $|x| \to \infty$. Prove that the operator $F(x) + G(-i\nabla)$ on $L^2(\mathbb{R}^d)$ has compact resolvent.

2.2. Let $d \geq 1$. Let $V \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$ with $\infty > p, q \geq \max(1, d/2)$ when $d \neq 2$ and $\infty > p, q > 1$ when $d = 2$. Prove that the operator $-\Delta + V(x)$ can be defined as a self-adjoint operator on $L^2(\mathbb{R}^d)$ with the quadratic form domain $H^1(\mathbb{R}^d)$. Moreover,

$$\sigma_{\text{ess}}(-\Delta + V) = [0, \infty).$$

2.3. Let $d = 1, 2$. Let $V \in L^1(\mathbb{R}^d)$ if $d = 1$ and $V \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $p > 1$ if $d = 2$. Prove that if

$$\int_{\mathbb{R}^d} V(x) \mathrm{d}x < 0$$

then $-\Delta + V$ has at least one negative eigenvalue. Hint: You may consider $u_\varepsilon(x) = e^{-\varepsilon|x|}$ when $d = 1$, and $u_\varepsilon(x) = e^{-(1+|x|)^\varepsilon}$ when $d = 2$.

2.4. Let $d \geq 1$. Let $\{u_n\}_{n=1}^N \subset H^1(\mathbb{R}^d)$ be an orthonormal family in $L^2(\mathbb{R}^d)$ and define $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$. Use Rumin's method to prove that

$$\sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n(x)|^2 \mathrm{d}x \geqslant K_d \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2}{d}} \mathrm{d}x.$$

Here the constant $K_d > 0$ depends only on d.

Homework Sheet 1

(Released 06.11.2020 - Discussed 12.11.2020)

1.1. Let $\Omega \subset \mathbb{R}^d$ be a Borel set, μ a locally finite Borel measure on Ω, and $a \in L^\infty_{\mathrm{loc}}(\Omega,\mu)$ a real-valued function. Consider the **multiplication operator** M_a on $L^2(\Omega,\mu)$ defined by

$$(M_a f)(x) = a(x)f(x), \quad D(M_a) = \{f \in L^2(\Omega,\mu), af \in L^2(\Omega,\mu)\}.$$

Prove that

(i) M_a is a self-adjoint operator and $\sigma(M_a) = \text{ess-range}(a) \subset \mathbb{R}$, namely

$$\lambda \in \sigma(M_a) \quad \text{iff } \mu(a^{-1}(\lambda-\varepsilon,\lambda+\varepsilon)) > 0, \quad \forall \varepsilon > 0.$$

(ii) λ is an eigenvalue of M_a iff $\mu(a^{-1}(\lambda)) > 0$. Moreover, the multiplicity of λ is $\dim L^2(a^{-1}(\lambda),\mu)$.

1.2. Let A be a self-adjoint operator A on a Hilbert space $\mathscr{H}$. Prove that $\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if there exists an orthonormal family $\{u_n\}_{n=1}^\infty \subset D(A)$ such that

$$\lim_{n\to\infty} \|(A-\lambda)u_n\| = 0.$$

Hint: You can use Spectral theorem to reduce to a multiplication operator.

1.3. Let A be a self-adjoint operator on a Hilbert space. Assume that A is bounded from below and its min-max values satisfies

$$\lim_{n\to\infty} \mu_n(A) = +\infty.$$

Prove that $(A+C)^{-1}$ is a compact operator for any constant $C > -\mu_1(A)$. (In this case we say that A has **compact resolvent**.)

1.4. Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$. Assume that A is bounded from below and let $\mu_n(A)$ be its min-max values. Prove that for all $N \in \mathbb{N}$,

$$\sum_{n=1}^N \mu_n(A) = \inf\Big\{\sum_{n=1}^N \langle u_n, Au_n\rangle : \{u_n\}_{n=1}^N \text{ an orthonormal family in } \mathscr{H}\Big\}.$$

1.5. (extra) Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$ such that $A > 0$ and that A has compact resolvent. Prove that $A \geq \varepsilon$ for a constant $\varepsilon > 0$.