(12.1) (a) ✓ (b) ✓

(c) If $\int \rho_0 < m \Rightarrow \mu = 0$

$$\Rightarrow \frac{5}{3} \rho_0^{2/3}(x) = \left[V - \rho_0 * \frac{1}{|x|} \right]_+$$

If we assume $\int \rho_0 < m$ and $\int \rho_0 < Z := \sum_k Z_k$ then consider integrating the TF equation against $\frac{1}{|x|^4} \mathbb{1}(|x| \geq R)$ with $R > 0$ large we obtain:

$$\frac{5}{3} \int_{|x| \geq R} \frac{\rho_0^{2/3}(x)}{|x|^4} dx \geq \int_{|x| \geq R} \left(V(x) - \rho_0 * \frac{1}{|x|} \right) \frac{1}{|x|^4} dx$$

For left-side, by Hölder's ineq:

$$\int_{|x| \geq R} \frac{\rho_0^{2/3}(x)}{|x|^4} \leq \left(\int_{|x| \geq R} \rho_0 \right)^{2/3} \left(\int_{|x| \geq R} \frac{1}{|x|^{12}} \right)^{1/3} \leq \frac{C}{R^3}$$

For right-side, we use the triangle inequality

$$V(x) = \sum_k \frac{Z_k}{|x - R_k|} \geq \sum_k \frac{Z_k}{|x| - R_k} \geq \frac{Z - \varepsilon_R}{|x|}$$

for all $|x| \geq R$ and $\varepsilon_R \to 0$ as $R \to \infty$

$$\Rightarrow \int_{|x| \geq R} \frac{V(x)}{|x|^4} \geq (Z - \varepsilon_R) \int_{|x| \geq R} \frac{1}{|x|^5}$$

By Newton Theorem:

$$\int_{|x|\geq R} \frac{1}{|x|^4} \cdot \frac{1}{|x-y|}\, dx = \int_{|x|\geq R} \frac{1}{|x|^4} \cdot \frac{dx}{\max(|x|,|y|)} \leq \int_{|x|\geq R} \frac{dx}{|x|^5}$$

y

$$\Rightarrow \int_{|x|\geq R} \left(\rho_0 * \frac{1}{|x|}\right) \frac{1}{|x|^4} = \int_{\mathbb{R}^3}\int_{|x|\geq R} \frac{1}{|x|^4} \frac{1}{|x-y|} \rho_0(y)\,dy$$

$$\leq \left(\int_{\mathbb{R}^3} \rho_0\right) \int_{|x|\geq R} \frac{1}{|x|^5}\,dx$$

Thus:

$$\int_{|x|\geq R} \left(V(x) - \rho_0 * \frac{1}{|x|}\right) \frac{1}{|x|^4}\,dx$$

$$\geq \left(Z - \varepsilon_R - \int_{\mathbb{R}^3}\rho_0\right) \int_R \frac{1}{|x|^5}\,dx$$

$$\geq \frac{c_0}{R^2} \quad \text{for a constant } c_0 > 0$$

as $Z - \int_{\mathbb{R}^3}\rho_0 > 0$ and $\varepsilon_R \to 0$

All this gives:

$$\frac{C}{R^3} \geq \int_{|x|\geq R} \frac{\rho_0^{2/3}}{|x|^4} \geq \int_{|x|\geq R} \left(V(x) - \rho_0 * \frac{1}{|x|}\right) \frac{1}{|x|^4} \geq \frac{c_0}{R^2}$$

for R large, which is a contradiction.

Thus we have: $\int_{\mathbb{R}^3} \rho_0 \geq \min(m, Z)$, $Z = \sum_k Z_k$.

In particular, if $m \leq Z$, then $\int \rho_0 \geq m \Rightarrow \int \rho_0 = m$.

It remains to prove that if $m > Z$, then: $\int \rho_0 = Z$.

$$\frac{5}{3}\rho_0^{2/3}(x) = \left[V(x) - \rho_0 * \frac{1}{|x|} - \mu\right]_+ \quad , \mu \geq 0$$

<u>Maximum principle</u>. $\sum_k \frac{1}{|x-R_k|}$ continuous

<u>Claim</u>: $\Phi(x) = V(x) - \rho_0 * \frac{1}{|x|} \geq 0$ a.e. $x \in \mathbb{R}^3$.

<u>Proof</u>. Assume $A = \{x : \Phi(x) < 0\} \neq \emptyset$

$\Rightarrow$ A open set and $R_k \notin A$, $\forall k$.

$$+\Delta\Phi = \rho_0 * \underbrace{\left(-\Delta\frac{1}{|x|}\right)}_{4\pi\delta} = 4\pi\rho_0(x) = 0$$

on A

Thus Φ is harmonic on A

$\lim_{x\to\infty} \Phi(x) = 0$ $\Rightarrow$ max attained inside

$\Rightarrow$ contradiction

Thus $A = \emptyset \Rightarrow \Phi \geq 0$ a.e. $x \in \mathbb{R}^3$.

<u>Then</u> $|x|\,\Phi(x) \xrightarrow{|x|\to\infty} Z - \int \rho_0 \geq 0$

$\Rightarrow \int \rho_0 \leq Z$. and $\int \rho_0 \geq \min(m, Z) = Z$

$\Rightarrow \int \rho_0 = Z$

<u>Remark</u> In this case $\int \rho_0 = Z < m \Rightarrow \mu = 0$.