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Fourier Analysis and nonlinear PDE

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Contents

Introduction

1st lecture

Fourier transform:

$$\hat{f}(k) := \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k\cdot x} f(x)\,\mathrm{d}x$$

for $f \in L^1(\mathbb{R}^d)$, $p := 2\pi k$ momentum.

Heat equation:

$$\begin{cases} \partial_t u = \Delta u \\ u(0,x) = u_0(x) \end{cases} \iff u(t,x) = \mathrm{e}^{t\Delta} u_0(x) \iff u(t,x) = \int_{\mathbb{R}^d} G(t,x,y) u_0(y)\,\mathrm{d}y\,,$$

with the *heat kernel*:

$$G(t,x,y) := \frac{1}{(4\pi t)^{\frac{d}{2}}} \mathrm{e}^{-\frac{|x-y|^2}{4t}}$$

Smoothing effect: $u_0 \in L^2(\mathbb{R}^d) \implies u(t,\cdot) \in C^\infty(\mathbb{R}^d)$, $\forall t > 0$.

Schrödinger equation: "heat equation with imaginary time"

$$\begin{cases} \mathrm{i}\,\partial_t u = -\Delta u \\ u(0,x) = u_0(x) \end{cases} \iff u(t,x) = \mathrm{e}^{\mathrm{i}t\Delta} u_0(x) = \frac{1}{(4\pi \mathrm{i} t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4\mathrm{i}t}} u_0(y)\,\mathrm{d}y$$

$u_0 \in L^1(\mathbb{R}^d) \implies u(x,t)$ is well-defined. Since $\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2}$, the solution is well-defined $\forall u_0 \in L^2(\mathbb{R}^d)$. $u(t,\cdot)$ is in general not smoother than u_0 but

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{1}{|4\pi t|^{\frac{d}{2}}} \|u_0\|_{L^1}\,. \qquad \text{(dispersive estimate)}$$

Many other linear equations: e.g. wave equation, transport equation, transport-diffusion equation:

$$\begin{cases} \partial_t f + v\cdot\nabla f - \alpha\Delta f = g \\ f(0,x) = f_0 \end{cases}$$

(v, α, g, f_0 given), $\alpha = 0$ transport.

We will focus on some nonlinear equations

$$\begin{cases} \Box u = F(u) \\ u(0,x) = u_0 \end{cases},$$

with $\Box$ linear. Important case:

Incompressible Navier-Stokes equation:

$$\begin{cases} \partial_t u + u\cdot\nabla u - \alpha\Delta u = -\nabla P + f \\ \operatorname{div}(u) = \sum_j \partial_{x_j} u_j = 0 \\ u(0,x) = u_0(x) \end{cases}$$

(system: $u = (u_j)_{j=1}^d$)

1. weak solution: $u \in H^{\frac{d}{2}-1}$ (Sobolev spaces)
   Leray: local existence
   Stability condition: If u is a solution, then

$$\|u(t,\cdot)\|_{L^2}^2 + 2\alpha \int_0^t \|\nabla_x u(s,\cdot)\|_{L^2}^2\,\mathrm{d}s \le \|u_0\|_{L^2}^2, \quad \forall t\,.$$

2. 2 dimensions:
   - $\exists$ global solution
   - Leray solution is unique

   $\rightsquigarrow$ global well-posed

3. 3 dimensions:
   - local existence ($d = 2, 3$) of a weak solution
   - If $\|u_0\|_{H^{\frac{1}{2}}}$ is small enough (small data) $\rightsquigarrow$ global existence

4. Open:
   - global well-posedness for large data?
   - regularity of solution (C^∞ data $\implies$ C^∞ solution?)

Clay's problem

(A) (global existence of smooth solutions): If u_0, p are C^∞, $f = 0$, then $\exists$ global solution $u \in C^\infty$ such that

$$\|u(t,\cdot)\|_{L^2}^2 \leq C\,, \quad \forall t > 0$$

(B) ($\exists$ blow up solution) $\exists u_0, f \in C^\infty$, but $\nexists$ global solution $u \in C^\infty$ satisfies the growing condition.

Two recent paper on the non-uniqueness

(2018) Buckmester-Vical: non-uniqueness of weak solution
Counter example: $f = 0$, $u_0, p \in C^\infty$, no Leray condition

(2022) Albritton, Brué Colombo: non-uniqueness with Leray condition, but $f \neq 0$

One blow-up result of T. Tao: Euler equation ($\alpha = 0$)

Back to the Fourier transform:

Non-commutative property: p and x do not commute!

"uncertainty principle": Heisenberg:

$$\text{"}\,\Delta p \Delta x \geq \hbar^2\,\text{"} \iff \left(\int_{\mathbb{R}^3} |2\pi k|^2 |\hat{u}(k)|^2 \,\mathrm{d}k\right)\left(\int_{\mathbb{R}^3} |x|^2 |u(x)|^2 \,\mathrm{d}x\right) \geq \frac{9}{4}\|u\|_{L^2}^4$$

follows $[p, x] = [-\mathrm{i}\nabla_x, x] = \mathrm{i}\frac{d}{2}$
Stability of atom:

$$\mathscr{E}(u) = \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|}\,\mathrm{d}x \qquad \text{("energy functional")}$$

Claim: $\mathscr{E}(u) \geq -C$, $\forall u$ "nice" and $\|u\|_{L^2(\mathbb{R}^3)} = 1$
Classical mechanics:

$$\inf_{(p,x)\in\mathbb{R}^3\times\mathbb{R}^3} \left(p^2 - \frac{1}{|x|}\right) = -\infty\,.$$

A proof:

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|}\,\mathrm{d}x = \int_{\mathbb{R}^3} \overline{u(x)}\frac{u(x)}{|x|}\,\mathrm{d}x = \int_{\mathbb{R}^3} \overline{\hat{u}(k)}\underbrace{\widehat{u\cdot\frac{1}{|x|}}(k)}_{\hat{u}*\underbrace{\widehat{\frac{1}{|x|}}(k)}_{\sim\frac{1}{|k|^2}}}\mathrm{d}k = C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\overline{\hat{u}(k)}\frac{\hat{u}(l)}{|k-l|^2}\,\mathrm{d}k\,\mathrm{d}l$$

$$\overset{\substack{\text{CS,}\\ \text{Young}\\ \downarrow}}{\leq} \frac{C}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(|\hat{u}(k)|^2\frac{(L+|k|)^2}{(L+|l|)^2|k-l|^2}+|\hat{u}(l)|^2\frac{(L+|l|)^2}{(L+|k|)^2|k-l|^2}\right)\mathrm{d}k\,\mathrm{d}l$$

$$= C\int_{\mathbb{R}^3}|\hat{u}(k)|^2(|k|^2+L)\underbrace{\int_{\mathbb{R}^3}\frac{1}{(L+|l|)^2|k-l|^2}\,\mathrm{d}l}_{\leq\varepsilon,\ \forall k\in\mathbb{R}^3,\ \text{ if } L \text{ large enough}}$$

Global theory ($\mathbb{R}^d$) good for linear equations, but local theory ($\Omega \subset \mathbb{R}^d$) is more helpful for nonlinear theory

$\Longrightarrow$ localization:

- x-space localization (1)
- p-space localization (2)

1. Caldron-Zugmund decomposition:

   Lemma 0.1. *If $\varphi\colon \mathbb{R}^d \to [0,\infty)$, $\alpha > 0$ large enough, $f \in L^1$, $\exists\{Q\}$ disjoint cubes in $\mathbb{R}^d$ such that*

   - $⨍_Q f = \frac{1}{|Q|}\int_Q \in [\alpha, 2^d\alpha]$, $\forall Q$
   - $f(x) \leq \alpha$ *a.e.* $x \in (\bigcup Q)^C$

2. Littlewood-Paley decomposition:

   Lemma 0.2 (Bernstein). *If supp $\hat{f} \subset \{2^{n-1} \leq |k| \leq 2^n\}$, $n \in \mathbb{Z}$, then*

   $$C^{-1}\|f\|_{L^p} \leq 2^{-n}\|\nabla f\|_{L^p} \leq C\|f\|_{L^p}\,, \quad \forall p \in [1,\infty]$$

   Theorem 0.3 (Littlewood-Paley). *$f = \sum_n f_n$, "supp $\hat{f}_n \subseteq 2^{n-1} \leq |k| \leq 2^n$". Then*

   $$C^{-1}\|f\|_{L^p(\mathbb{R}^d)} \leq \left\|\sum_n |f_n|^2\right\|_{L^p} \leq C\|f\|_{L^p(\mathbb{R}^d)}\,, \quad \forall p \in [1,\infty]$$

   "low vs. high frequency"

1 Basic Analysis

2nd lecture

1.1 Fourier transform

$$\hat{f}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k\cdot x} f(x)\,\mathrm{d}x$$

Theorem 1.1. *(1) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in C(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1(\mathbb{R}^d)}$.*

(2) If $f \in L^1 \cap L^2$, then

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2} \qquad \text{(Plancherel equality)}$$

Consequently, $\mathcal{F}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is well-defined and it is a unitary transformation.

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot x}\hat{f}(k)\,\mathrm{d}k \quad \left(= \lim_{n\to\infty}\int_{\mathbb{R}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot x}\hat{f}_n(k)\,\mathrm{d}k\right)$$

$\rightsquigarrow$ standard L^2-theory of Fourier transform

Extension of the Fourier transform to distributions

Duality argument: X space, X' dual space (space of linear, continuous functionals). If we have a continuous linear operator $A\colon X \to X$, then we can define the map $A^*\colon X' \to X'$ such that

$$\langle A^* u, v\rangle = \langle u, Av\rangle\ , \quad \forall u \in X',\ \forall v \in X\,.$$

Here, $\forall u \in X'$: $v \mapsto \langle u, Av\rangle$ is a linear and continuous map from X to $\mathbb{C}$, because $v_n \to v$ in $X \implies Av_n \to Av$ in $X \implies \langle u, Av_n\rangle \to \langle u, Av\rangle$
$\implies$ $\exists$ an element $u \in X'$ such that $v \mapsto \langle u, Av\rangle$ can be identified to z, namely $\langle z, v\rangle := \langle u, Av\rangle$, $\forall v \in X$. We set $A^*u = z$ to define the mapping A^*.

Here, A^* is continuous because if $u_n \to u$ in X', then

$$\langle A^* u_n, v\rangle = \langle u_n, Av\rangle \xrightarrow{n\to\infty} \langle u, Av\rangle\ , \forall v \in X \implies A^* u_n \to Au$$

Principle: $X \subset Y$ as topological spaces $\implies X' \supset Y'$
"critical case": $(L^2(\mathbb{R}^d))' = L^2(\mathbb{R}^d)$. But if we have $X \subset L^2(\mathbb{R}^d)$, then $X' \supset L^2(\mathbb{R}^d)$.

Schwartz space:

$$\mathscr{S}(\mathbb{R}^d) := \{f\colon \mathbb{R}^d \to \mathbb{C} \mid f \in C^\infty \text{ and } \sup_{x\in\mathbb{R}^d}(1+|x|^k)|\,\partial^\alpha f(x)| < \infty\,, \forall k \in \mathbb{N},\ \forall \alpha \in \mathbb{N}_0^d\}$$

Associated with the family of seminorms $\{\|\cdot\|_{k,\mathscr{S}}\}_{k=0}^\infty$ where

$$\|f\|_{k,\mathscr{S}} = \sup_{\substack{x\in\mathbb{R}^d\\ |\alpha|\le k}} (1+|x|^k)|\,\partial^\alpha f(x)|\,.$$

It is known that $\mathscr{S}(\mathbb{R}^d)$ is a Fréchet space with this family of seminorms. We can think of $\mathscr{S}$ as a metric space via:

$$d(f,g) := \sum_{k=0}^\infty \frac{1}{2^k}\frac{\|f-g\|_{k,\mathscr{S}}}{1+\|f-g\|_{k,\mathscr{S}}}\,.$$

Thus $f_n \to f \iff \|f_n - f\|_{k,\mathscr{S}} \to 0$, $\forall k \in \mathbb{N}_0$.

Exercise: Take $f, g \in \mathscr{S}(\mathbb{R}^d)$ such that $\int g = 1$. Define

$$g_\varepsilon(x) := \varepsilon^{-d} g(\frac{x}{\varepsilon})\,, \quad \varepsilon > 0\,.$$

Prove that $g_\varepsilon * f \xrightarrow{\varepsilon\downarrow 0} f$ in $\mathscr{S}$.

Theorem 1.2. *The Fourier transform satisfies* $\mathcal{F}\colon \mathscr{S} \to \mathscr{S}$ *and*

$$\forall s \in \mathscr{S}\, \forall k \geq 0\,, \exists N \geq 0: \quad \|\hat{f}\|_{k,\mathscr{S}} \leq C\|f\|_{N,\mathscr{S}} \quad (C \text{ independent of } f)$$

Moreover, $\mathcal{F}$ *is bijective.*

Proof. Take $f \in \mathscr{S}$, $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$. Then

$$\begin{aligned}\|(1+|\xi|^k)\, \partial_\xi^\alpha\, \hat{f}(\xi)\|_{L^\infty} &= \left\|(1+|\xi|^k)\widehat{(-2\pi \mathrm{i} x)^\alpha f(x)}(\xi)\right\|_{L^\infty} \\ &\overset{k \text{ even}}{\overset{\downarrow}{=}} \left\|\sum_\beta \widehat{\partial_x^\beta (-2\pi\mathrm{i}x)^\alpha f(x)}(\xi)\right\|_{L^\infty} \leq \|\cdots\|_{L^1} \leq C_k \|f\|_{N,\mathscr{S}}\end{aligned}$$

if $N > k + d + 1$.
Calculation rules used:

$$\partial_{\xi_j}\, \hat{f}(\xi) = \int_{\mathbb{R}^d} \partial_{\xi_j} \left(\mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x} f(x)\right) = \widehat{(-2\pi\mathrm{i}x_j) f(x)}(\xi)$$

$$(-2\pi\mathrm{i}\xi_j)\hat{f}(\xi) = \int_{\mathbb{R}^d} \underbrace{(-2\pi\mathrm{i}\xi_j)\mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x}}_{\partial_{x_j}(\mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x})} f(x)\,\mathrm{d}x = -\int \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot x}\, \partial_{x_j}\, f(x)\,\mathrm{d}x$$

$$\partial^\beta\, x^\alpha f(x) = \sum_{\substack{\alpha',\beta' \\ |\alpha'|\leq|\alpha|\leq k \\ |\beta'|\leq|\beta|\leq k}} x^{\alpha'}\, \partial_x^{\beta'}\, f(x)$$

$$\begin{aligned}\implies \|\,\partial_x^\beta\, x^\alpha f(x)\|_{L^1(\mathbb{R}^d)} &= \left\|\frac{1}{(|x|+1)^{d+1}}(1+|x|)^{d+1} \sum_{\alpha',\beta'} x^{\alpha'}\, \partial_x^{\beta'}\, f(x)\right\|_{L^1} \\ &\leq \left\|\frac{1}{(|x|+1)^{d+1}}\right\|_{L^1} \left\|(1+|x|)^{d+1} \sum_{\alpha',\beta'} x^{\alpha'}\, \partial_x^{\beta'}\, f(x)\right\|_{L^\infty} \leq C_{k,d}\|f\|_{N,\mathscr{S}}\end{aligned}$$

Consequently, $\mathcal{F}\colon \mathscr{S} \to \mathscr{S}$ is continuous. In fact, if $f_n \to f$ in $\mathscr{S} \implies \|f_n - f\|_{N,\mathscr{S}} \to 0$, $\forall N$

$$\implies \|\,\mathcal{F}(f_n - f)\|_{k,\mathscr{S}} \leq C\|f_n - f\|_{N_k,\mathscr{S}} \xrightarrow{n\to\infty}, \quad \text{for any } k \geq 0$$

$g \in \mathscr{S} \implies f = \check{g} \in \mathscr{S}$ (by following the previous proof: $\mathrm{i} \mapsto -\mathrm{i} \implies g = \hat{f}$ in $L^2 \implies g = \hat{f}$ in $\mathscr{S}$). ☐

Theorem 1.3. $\mathcal{F}\colon \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ *is well-defined as a linear and continuous map.*

Remark 1.4. The space $\mathscr{S}'(\mathbb{R}^d)$ is called tempered distributions.
Thus, $\forall f \in \mathscr{S}'$, $\hat{f}$ is well-defined.

Examples of $\mathscr{S}'(\mathbb{R}^d)$: If $f \in L^1(\mathbb{R}^d)$, then $f \in \mathscr{S}'(\mathbb{R}^d)$, via the identification of f with

$$\mathscr{S}' \ni T_f\colon \begin{array}{ccc} \mathscr{S}(\mathbb{R}^d) & \to & \mathbb{C} \\ \varphi & \mapsto & T_f(\varphi) \quad := \int f\,\varphi \end{array}.$$

More generally, if $f \in L^1_{\mathrm{loc}}$ and $|f(x)|$ grows at most polynomially at ∞, i.e.

$$\exists N: \quad |f(x)| \leq C_N(1+|x|)^N, \quad \forall |x| \geq R_N$$

then $f \in \mathscr{S}'(\mathbb{R}^d)$.

Be careful: $e^{|x|} \notin \mathscr{S}(\mathbb{R}^d)$.

Another example: Dirac delta function $\delta_0 \in \mathscr{S}'(\mathbb{R}^d)$ defined as

$$\delta_0(\varphi) = \varphi(0) \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d)\,.$$

And we have $\hat{\delta}_0(\xi) = 1$.
Last example: $f(x) = \frac{1}{|x|^\lambda}$, $0 < \lambda < d \implies f \in \mathscr{S}'(\mathbb{R}^d)$

$$\widehat{\frac{C_\lambda}{|x|^\lambda}} = \frac{C_{d-1}}{|\xi|^{d-1}}, \;\; C_\lambda = \pi^{-\frac{d}{2}}\Gamma\left(\frac{\lambda}{2}\right), \;\; \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \;\; \Gamma(n) = (n-1)!$$

3rd lecture

Recall an abstract result:

Lemma 1.5. *X topological vector space, $A\colon X \to X$ linear and continuous, then $\exists A^*\colon X' \to X'$ linear continuous by*

$$\langle x, Ay\rangle_{X',X} = \langle A^*x, y\rangle_{X',X}\ .$$

Proof. We already proved $\exists A^*x$, $\forall x \in X'$ and A^* is linear. Also obvious that if $x_n \to x$ in X', then $A^*x_n \to A^*x$ in X':

$$\langle A^*x_n, y\rangle = \langle x_n, Ay\rangle \xrightarrow{n\to\infty} \langle x, Ay\rangle = \langle A^*, y\rangle\,, \quad \forall y \in X$$

But we need a more precise argument for the continuity of A^* on X'.

Discussion of the weak-$*$-topology:

Given X topological vectorspace (TVS), X' its dual

Definition 1.6. The weak-$*$-topology on X' is the weakest topology on X' such that

$$X' \ni f \mapsto f(x)$$

is a continuous map $X' \to \mathbb{C}$, $\forall x \in X$.

What are the open sets in X'?

- $\forall x \in X$, $\forall \varepsilon > 0$,
$$\mathcal{O}_{x,\varepsilon} = \{f \in X' \,|\, |f(x)| < \varepsilon\}$$
is an open set in X', because $W_x\colon X' \ni f \to f(x)$ is continuous and so
$$W_x^{-1}(B_\varepsilon) = \{f \in X' \,|\, W_x(f) \in B_\varepsilon\} = \mathcal{O}_{x,\varepsilon}$$
- Any open set $\mathcal{O} \subseteq X'$ with $0 \in \mathcal{O}$, can be written as $\mathcal{O} = \bigcup_{i\in I} \mathcal{O}_{x_i,\varepsilon_i}$

We need to prove $(A^*)^{-1}(\mathcal{O})$ is open for $\mathcal{O}$ open in X'. It suffices to prove that $(A^*)^{-1}(\mathcal{O}_{x,\varepsilon})$ is open, $\forall x \in X$, $\forall \varepsilon > 0$. We have:

$$\begin{aligned}(A^*)^{-1}(\mathcal{O}_{x,\varepsilon}) &= \{f \in X' \,|\, A^*f \in \mathcal{O}_{x,\varepsilon}\} = \left\{f \in X' \,\middle|\, |\langle A^*f, x\rangle_{X',X}| < \varepsilon\right\} = \left\{f \in X' \,\middle|\, |\langle f, Ax\rangle_{X',X}| < \varepsilon\right\} \\ &= \mathcal{O}_{Ax,\varepsilon}\end{aligned}$$

□

1.2 Distributions

DEFINITION OF DISTRIBUTIONS $\mathscr{D}'(\mathbb{R}^d)$

Test functions: $\mathscr{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$, where $\varphi_n \xrightarrow{n\to\infty} \varphi$ in $\mathscr{D}(\mathbb{R}^d)$ iff

- $\exists K \subset \mathbb{R}^d$ compact such that $\operatorname{supp}\varphi_n \subset K$, $\forall n \in \mathbb{N}$
- $\|\partial^\alpha \varphi_n - \partial^\alpha \varphi\|_\infty \xrightarrow{n\to\infty} 0$, $\forall \alpha \in \mathbb{N}_0^d$

We define $\mathscr{D}'(\mathbb{R}^d)$ as the dual space of $\mathscr{D}(\mathbb{R}^d)$, i.e.

$$\mathscr{D}'(\mathbb{R}^d) = \{f \colon \mathscr{D}(\mathbb{R}^d) \to \mathbb{C} \text{ linear and continuous}\}$$

Actually

$$T \in \mathscr{D}'(\mathbb{R}^d) \iff \forall \varphi_n \xrightarrow{n\to\infty} \varphi \text{ in } \mathscr{D}(\mathbb{R}^d): \; T(\varphi_n) \xrightarrow{n\to\infty} T(\varphi)$$

(see Analysis, Lieb-Loss, Chapter 6)

Example 1.7. For $f \in L^1_{\text{loc}}$, we define $T_f \in \mathscr{D}'(\mathbb{R}^d)$ by

$$T_f(\varphi) = \int_{\mathbb{R}^d} f(x)\,\varphi(x)\,\mathrm{d}x, \;\; \forall \varphi \in \mathscr{D}(\mathbb{R}^d)\,.$$

If $\varphi_n \xrightarrow{n\to\infty} \varphi$ in $\mathscr{D}(\mathbb{R}^d)$, then $\exists K$ compact such that $\operatorname{supp}\varphi_n \subset K$, $\forall n \in \mathbb{N}$ and $\varphi_n \to \varphi$ in L^∞, and so:

$$\begin{aligned}|T_f(\varphi_n) - T_f(\varphi)| = \left|\int_{\mathbb{R}^d} f(x)(\varphi_n(x) - \varphi(x))\,\mathrm{d}x\right| &\le \int_K |f(x)||\varphi_n(x) - \varphi(x)|\,\mathrm{d}x \\ &\le \|f\|_{L^1(K)}\|\varphi_n - \varphi\|_{L^\infty} \xrightarrow{n\to\infty} 0\end{aligned}$$

Note that $f(x) = e^{|x|} \notin \mathscr{S}'(\mathbb{R}^d)$, but $f \in \mathscr{D}'(\mathbb{R}^d)$.

Lemma 1.8 (Fundamental Lemma of calculus of variations). *$T\colon f \mapsto T_f$ from $L^1_{\text{loc}}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ is injective. Put differently, if $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} f(x)\,\varphi(x)\,\mathrm{d}x = 0 \;\; \forall \varphi \in \mathscr{D}(\mathbb{R}^d) \implies f = 0\,.$$

Proof. First, take $f \in L^1(\mathbb{R}^d)$. Choose $\varphi_{\varepsilon,y}(x) = \varepsilon^{-d}\,g(\frac{y-x}{\varepsilon})$, $g \in C_c^\infty$, $\int_{\mathbb{R}^d} g = 1$, then

$$0 = \int_{\mathbb{R}^d} f(x)\,\varphi_{\varepsilon,y}(x)\,\mathrm{d}x = \left(f * \left(\varepsilon^{-d}\,g\left(\frac{y - \bullet}{\varepsilon}\right)\right)\right)(y) \xrightarrow{\varepsilon\to 0} f \text{ in } L^1 \implies f = 0$$

Second, take $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then for $\chi \in C_c^\infty(\mathbb{R}^d)$:

$$0 = \int_{\mathbb{R}^d} (f\chi)\,\varphi\,, \;\; \forall \varphi \in C_c^\infty$$

By the first step: $f\chi = 0$, $\forall \chi \in C_c^\infty$, so $f = 0$ a.e.. □

1.3 Sobolev spaces

HOMOGENEOUS SOBOLEV SPACES

Definition 1.9 (Homogeneous Sobolev spaces). For $s \in \mathbb{R}$, denote

$$\dot{H}^s(\mathbb{R}^d) = \left\{f \in \mathscr{S}'(\mathbb{R}^d) : \,\middle|\, \hat{f} \in L^1_{\text{loc}}(\mathbb{R}^d) \text{ and } \|f\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} |2\pi k|^{2s}|\hat{f}(k)|^2\,\mathrm{d}k < \infty\right\}$$

By this way, $\dot{H}^s$ is a normed space and the norm comes from the inner product

$$\langle f, g\rangle_{\dot{H}^s} = \int_{\mathbb{R}^d} |2\pi k|^2\,\overline{\hat{f}(k)}\hat{g}(k)\,\mathrm{d}k$$

Theorem 1.10. *$\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space iff $s < \frac{d}{2}$ (s can be negative).*

Proof. Step 1: Assume $s < \frac{d}{2}$. Take a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\dot{H}^s(\mathbb{R}^d)$. From the definition of the norm:

$$\|f_n - f_m\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^d} |2\pi k|^{2s} |\hat{f}_n(k) - \hat{f}_m(k)|^2 \,\mathrm{d}k \xrightarrow{m,n\to\infty} 0$$

Hence, $\{\hat{f}_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^d, |k|^{2s}\,\mathrm{d}k)$. Since L^2 is complete, $\mathscr{S}' \ni \hat{f}_n \xrightarrow{n\to\infty} g \in L^2(\mathbb{R}^d, |k|^{2s}\,\mathrm{d}k)$.
We want to prove that $g = \hat{f}$ for some $f \in \dot{H}^s(\mathbb{R}^d) \implies f_n \to f$ in $\dot{H}^s(\mathbb{R}^d)$. We want to first show that $g \in \mathscr{S}'$. We decompose $g \in L^2(\mathbb{R}^d, |k|^{2s}\,\mathrm{d}k) \subset L^1_{\mathrm{loc}}$:

$$g = \mathbb{1}_{B_1} g + \mathbb{1}_{B_1^C} g$$

We have

$$\int_{\mathbb{R}^d} |\mathbb{1}_{B_1} g(k)|\,\mathrm{d}k = \int_{B_1} |g(k)|\,\mathrm{d}k \le \underbrace{\left(\int_{B_1} |k|^{2s}|g(k)|^2\,\mathrm{d}k\right)^{\frac{1}{2}}}_{<\infty} \underbrace{\left(\int_{B_1} \frac{1}{|k|^{2s}}\,\mathrm{d}k\right)^{\frac{1}{2}}}_{<\infty,\text{ since } d>2s}$$

Moreover,

$$\mathbb{1}_{B_1^C} g(k) \in \mathbb{1}_{B_1^C} L^2(\mathbb{R}^d, |k|^{2s}\,\mathrm{d}k) \subset L^2(\mathbb{R}^d, (1+|k|^{2s})\,\mathrm{d}k)$$

$$\implies \int_{\mathbb{R}^d} |\mathbb{1}_{B_1^C} g(k)|^2 (1+|k|^{2s})\,\mathrm{d}k < \infty \implies \mathbb{1}_{B_1^C} g \in L^2(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$$

$$\implies g \in L^1 + L^2 \implies f = \check{g} \in L^\infty + L^2 \in \dot{H}^s(\mathbb{R}^d)$$

Step 2: Assume $s \ge \frac{d}{2}$. We assume for contradiction that $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space. Note that the space $W := \{f \in \dot{H}^s \mid \|f\|_W < \infty\} \subset \dot{H}^s(\mathbb{R}^d)$,

$$\|f\|_W = \|\hat{f}\|_{L^1(B_1)} + \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

is a Banach space with:

$$\|f\|_{\dot{H}^s} \le \|f\|_W \implies \mathrm{id}\colon W \to \dot{H}^s \text{ is continuous and surjective}$$

$$\overset{\text{open mapping}}{\implies} \|f\|_W \le C\|f\|_{\dot{H}^s}$$

$$\implies \|\hat{f}\|_{L^1(B_1)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^d)}, \quad \forall f \in \dot{H}^s(\mathbb{R}^d)$$

$$\implies \int_{B_1} |\hat{f}(k)|\,\mathrm{d}k \le C\left(\int_{\mathbb{R}^d} |k|^{2s}|\hat{f}(k)|^2\,\mathrm{d}k\right)^{\frac{1}{2}}, \quad \forall f \in \dot{H}^s(\mathbb{R}^d)$$

Think of the case $2s = d$. We want to have $\int_{B_1} |\hat{f}(k)|\,\mathrm{d}k$ large.
good candidate: $\hat{f}(k) = \frac{1}{|k|^d}$, but $|k|^{2s}|\hat{f}(k)|^2 = \frac{1}{|k|^d}$ also singular. A safer choice is:

$$\hat{f}(k) = \sum_{n=1}^{\infty} \frac{1}{|k|^d n^{1+\varepsilon}} \mathbb{1}(2^{-(n+1)} \le |k| \le 2^{-n}),$$

for $\varepsilon > 0$. Then:

$$\begin{aligned}\int_{B_1} |\hat{f}(k)|\,\mathrm{d}k &= \sum_n \int_{2^{-(n+1)}\le|k|\le 2^{-n}} \frac{1}{|k|^d n^{1+\varepsilon}}\,\mathrm{d}k \ge \sum_n \frac{1}{n^{1+\varepsilon}} \frac{|\{2^{-(n+1)} \le |k| \le 2^{-n}\}|}{(2^{-n})^d} \\ &\ge \frac{1}{C}\sum_n \frac{1}{n^{1+\varepsilon}} \xrightarrow{\varepsilon\to 0} \infty,\end{aligned}$$

but

$$\int_{\mathbb{R}^d} |k|^{2s} |\hat{f}(k)|^2 \,\mathrm{d}k = \sum_n \int_{\mathbb{R}^d} |k|^{2s} \frac{1}{(|k|^d n^{1+\varepsilon})^2} \mathbb{1}(2^{-(n+1)} \leq |k| \leq 2^{-n}) \,\mathrm{d}k$$
$$\leq C \sum_n \frac{1}{n^{2+2\varepsilon}} < \underbrace{C\frac{\pi^2}{6}}_{\text{independent of } \varepsilon} < \infty \quad ↯$$

□

4th lecture

DUALITY

Heuristically $\dot{H}^{-s}(\mathbb{R}^d) = \left(\dot{H}(\mathbb{R}^d)\right)'$

Theorem 1.11. *If $|s| < \frac{d}{2}$, then $\dot{H}(\mathbb{R}^d)^{-s} = \left(\dot{H}^s(\mathbb{R}^d)\right)'$.*

Proof. Step 1: $\dot{H}(\mathbb{R}^d)^{-s} \subset \left(\dot{H}^s(\mathbb{R}^d)\right)'$
Take $f \in \dot{H}^{-s}(\mathbb{R}^d)$. Construct $L_f : \dot{H}^s \to \mathbb{C}$ by $L_f(g) = \int_{\mathbb{R}^d} \overline{f} g$.

$$|L_f(g)| = \left|\int_{\mathbb{R}^d} \overline{f} g\right| \overset{\text{Plancherel}}{=} \left|\int_{\mathbb{R}^d} \overline{\hat{f}}(k)\hat{g}(k)\,\mathrm{d}k\right| = \left|\int_{\mathbb{R}^d} \underbrace{|k|^{-s}\overline{\hat{f}}(k)}_{\in L^2} \underbrace{|k|^s \hat{g}(k)}_{\in L^2}\right|$$
$$\overset{\text{C.S.}}{\leq} \left\| |k|^{-s}\hat{f}(k)\right\|_{L^2} \| |k|^s \hat{g}(k)\|_{L^2} = \|f\|_{\dot{H}^{-s}} \|g\|_{\dot{H}^s}$$

Step 2: $\left(\dot{H}^s(\mathbb{R}^d)\right)' \subset H^{-s}(\mathbb{R}^d)$
Take $L \in \left(\dot{H}^s(\mathbb{R}^d)\right)'$. We want to find $f \in \dot{H}^{-s}(\mathbb{R}^d)$ such that $L(g) = \int_{\mathbb{R}^d} \overline{f} g$, $\forall g \in \dot{H}^s(\mathbb{R}^d)$. Note that $\varphi \mapsto L(\widecheck{|k|^{-s}\hat{\varphi}})$ is an element of $\left(L^2(\mathbb{R}^d)\right)'$:

$$\begin{aligned} \varphi \in L^2 &\implies \hat{\varphi} \in L^2 \\ &\implies \int_{\mathbb{R}^d} |k|^{2s} \big| |k|^{-s}\hat{\varphi}(k)\big|^2 \,\mathrm{d}k = \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 \,\mathrm{d}k = \|\varphi\|_{L^2} \\ &\implies \widecheck{|k|^{-s}\hat{\varphi}} \in \dot{H}^{-s} \end{aligned} \qquad \text{(well-defined)}$$

$$|L(\widecheck{|k|^{-s}\hat{\varphi}})| \leq \|L\|_{(\dot{H}^s)'} \|\widecheck{|k|^{-s}\hat{\varphi}}\|_{\dot{H}^s} = |2\pi|^s \|L\|_{(\dot{H}^s)'} \|\varphi\|_{L^2} \qquad \text{(bounded)}$$

By Riesz' representation theorem on $L^2(\mathbb{R}^d)$, $\exists! a \in L^2(\mathbb{R}^d)$ such that

$$L(\widecheck{|k|^{-s}\hat{\varphi}}) = \int_{\mathbb{R}^d} \overline{a}\,\varphi = \int_{\mathbb{R}^d} \overline{\hat{a}}\hat{\varphi} = \int_{\mathbb{R}^d} |k|^s \overline{\hat{a}(k)} |k|^{-s}\hat{\varphi}(k)\,\mathrm{d}k = \int_{\mathbb{R}^d} \overline{\widecheck{|k|^s\hat{a}(k)}} \cdot \widecheck{|k|^{-s}\hat{\varphi}(k)}\,\mathrm{d}k$$
$$= \int_{\mathbb{R}^d} \overline{f(x)} \widecheck{|k|^{-s}\hat{\varphi}(k)},$$

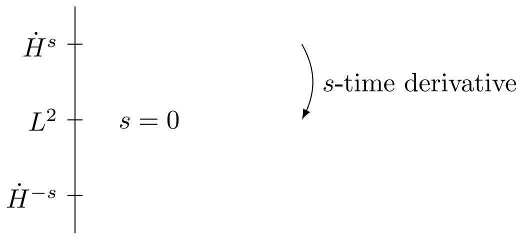
where

$$\hat{f}(k) = \underbrace{|k|^s \underbrace{\hat{a}(k)}_{L^2}}_{\in L^1_{\text{loc}}} \implies f \in \dot{H}^{-s}(\mathbb{R}^d).$$

□

Remark 1.12. (1) In general, we can't compare $\dot{H}^r$ and $\dot{H}^s$ if $r < s$. Later, for "inhomogeneous Sobolev spaces" we have $H^r(\mathbb{R}^d) \subseteq H^s(\mathbb{R}^d)$ for $s < r$.

(2) The power s in $\dot{H}^s(\mathbb{R}^d)$ should be thought of as the order of derivatives:



<u>Derivatives of Distributions</u>

$f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $T_f \in \mathscr{D}'(\mathbb{R}^d)$ with $T_f(\varphi) = \int_{\mathbb{R}^d} f\,\varphi$, $\forall\,\varphi \in \mathscr{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$. If $f \in C^\infty$, then

- $\partial_{x_1} f \leftrightarrow T_{\partial_{x_1} f}\,\varphi = \int (\partial_{x_1} f)\,\varphi = -\int f\,\partial_{x_1}\varphi$
- $\partial^\alpha f \leftrightarrow T_{\partial^\alpha f}(\varphi) = \int (\partial^\alpha f)\,\varphi = (-1)^{|\alpha|}\int f\,\partial^\alpha \varphi$

Definition 1.13. If $T \in \mathscr{D}'(\mathbb{R}^d)$ is a distribution, then its derivative $\partial^\alpha T \in \mathscr{D}'(\mathbb{R}^d)$ is defined by

$$\partial^\alpha T(\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi), \quad \forall\,\varphi \in C_c^\infty\,.$$

(It is easy to see that ∂^α is a distribution: $\varphi_n \xrightarrow{n\to\infty} \varphi$ in $\mathscr{D}(\mathbb{R}^d)$, then $\partial^\alpha \varphi_n \xrightarrow{n\to\infty} \partial^\alpha$ in $\mathscr{D}(\mathbb{R}^d)$, so $\partial^\alpha T \in \mathscr{D}'(\mathbb{R}^d)$ again).

<u>Restriction to tempered distributions</u>: Take $f \in \mathscr{S}(\mathbb{R}^d)$. Then it can be identified ("$\leftrightarrow$") with $\tilde{T}_f$:

$$f \leftrightarrow \tilde{T}_f(\varphi) = \int_{\mathbb{R}^d} \overline{f}\,\varphi = \int_{\mathbb{R}^d} \overline{\hat{f}(k)}\hat{\varphi}(k)\,\mathrm{d}k\,, \quad \forall\,\varphi \in \mathscr{S}(\mathbb{R}^d)\,.$$

Similarly:

$$\partial^\alpha f \leftrightarrow \tilde{T}_{\partial^\alpha f}(\varphi) = \int_{\mathbb{R}^d} \overline{\partial^\alpha f}\,\varphi = \int_{\mathbb{R}^d} \overline{\widehat{\partial^\alpha f}(k)}\hat{\varphi}(k)\,\mathrm{d}k = \int_{\mathbb{R}^d} (-2\pi \mathrm{i}k)^\alpha \overline{\hat{f}(k)}\hat{\varphi}(k)\,\mathrm{d}k$$

Thus we can identify the derivative ∂^α with the multiplication $(-2\pi\mathrm{i}k)^\alpha$ in the Fourier variable. Moreover, the formula

$$\tilde{T}_{\partial^\alpha f}(\varphi) = \int_{\mathbb{R}^d} \overline{\hat{f}(k)}\overline{(2\pi\mathrm{i}k)^\alpha}\hat{\varphi}(k)\,\mathrm{d}k = (-1)^{|\alpha|}\left\langle \hat{f}, (2\pi\mathrm{i}k)^\alpha\hat{\varphi}(k)\right\rangle$$

is well-defined even if $f \in \mathscr{S}'(\mathbb{R}^d)$.

Theorem 1.14. *Let $s \in \mathbb{N} = \{1, 2, \ldots\}$.*

(1) $\dot{H}^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}' \;\middle|\; \partial^\alpha f \in L^2(\mathbb{R}^d),\ \forall |\alpha| = s \right\}$

(2) $\dot{H}^{-s}(\mathbb{R}^d) = \left\{ f \in \mathscr{S}' \;\middle|\; f = \sum_{|\alpha|=s} \partial^\alpha g_\alpha,\ g_\alpha \in L^2(\mathbb{R}^d) \right\}$

Proof. (1) Take $f \in \dot{H}^s(\mathbb{R}^d)$. Then $f \in \mathscr{S}'$, $\hat{f} \in L^1_{\text{loc}}$ and $\int |k|^{2s}|\hat{f}(k)|^2\,\mathrm{d}k < \infty$. Then $\widehat{\partial^\alpha f}(k) = (-2\pi\mathrm{i}k)^\alpha \hat{f}(k)$ in $\mathscr{S}'$, but

$$\|(-2\pi\mathrm{i}k)^\alpha \hat{f}(k)\|_{L^2} < \infty \implies (-2\pi\mathrm{i}k)^\alpha \hat{f}(k) \in L^2 \implies \partial^\alpha f = \widecheck{(-2\pi\mathrm{i}k)^\alpha \hat{f}(k)} \in L^2$$

Reversely, if $f \in \mathscr{S}'$ with $\partial^\alpha f \in L^2$, $\forall |\alpha| = s$, we get immediately

$$\int_{\mathbb{R}^d} |k|^{2s}|\hat{f}(k)|^2\,\mathrm{d}k \le \sum_{|\alpha|=s} \|\,\partial^\alpha f\|_{L^2} < \infty$$

(2) Take $f \in \dot{H}^{-s}$. Then $\int |k|^{-2s}|\hat{f}(k)|^2 \,\mathrm{d}k < \infty$. Note that $|k|^{2s} = \sum_{|\alpha|=s} c_\alpha k^{2\alpha}$. Think of $f = \sum_{|\alpha|=s} \partial^\alpha g_\alpha$, for some $g_\alpha \in L^2$ to be chosen, then

$$\hat{f}(k) = \sum_\alpha (-2\pi \mathrm{i} k)^\alpha \widehat{g_\alpha}(k) \implies \underbrace{|k|^{-s}\hat{f}(k)}_{\in L^2} = (-2\pi\mathrm{i})^{|\alpha|} \sum_\alpha \frac{k^\alpha}{|k|^s} \widehat{g_\alpha}(k)$$

Thus, we have to choose $\widehat{g_\alpha}(k) := c_\alpha \frac{k^\alpha}{|k|^{2s}} \frac{\hat{f}(k)}{(-2\pi\mathrm{i})^s}$, then

$$\int |\widehat{g_\alpha}|^2 \le \int |c_\alpha|^2 |k|^{-2s} |\hat{f}(k)|^2 \,\mathrm{d}k < \infty \implies \widehat{g_\alpha} \in L^2 \implies g_\alpha \in L^2 .$$

We can check:

$$\widehat{\sum_{|\alpha|=s} \partial^\alpha g_\alpha} = \sum_{|\alpha|=s} c_\alpha \frac{k^\alpha}{|k|^{2s}} \frac{\hat{f}(k)}{(-2\pi\mathrm{i})^s} (-2\pi\mathrm{i}k)^\alpha = \hat{f}(k) \sum_{|\alpha|=s} c_\alpha \frac{k^{2\alpha}}{|k|^{2s}} = \hat{f}(k) .$$

The remaining inclusion is an exercise.

□

Fractional Sobolev spaces

$\dot{H}^s(\mathbb{R}^d)$, $s \notin \mathbb{Z}$.

Theorem 1.15. *Take $s \in (0,1)$, then*

$$\dot{H}^s(\mathbb{R}^d) = \left\{ f \in L^2_{\mathrm{loc}} \;\middle|\; \|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = C_{d,s} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y < \infty \right\} .$$

Proof. Take $f \in \dot{H}^s(\mathbb{R}^d)$. Then

$$f = \underbrace{\widecheck{\mathbb{1}_{B_1}\hat{f}}}_{\in L^2_{\mathrm{loc}}} + \underbrace{\widecheck{\mathbb{1}_{B_1^C}\hat{f}}}_{\in L^2}, .$$

We have:

$$\begin{aligned} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y &\overset{x-y=z}{\overset{\downarrow}{=}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(y+z)-f(y)|^2}{|z|^{d+2s}} \,\mathrm{d}z \,\mathrm{d}y \\ &\overset{\text{Plancherel in } y}{\overset{\downarrow}{=}} \int_{\mathbb{R}^d} \frac{1}{|z|^{d+2s}} \int_{\mathbb{R}^d} \left| \mathrm{e}^{2\pi\mathrm{i}k\cdot z}\hat{f}(k) - \hat{f}(k) \right|^2 \mathrm{d}k \,\mathrm{d}z \\ &= \underbrace{\int_{\mathbb{R}^d} \frac{1}{|z|^{d+2s}} \frac{|\mathrm{e}^{2\pi\mathrm{i}k\cdot z} - 1|}{|k|^{2s}} \,\mathrm{d}z}_{=:\, C_{d,s}} \underbrace{\int_{\mathbb{R}^d} |k|^{2s} |\hat{f}(k)|^2 \,\mathrm{d}k}_{=\|f\|_{\dot{H}^s(\mathbb{R}^d)<\infty}} \end{aligned}$$

$C_{d,s}$ is independent of k, which can be shown, using scaling, see exercise. The other inclusion goes exactly the same way. □

⚠ The new norm $\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}$ is a NON-local object: This is also what happens for the relativistic kinetic energy $\sqrt{m^2-\Delta}$ or $\sqrt{-\Delta}$. 5th lecture

Sobolev embedding

Theorem 1.16. *Let $0 < s < \frac{d}{2}$. Then*

$$\dot{H}^s \subset L^p(\mathbb{R}^d) \;\; \textit{for } p = \frac{2d}{d-2s}$$

with continuous embedding:

$$\|f\|_{L^p} \le C_d \|f\|_{\dot{H}^s(\mathbb{R}^d)} , \quad \forall f \in \dot{H}^s(\mathbb{R}^d)$$

Remark 1.17. $\exists ! p$ for the theorem to hold! This can be seen from a "scaling argument" (dimension analysis). For any $f \in \mathscr{S}(\mathbb{R}^d) \subset \dot{H}^s(\mathbb{R}^d)$, denote $f_\ell(x) = \ell^{\frac{d}{2}} f(\ell x)$. Then

$$\|f_\ell\|_{L^2}^2 = \int_{\mathbb{R}^d} \ell^d |f(\ell x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}^d} |f(y)|^2 \, \mathrm{d}y = \|f\|_{L^2}^2 \, .$$

So:

$$\|f_\ell\|_{L^p} = \Big(\int_{\mathbb{R}^d} \ell^{\frac{d}{2}p} |f(\ell x)|^p \, \mathrm{d}x \Big)^{\frac{1}{p}} = \big(\ell^{\frac{d}{2}p - d} |f(y)|^p \, \mathrm{d}y \big)^{\frac{1}{p}} = \ell^{\frac{d}{2} - \frac{d}{p}} \|f\|_{L^p}$$

$$\hat{f}_\ell(k) = \int_{\mathbb{R}^d} \ell^{\frac{d}{2}} f(\ell x) \mathrm{e}^{-2\pi \mathrm{i} k \cdot x} \, \mathrm{d}x = \ell^{\frac{d}{2} - d} \int_{\mathbb{R}^d} f(y) \mathrm{e}^{-2\pi \mathrm{i} \frac{k}{\ell} \cdot y} \, \mathrm{d}y = \ell^{-\frac{d}{2}} \hat{f}(k/\ell)$$

$$\implies \|f_l\|_{\dot{H}^s(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |2\pi k|^{2s} \ell^{-d} |\hat{f}(k/\ell)|^2 \, \mathrm{d}k \Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^d} |2\pi \xi \ell|^{2s} |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \Big)^{\frac{1}{2}} = \ell^s \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

Key observation: If $A, B > 0$ and $A\ell^\alpha \le B\ell^\beta$, $\forall \ell > 0$, for some $\alpha, \beta \in \mathbb{R}$, then $\alpha = \beta$ (necessary condition), since

$$\begin{cases} \text{If } \alpha < \beta, \text{ take } \ell \to 0 \text{ such that } \ell^\alpha \gg \ell^\beta \; ↯ \\ \text{If } \alpha > \beta, \text{ take } \ell \to \infty \text{ such that } \ell^\alpha \gg \ell^\beta \; ↯ \end{cases}$$

Consequently, to have $\|f_\ell\|_{L^p(\mathbb{R}^d)} \le C \|f_\ell\|_{\dot{H}(\mathbb{R}^d)}$, $\forall \ell > 0$, then a necessary condition is

$$\frac{d}{2} - \frac{d}{p} = s \iff p = \frac{d}{\frac{d}{2} - s} = \frac{2d}{d - 2s} \, .$$

(e.g. if $s = 1$, $\dot{H}(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, $\dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4)$: "Sobolev inequality becomes weaker in higher dimenions".

Proof. (of the Sobolev inequality, Chemin-Xu's proof of 1977):

$$\begin{aligned} K := \|f\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^d} |2\pi k|^{2s} |\hat{f}(k)|^2 \, \mathrm{d}k \\ &= \int_{\mathbb{R}^d} \Big(\int_0^\infty \mathbb{1}_{\{|2\pi k|^{2s} > E\}} \, \mathrm{d}E \Big) |\hat{f}(k)|^2 \, \mathrm{d}k && \text{(Layer-cake representation)} \\ &= \int_0^\infty \Big(\int_{\mathbb{R}^d} \mathbb{1}_{\{|2\pi k|^{2s} > E\}} |\hat{f}(k)|^2 \, \mathrm{d}k \Big) \mathrm{d}E && \text{(Fubini)} \\ &= \int_0^\infty \Big(\int_{\mathbb{R}^d} |f^{E_+}(x)|^2 \, \mathrm{d}x \Big) \mathrm{d}E, \quad \text{with } \widehat{f^{E_+}}(k) := \mathbb{1}_{\{|2\pi k|^{2s} > E\}} \hat{f}(k) && \text{(Plancherel)} \\ &= \int_{\mathbb{R}^d} \Big(\int_0^\infty |f^{E_+}(x)|^2 \, \mathrm{d}E \Big) \mathrm{d}x && \text{(Fubini)} \end{aligned}$$

How can we relate $f^{E_+}(x)$ to $f(x)$? We can write

$$f^{E_+} = f(x) - f^{E_-}(x) \, , \text{ where } \widehat{f^{E_-}}(k) = \mathbb{1}_{\{|2\pi k|^{2s} \le E\}} \hat{f}(k)$$

(then $\widehat{f^{E_+}}(k) + \widehat{f^{E_-}}(k) = \hat{f}(k)$).
Key step: By the triangle inequality:

$$|f^{E_+}(x)| = |f(x) - f^{E_-}(x)| \ge |f(x)| - |f^{E_-}(x)|$$

and by Hölder:

$$\begin{aligned} |f^{E_-}(x)| &= \Big| \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} k \cdot x} \mathbb{1}_{\{|2\pi k|^{2s} \le E\}} \hat{f}(k) \, \mathrm{d}k \Big| \le \Big(\int_{\mathbb{R}^d} |2\pi k|^{2s} |\hat{f}(k)|^2 \, \mathrm{d}k \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|2\pi k|^{2s} \le E\}}}{|2\pi k|^{2s}} \, \mathrm{d}k \Big)^{\frac{1}{2}} \\ &= C K^{\frac{1}{2}} E^{\frac{d-2s}{4s}} \end{aligned}$$

$$\implies |f^{E_+}(x)| \geq |f(x)| - CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}} \implies |f^{E_+}(x)|^2 \geq \left(|f(x)| - CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}}\right)_+^2 ,$$

where $(\cdot)_+$ denotes the positive part. Conclude:

$$K = \int_{\mathbb{R}^d} \int_0^\infty |f^{E_+}(x)|^2 \,\mathrm{d}E \,\mathrm{d}x \geq \int_{\mathbb{R}^d} \int_0^\infty \left(|f(x)| - CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}}\right)_+^2 \mathrm{d}E \,\mathrm{d}x$$

Integrate over $\{E \in (0,\infty) \,|\, |f(x)| > CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}}\} = \left\{E \in (0,\infty) \,\middle|\, E < C\left(\frac{|f(x)|}{K^{\frac{1}{2}}}\right)^{\frac{4s}{d-2s}}\right\}$:

$$\begin{aligned}
\int_0^\infty \left(|f(x)| - CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}}\right)_+^2 \mathrm{d}E &= \int_0^{C\left(\frac{|f(x)|}{K^{\frac{1}{2}}}\right)^{\frac{4s}{d-2s}}} \left(|f(x)| - CK^{\frac{1}{2}} E^{\frac{d-2s}{4s}}\right)^2 \mathrm{d}E \\
&= C|f(x)|^2 \left(\frac{|f(x)|}{K^{\frac{1}{2}}}\right)^{\frac{4s}{d-2s}}
\end{aligned}$$

Hence;

$$\begin{aligned}
K &\geq C \int_{\mathbb{R}^d} |f(x)|^2 \left(\frac{|f(x)|}{K^{\frac{1}{2}}}\right)^{\frac{4s}{d-2s}} \mathrm{d}x = \frac{C}{K^{\frac{2s}{d-2s}}} \int |f(x)|^{\frac{2d}{d-2s}} \,\mathrm{d}x \\
\implies K^{\frac{d}{d-2s}} &\geq C \int_{\mathbb{R}^d} |f(x)|^{\frac{2d}{d-2s}} \,\mathrm{d}x \\
\implies \left(\int_{\mathbb{R}^d} |f(x)|^{\frac{2d}{d-2s}}\right)^{\frac{d-2s}{2d}} &\leq CK^{\frac{1}{2}} = C\|f\|_{\dot{H}^s}
\end{aligned}$$

□

<u>Inhomogeneous Sobolev spaces</u>

For $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}' \,\middle|\, \hat{f} \in L^2_{\mathrm{loc}},\ \int_{\mathbb{R}^d} (1 + |2\pi k|^2)^s |\hat{f}(k)|^2 \,\mathrm{d}k =: \|f\|_{H^s}^2 < \infty \right\}$$

Obviously:

- $H^s(\mathbb{R}^d)$ is a Hilbert space $\forall s \in \mathbb{R}$
- $H^s \subset H^t$ if $t < s$
- $H^{-s} = (H^s)'$

Theorem 1.18 (Sobolev embedding). *If $0 < s < \frac{d}{2}$, then $H^s(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$, $\forall 2 \leq p \leq \frac{2d}{d-2s}$ continuously embedded.*

Proof. exercise □

<u>General discussion of compactness</u>

(i) Let X be a Banach space. Then a set $A \subset X$ is *compact* if every sequence $\{f_n\}_{n=1}^\infty \subseteq A$ has a convergent subsequence $f_{n_k} \xrightarrow{k\to\infty} f \in A$.
<u>Remark</u>: If $A = Y \cap \overline{B(0,1)}$, where Y is a subspace of X, then A is compact iff $\dim Y < \infty$.

(ii) Let X be a Hilbert space. Then $A \subset X$ is *weakly compact*, iff $\forall \{f_n\}_{n=1}^\infty \subset A$, $\exists$ a subsequence $f_{n_k} \rightharpoonup f \in A$, for $k \to \infty$, i.e.

$$\langle f_{n_k}, \varphi \rangle_X \to \langle f, \varphi \rangle_X\,, \quad \forall \varphi \in X\,.$$

More generally, if $A \subset L^p(\mathbb{R}^d)$ (or $L^p(\Omega)$), with $1 \le p < \infty$, then A is compact if $\forall \{f_n\}_{n=1}^\infty \subset A$, there exists a subsequence $f_{n_k} \rightharpoonup f$ in L^p, i.e. for $\frac{1}{p} + \frac{1}{q} = 1$ and all $g \in L^q$:

$$\int_{\mathbb{R}^d} g(x) f_{n_k}(x)\,\mathrm{d}x \xrightarrow{k\to\infty} \int_{\mathbb{R}^d} g(x) f(x)\,\mathrm{d}x$$

Theorem 1.19 (Banach-Alaoglu). *If $\{f_n\}_{n=1}^\infty$ is a bounded sequence in a Hilbert space, then there exists a weakly convergent subsequence $f_{n_k} \rightharpoonup f$ in the Hilbert space. Put differently, $\overline{B(0,1)}$ is a weak-compact set.*

Proof. Tutorial □

Theorem 1.20 (Banach-Steinhaus). *If $f_n \rightharpoonup f$ in a Hilbert space, then*

$$\|f\| \le \liminf_{n\to\infty} \|f_n\| \le \limsup_{n\to\infty} \|f_n\| < \infty$$

Moreover,

$$f_n \xrightarrow{n\to\infty} f \iff \begin{cases} f_n \rightharpoonup f \\ \|f_n\| \to \|f\| \end{cases}$$

Remark 1.21. The Banach-Alaoglu and Banach-Steinhaus theorems also hold for $L^p(\Omega)$, $\forall 1 < p < \infty$.

Definition 1.22. If $T\colon X \to Y$ is a linear operator between two Banach spaces, then we say that T is a *compact operator* iff

$$\forall x_n \rightharpoonup x \text{ in } X \implies Tx \xrightarrow{n\to\infty} Tx \text{ in } Y\,.$$

Theorem 1.23 (Sobolev compact embedding). *Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $t < s$. Then multiplication by φ is a compact operator $H^s(\mathbb{R}^d) \to H^t(\mathbb{R}^d)$, i.e. if $f_n \rightharpoonup f$ weakly in $H^s(\mathbb{R}^d)$, then $\varphi \cdot f_n \xrightarrow{n\to\infty} \varphi \cdot f$ in $H^t(\mathbb{R}^d)$.*

Proof. We take $f_n \rightharpoonup f$ in H^s and consider

$$\begin{aligned} \|\varphi f_n\|_{H^t}^2 &= \int_{\mathbb{R}^d} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k \\ &= \int_{|k|\le R} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k + \int_{|k|\ge R} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k \end{aligned}$$

We have:

$$\begin{aligned} \int_{|k|\ge R} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k &\le (1 + |2\pi R|^2)^{t-s} \int_{\mathbb{R}^d} (1 + |2\pi k|^2)^s |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k \\ &\le \underbrace{\sup_{n\in\mathbb{N}} \|\varphi f_n\|_{H^s(\mathbb{R}^d)}}_{<\infty} (1 + |2\pi R|^2)^{t-s} \xrightarrow{R\to\infty} 0 \end{aligned}$$

Here we used, that $\varphi\colon H^s \to H^s$ is a bounded operator, i.e. $\|\varphi \cdot f\|_{H^s} \le C\|f\|_{H^s}$ and that a weakly convergent sequence is norm bounded. On the other hand, $\forall R > 0$:

$$\int_{|k|\le R} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k \le C_{t,R} \int_{|k|\le R} |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k$$

We note that $\forall |k| \le R$:

$$|\widehat{\varphi f_n}(k)| = \left| \int_{\mathbb{R}^d} \hat{\varphi}(k - \xi) \hat{f}_n(\xi)\,\mathrm{d}\xi \right| = \left| \int_{\mathbb{R}^d} \underbrace{\frac{\hat{\varphi}(k-\xi)}{(1 + |2\pi\xi|^2)^s}}_{\in H^s(\mathbb{R}^d)} (1 + |2\pi\xi|^2)^s \hat{f}_n(\xi)\,\mathrm{d}\xi \right| \xrightarrow{n\to\infty} 0$$

Claim: $\forall n$: $\|\widehat{\varphi f_n}(k)\|_{L^\infty(B_R)} \le C_R$, since $\{f_n\}$ is bounded in H^s and $\varphi \in \mathscr{S}(\mathbb{R}^d)$ (exercise) Thus:

$$\int_{|k|\le R} (1 + |2\pi k|^2)^t |\widehat{\varphi f_n}(k)|^2\,\mathrm{d}k \xrightarrow{n\to\infty} 0$$

by dominated convergence. □

2nd tutorial

Theorem. *Let X be a Banach space. Then $\overline{B_1(0)}$ is compact iff $\dim X < \infty$.*

$\rightsquigarrow$ Weak topology

Definition. Let X be a Banach space. Then $x \rightharpoonup x$ weakly in X iff $L(x_n) \to L(x)$, $\forall L \in X^*$.

Remark. If $x_n \to x$ strongly, i.e. $\|x_n - x\| \to 0$, then $x_n \rightharpoonup x$ weakly, since

$$|L(x_1) - L(x_2)| \leq \|L\|\|x_n - x\| \to 0\,.$$

The weak topology does not coincide with the strong topology in infinite dimensions.

Definition. Let X be a Banach space. Then $L_n \xrightarrow[w^*]{n\to\infty} L$ weakly in X^*, iff $L_n(x) \to L(x)$, $\forall x \in X$.

Theorem (Banach-Alaoglu). *Let X be a seperable Banach-space. Then every sequence $\{L_n\}_{n\in\mathbb{N}} \subseteq X^*$ such that $\|L_n\|$ is bounded in n has a convergent subsequence in X^*.*

Remark. X is seperable iff there exists a countable dense subset. The assumption that X is seperable can be relaxed, however, then Tychonoff/Zorn's Lemma is needed for the proof. E.g. $L^p(\mathbb{R}^d)$, $p < \infty$ is seperable.

Proof. (Compactness in $\mathbb{C}$ and Cantor's diagonal argument)

- Since X is seperable, there exists $\{x_n\}_n$ dense in X.
- Since $\{L_n(x_1)\}$ is a bounded sequence in $\mathbb{C}$, there exists a subsequence $\{L_{n_{k_1}}(x_1)\}$ which converges when $k_1 \to \infty$.
- Since $\{L_{n_{k_1}}(x_2)\}$ is a bounded sequence in $\mathbb{C}$, there exists a subsequence $\{L_{n_{k_2}}(x_2)\}$ which converges when $k_2 \to \infty$.

By induction, $\forall m \in \mathbb{N}$, there exists a subsequence $\{L_{k_m}(x_m)\}$ which converges for $k_m \to \infty$. We take a common subsequence $\{L_{n_k}\}$ of all $\{L_{n_{k_m}}\}$ (which is doable by Cantors diagonal argument). Finally, we construct $L \in X^*$, such that $L(x_m) = \lim_{k\to\infty} L_{n_k}(x_m)$, $\forall m$.
Easy thought: $\forall x \in X\, \exists$ sequence $\{y_m\} \subseteq \{x_m\}$ such that $y_m \to x$ as $m \to \infty$. We want to define $L(x) := \lim_{m\to\infty} L(y_m)$.
Why is it well-defined? $\lim_{k\to\infty} L_{n_k}(y_m) =: \{L(y_m)\}$ is a Cauchy sequence. In general, if a bounded linear operator is defined on a dense subset of a Banach space and maps into Banach space, there exists a unique linear continuous extension. ☐

In general, if X is not seperable, then to prove the Banach-Alaoglu theorem, we need to do "induction" in uncountable families $\rightsquigarrow$ Zorn lemma/Axiom of choice.

Remark. If X is a Hilbert space:

- We can identify X^* to X by Riesz's representation theorem.
  $\Longrightarrow$ The Banach-Alaoglu theorem gives us the compactness in the weak topology, namely, for every bounded sequence $\{x_n\} \subset X$, there exists a subsequence $x_{n_k} \rightharpoonup x$ in X, i.e.
  $$\langle x_{n_k}, \varphi\rangle \to \langle x, \varphi\rangle \;, \forall\, \varphi \in X\,.$$
- When X is a Hilbert space, we can relax the seperability condition easily. Indeed, if $\{x_n\} \subset X$ is a bounded sequence, then consider $Y = \overline{\text{span}(x_n)} \subset X$. Then Y is a seperable Hilbert space, so Banach-Alaoglu for seperable spaces applies:
  $$\exists \text{ subsequence } x_{n_k} \rightharpoonup x \text{ in } Y$$
  But this means $x_{n_k} \rightharpoonup x$ in X:
  Key: $X = Y \oplus Y^\perp$. Then $\forall\, \varphi \in X$, $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \in Y$, $\varphi_2 \in Y^\perp$:
  $$\langle x_{n_k}, \varphi\rangle - \langle x, \varphi\rangle = \langle x_{n_k} - x, \varphi_1\rangle + \underbrace{\langle x_{n_k}, \varphi_2\rangle}_{=0} \xrightarrow{k\to\infty} 0$$

Definition. Let X be a Banach space. We call X a *reflexive space* iff $X^{**} = X$.

Remark. $X \subset X^{**}$ is the trivial direction: Let $x \in X$, then $X^* \ni L \mapsto L(x)$ is linear and continuous, so x can be interpreted as an element of X^{**} (canonical embedding). The key assumption here is that X contains all elements in X^{**}.

Remark. If X is reflexive, then by the Banach-Alaoglu theorem, we have the weak-compactness result for X, namely $\forall \{x_n\}_n$ bounded in X, there exists a subsequence $x_{n_k} \rightharpoonup x$ in X. The reason is that $x_{n_k} \rightharpoonup x$ weakly in X iff $x_{n_k} \to x$ weak-$*$ in X^{**}.

$\rightsquigarrow$ Brezis Functional Analysis, Sobolev spaces and PDE

Theorem (Banach-Steinhaus/Uniform boundedness principle). *Consider a family F of linear and continous mappings $\{T\}_{T\in F}$: $T\colon X \to Y$, X Banach, Y normed. If, $\forall x \in X$,*

$$\sup_{T\in F} \|T(x)\|_Y < \infty\,,$$

then

$$\sup_{T\in F} \|T\| < \infty\,.$$

Remark. (1) Consequently, if $L_n \to L$ weak-$*$ in X^* (Banach), then

$$\sup_n \|L_n\| < \infty\,.$$

This is because $\forall x \in X$, $L_n(x) \to L(x) \implies \sup_n \|L_n(x)\| < \infty$, so the claim follows by Banach-Steinhaus.

(2) If $x_n \to x$ weakly in X, then $\sup_n \|x_n\| < \infty$. Use $X \subset X^{**}$ and Hahn-Banach.

Proof.
- One proof (direct): Analysis Lieb-Loss Theorem 2.12 (for L^p)
- The standard proof: Define

$$X_n = \left\{ x \in X \,\middle|\, \sup_{T\in F} \|T(x)\|_Y \le n \right\}\,.$$

Then X_n is closed and $\bigcup_{n=1}^{\infty} X_n = X$. By the Baire category theorem, $\exists n$, $\exists$ open ball $B_{x_0}(r_0) \subset X_n$, i.e.

$$|T_n(x)| \le n,\ \ \forall x \in B_{x_0}(r_0) \implies \|T(x-x_0)\| \le n + \|T_{x_0}\|,\ \ \forall y := x - x_0 \in B_0(r_0)$$
$$\implies \|Ty\| \le C\ \ \forall y \in B_0(r_0) \implies \|T\| \le \frac{C}{r_0}\,,\ \ \forall T \in F$$

□

6th lecture

Theorem 1.24 ($s > \frac{d}{2}$). *$H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ with continuous embedding.*

Proof.

$$\|f\|_{L^\infty} \le \|\hat{f}\|_{L^1} = \int_{\mathbb{R}^d} |\hat{f}(k)|\,\mathrm{d}k \le \left(\int_{\mathbb{R}^d} (1+|k|^2)^s |\hat{f}(k)|^2\,\mathrm{d}k\right)^{\frac12} \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(1+|k|^2)^s}\right)^{\frac12}$$
$$\le C_{d,s}\|f\|_{H^s(\mathbb{R}^d)}$$

□

If we work harder, we find that $f \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, is Hölder continuous:

$$\sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \le C < \infty \quad \text{for some } \alpha > 0$$

(exercise)

Theorem 1.25 **($s = \frac{d}{2}$).** *We have the Moser-Trudinger inequality:* $\exists c_0, C > 0$*:*

$$\forall \|f\|_{H^s(\mathbb{R}^d)} \leq 1, \quad \int_{\mathbb{R}^d} \big(\exp(c_0 |f(x)|^2) - 1\big)\,\mathrm{d}x \leq C$$

Proof. exercise ☐

Remark 1.26. The Moser-Trudinger inequality fails if we only assume $\|f\|_{\dot{H}^{\frac{d}{2}}(\mathbb{R}^d)} \leq 1$. However, this is ok, if we work on bounded sets.

Theorem 1.27 **($s = \frac{d}{2}$, original Moser-Trudinger version).** *Let* $f \in \dot{H}^{\frac{d}{2}}$ *with* $\|f\|_{\dot{H}^{\frac{d}{2}}(\mathbb{R}^d} \leq 1$ *and* $\operatorname{supp} f \subset B_R$*, then*

$$\int_{B_R} \big(\exp(c_0 |f(x)|^2 - 1)\,\mathrm{d}x\big) \leq C_R$$

This version follows from the previous version, since

$$\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} \geq \|f\|_{L^2(\mathbb{R}^d)} \quad \text{if } \operatorname{supp} f \subset B_R$$

$$\iff \|f\|_{\dot{H}^{\frac{d}{2}}(\mathbb{R}^d)} \sim_R \|f\|_{H^s(\mathbb{R}^d)}$$

example: $H^1(\mathbb{R}^2) \not\subset L^\infty(\mathbb{R}^2)$ but "almost".

Theorem 1.28 **($s = \frac{d}{2}$, BMO - bounded mean osscilation).** *Let* $f \in \dot{H}^{\frac{d}{2}} \cap L^1_{\text{loc}}$*, then*

$$\|f\|_{BMO} := \sup_{B \ Ball} ⨍_B |f - \overline{f}_B|\,\mathrm{d}x \leq C\|f\|_{\dot{H}^{\frac{d}{2}}(\mathbb{R}^d)},$$

where

$$\overline{f}_B = ⨍_B f = \frac{1}{|B|}\int_B f\,.$$

Proof. We can even control the variant:

$$⨍_B |f - \overline{f}_B|^2\,\mathrm{d}x \lesssim ⨍_B |(f - \overline{f}_B)^{E_-}|^2_+ + ⨍_B |(\underbrace{f - \overline{f}_B}_{g})^{E_+}|^2 \leq \|(f - \overline{f}_B)^{E_-}\|^2_{L^\infty} + R^{-d}\|f^{E_+}\|^2_{L^2(2B)}$$

$$\leq \|\nabla f^{E_-}\|^2_{L^\infty} R^2 + R^{-d}\int \frac{|k|^{2s}}{E^{2s}}|\hat{f}(k)|\,\mathrm{d}k \leq \underbrace{\Big(\int_{|k|\leq E} |k||\hat{f}(k)|\,\mathrm{d}k\Big)^2 R^2}_{\substack{\leq (\int_{|k|\leq E}|k|^{2-2s})(\int_{|k|\leq E}|k|^{2s}|\hat{f}(k)|^2)R^2 \\ \leq (E^2R^2 + \frac{1}{R^dE^d})\|f\|^2_{\dot{H}}}} + R^{-d}\frac{\|f\|^2_{\dot{H}}}{E^d}$$

$$\leq 2\|f\|^2_{\dot{H}}$$

by taking $E = \frac{1}{R}$.
Using: $f \in \dot{H}^{\frac{d}{2}}(\mathbb{R}^d)$, $f = f^{E_+} + f^{E_-}$, $\widehat{f^{E_+}}(k) = \hat{f}(k)\mathbb{1}(|2\pi k|^2 > E)$.
This proof will be discussed again in the 3rd tutorial.

☐

Hardy inequality

If $d \geq 3$, then

$$\|f\|^2_{\dot{H}^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla f|^2 \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{|f(x)|^2}{|x|^2}\,\mathrm{d}x$$

The constant $\frac{(d-2)^2}{4}$ on the RHS is optimal, but there exists no optimizer $f \in \dot{H}^1$.

Proof. ($d = 3$) Candidate of optimizer solves:

$$-\Delta f = \frac{1}{4}\frac{f(x)}{|x|^2}$$

This has a solution $f(x) = \frac{1}{|x|^{\frac{1}{2}}}$ (or $1/|x|^{\frac{d-2}{2}}$ in $d \geq 3$ dimensions).($f \sim \frac{1}{|x|^\alpha}$ at $|x| \to \infty$, $\Delta f \sim \frac{1}{|x|^{\alpha+2}}$). But $\frac{1}{|x|^{\frac{1}{2}}} \notin \dot{H}^1(\mathbb{R}^d)$. This is, however, not a problem to run the "ground state substitution", i.e. $g(x) = |x|^{1/2} f(x)$. Then

$$\int_{\mathbb{R}^d} |\nabla f|^2 - \int_{\mathbb{R}^d} \frac{1}{4|x|^2}|f(x)|^2 = \int_{\mathbb{R}^d} |\nabla g|^2 \geq 0\,.$$

□

Theorem 1.29 (Perron-Frobenius principle for Schrödinger operator). *Assume $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $(-\Delta - V(x))\psi(x) = E\psi(x)$ holds for a constant $E \in \mathbb{R}$ and $\forall x \in \Omega \subset \mathbb{R}^d$, $\psi(x) > 0$. Then*

$$-\Delta - V(x) \geq E \text{ in } L^2(\Omega) \text{ with Dirichlet boundary condition}$$
$$\iff \int_\Omega |\nabla \varphi|^2 - \int_\Omega V(x)|\varphi(x)|^2 \geq E \int_\Omega |\varphi(x)|^2 \,\mathrm{d}x\,, \quad \forall \varphi \in C_c^\infty$$

This theorem can be proved by the same ground state substitution technique, namely $\forall \varphi \in C_c^1(\Omega)$, define $g(x) = \frac{\varphi(x)}{\psi(x)}$.

$$\rightsquigarrow \text{LHS} - \text{RHS} \to \int |\nabla g|^2 \geq 0$$

(exercise)

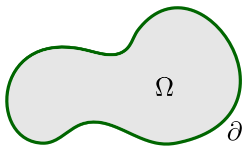
TRACE OPERATOR

1D case: We know that $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, even Hölder continuous. This means that if $f \in H^1(\mathbb{R})$, $f(0)$ is completely well-defined.

2D case: $H^1(\mathbb{R}^2) \not\subseteq L^\infty(\mathbb{R}^2)$. $f|_\Omega := \mathbb{1}_\Omega f \in L^2(\Omega)$ easily makes sense, but $f|_\Omega$ is not easily defined! However, this is doable thanks to the trace operator.
In general, if $\Omega \subset \mathbb{R}^d$, $\partial\Omega$ "smooth enough", then

$$\forall f \in H^1(\mathbb{R}^d) \rightsquigarrow f|_{\partial\Omega} \in L^2(\partial\Omega)$$

uniquely defined (exercise).



Theorem 1.30. *Define $B\colon \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{d-1})$ as follows:*

$$\varphi(x_1, \ldots, x_d) \mapsto (B\varphi)(x_2, \ldots, x_d) = \varphi(0, x_2, \ldots, x_d)$$

Then, $\forall s \in \mathbb{R}$, the mapping B extends continuously

$$H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})\,,$$

i.e. $\|B\varphi\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} \leq C_{d,s}\|\varphi\|_{H^s(\mathbb{R}^d)}$.

Proof. Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and for $x \in \mathbb{R}^d$, denote $x = (x_1, \ldots, x_d) = (x_1, x')$, $x' \in \mathbb{R}^{d-1}$ ($k = (k_1, k')$ respectively). Then $(B\varphi)(x') = \varphi(0, x')$, so

$$\widehat{B\varphi}(k') = \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-2\pi \mathrm{i} k'\cdot x'}\,\varphi(0, x')\,\mathrm{d}x' = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k'\cdot x'} \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i} k_1\cdot x_1}\,\varphi(x_1, x')\,\mathrm{d}x_1}_{\text{Fourier transform in the first variable}}\,\mathrm{d}k_1\,\mathrm{d}x'$$
$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \hat{\varphi}(k_1, k')\,\mathrm{d}k_1\,.$$

Here we used the fact that for $f \in \mathscr{S}(\mathbb{R})$:

$$\int_{\mathbb{R}} \hat{f}(k_1)\,\mathrm{d}k_1 = f(0) \left(\iff \hat{f}(0) = \int_{\mathbb{R}} f(x)\,\mathrm{d}x \right).$$

Consequently:

$$\begin{aligned}
&\|B\varphi\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} \\
&= \int_{\mathbb{R}^{d-1}} (1+|2\pi k'|^2)^{s-\frac{1}{2}} |\widehat{B\varphi}(k)|^2\,\mathrm{d}k' \lesssim \int_{\mathbb{R}^{d-1}} (1+|k'|^2)^{s-\frac{1}{2}} \left| \int_{\mathbb{R}} \hat{\varphi}(k_1,k')\,\mathrm{d}k_1 \right|^2 \mathrm{d}k' \\
&\leq \int_{\mathbb{R}^{d-1}} (1+|k'|^2)^{s-\frac{1}{2}} \left| \int_{\mathbb{R}} \hat{\varphi}(k_1,k')\,\mathrm{d}k_1 \right|^2 \mathrm{d}k' \\
&\leq \int_{\mathbb{R}^{d-1}} (1+|k'|^2)^{s-\frac{1}{2}} \left(\int_{\mathbb{R}} (1+k_1^2+|k'|^2)^s |\hat{\varphi}(k_1,k')|^2\,\mathrm{d}k_1 \right) \underbrace{\int_{\mathbb{R}} (1+k_1^2+|k'|^2)^{-s}\,\mathrm{d}k_1}_{\lesssim (1+|k'|^2)^{-s+\frac{1}{2}}} \mathrm{d}k' \quad (\text{ok if } s > \tfrac{1}{2}) \\
&\lesssim \int_{\mathbb{R}^d} (1+k_1^2+|k'|^2)^s |\hat{\varphi}(k,k')|^2\,\mathrm{d}k_1\,\mathrm{d}k' = \|\varphi\|^2_{H^s(\mathbb{R}^d)}
\end{aligned}$$

Note, that

$$\int_{\mathbb{R}} \frac{1}{k_1^2+1}\,\mathrm{d}k_1 \overset{k_1=\sqrt{L}y}{\underset{\downarrow}{=}} \int_{\mathbb{R}} \frac{1}{L(1+y^2)} \sqrt{L}\,\mathrm{d}y = L^{-\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{1+y^2}\,\mathrm{d}y \right) < \infty$$

and similarly, if $s > \frac{1}{2}$:

$$\int_{\mathbb{R}} \frac{1}{(k_1^2+L)^s}\,\mathrm{d}k_1 = cL^{-s+\frac{1}{2}}$$

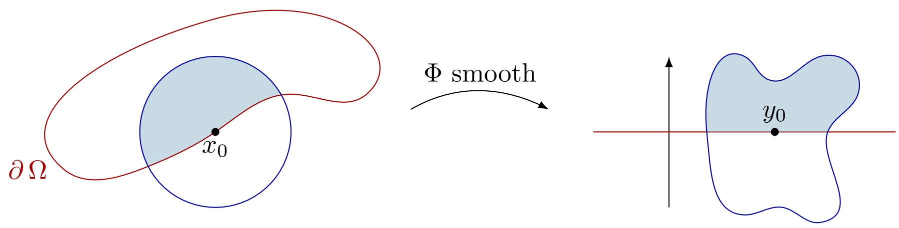
Thus $\|B\varphi\|_{H^{s-\frac{1}{2}}(\mathbb{R}^d)} \leq C_{d,s}\|\varphi\|_{H^s(\mathbb{R}^d)}$, $\forall \varphi \in \mathscr{S}(\mathbb{R}^d)$ and $s > \frac{1}{2}$. This allows us to extend the action of B to all $\varphi \in H^s$, $\forall s > \frac{1}{2}$. For general s, we need a duality argument (book). □

Theorem 1.31 (General domain $\Omega \subset \mathbb{R}^d$). *Assume $\Omega \subseteq \mathbb{R}^d$ is open, bounded and $\partial\Omega$ is smooth. Then the trace operator*

$$B\colon H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$$

is well-defined and bounded. Here, $H^s(\Omega) = \{f|_\Omega \mid f \in H^s(\mathbb{R}^d)\}$.

The smoothness condition on $\partial\Omega$ gives:



Since Ω is bounded, $\partial\Omega$ is compact an can be covered by finitely many of those balls. Then apply the previous trace theory to the flat boundary case.

2 Some basic evolution equations

7th lecture

2.1 Heat equation

$$\begin{cases} \partial_t f(t,x) = \Delta f(t,x), & t>0,\ x\in\mathbb{R}^d \\ f(0,x) = f_0(x) \end{cases}$$

Basic calculation: Take the Fourier transform in $x \in \mathbb{R}^d$:

$$\begin{cases} \partial_t \hat{f}(t,k) = -|2\pi k|^2 \hat{f}(t,k), & t>0,\ k\in\mathbb{R}^d \\ \hat{f}(0,k) = \widehat{f_0}(k) \end{cases}$$

Then $g(t) = \hat{f}(t,k)$ solves the ODE

$$\begin{cases} \partial_t g = -|2\pi k|^2 g \\ g(0) = g_0 \text{ given} \end{cases} \implies g(t) = e^{-t|2\pi k|^2} g_0$$

Hence, for $k \in \mathbb{R}^d$, $t > 0$:

$$\hat{f}(t,k) = \mathrm{e}^{-t|2\pi k|^2}\widehat{f_0}(k) = \widehat{\mathcal{F}^{-1}(\mathrm{e}^{-t|2\pi k|^2})}\cdot \widehat{f_0}(k) = \widehat{\mathcal{F}^{-1}(\mathrm{e}^{-t|2\pi k|^2}) * f_0}(k)$$

Using that

$$\widehat{\mathrm{e}^{-\pi|x|^2}}(k) = \mathrm{e}^{-\pi|k|^2} \implies \widehat{\mathrm{e}^{-t|2\pi x|^2}}(k) = \widehat{\mathrm{e}^{-\pi(\sqrt{4\pi t}|x|)^2}}(k) = \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\pi\frac{|k|^2}{4\pi t}} = \frac{1}{(4\pi t)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|k|^2}{4t}}\,,$$

we obtain:

$$f(t,x) = \mathcal{F}^{-1}(\mathrm{e}^{-t|2\pi k|^2}) * f_0(x) = \int_{\mathbb{R}^d} K(x-y) f_0(y)\,\mathrm{d}y =: (\mathrm{e}^{t\Delta})f_0(x), \quad \hat{K}(k) = \mathrm{e}^{-t|2\pi k|^2}$$

<u>Heat kernel</u>: $K(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\mathrm{e}^{-\frac{|x|^2}{4t}}$

Theorem 2.1. *(1) For all $f_0 \in \mathscr{S}'(\mathbb{R}^d)$, there exists a unique solution of the heat equation $f(t,x) = (\mathrm{e}^{t\Delta} f_0)(x)$. If f_0 is regular enough, then*

$$f(t,x) = (\mathrm{e}^{t\Delta} f_0)(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} f_0(y)\,\mathrm{d}y\,,$$

e.g. $f_0 \in L^2(\mathbb{R}^d)$.

(2) (Smoothing effect) If $f_0 \in H^{s_0}(\mathbb{R}^d)$ for some $s_0 \in \mathbb{R}$, then $f_t \in \bigcap_{s>0} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

(3) (Asymptotic behaviour) If $f_0 \in L^2(\mathbb{R}^d)$, then $f_t \xrightarrow{t\downarrow 0} f_0$ in L^2 and $\|f_t\|_{L^2(\mathbb{R}^d)} \xrightarrow{t\to\infty} 0$.

Proof. (1) **Existence**: If $f_0 \in L^2$, then $\widehat{f_0}$ makes sense and we can run the previous argument. If $f_0 \in \mathscr{S}'$, then we need to be precise with the meaning of solution.

Definition 2.2. We say that the family $\{f(t,\bullet)\}_{t>0} \subseteq \mathscr{S}'$ is a distributional solution of the heat equation iff

$$\begin{cases} \partial_t \langle f(t,\bullet),\varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle f(t,\bullet),\Delta\varphi\rangle_{\mathscr{S}',\mathscr{S}}\,, & \forall\varphi\in\mathscr{S}(\mathbb{R}^d),\ \forall t>0 \\ \langle f(0,\bullet),\varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle f_0,\varphi\rangle_{\mathscr{S}',\mathscr{S}} \end{cases}$$

Uniqueness: We want to prove that if $f_0 = 0$, then $f_t \equiv 0$ is the unique solution (this is enough by linearity). This basically follows from the fact

$$\begin{cases} \partial_t \hat{f}(t,k) = -|2\pi k|^2\hat{f}(t,k) \\ \hat{f}(0,k) = 0 \end{cases} \implies \hat{f}(t,k) = 0\,.$$

This is ok, if f is regular enough. For general distributions, this argument can be done by the weak formulation: For $y \in \mathbb{R}^d$, define $\varphi_y(x) := \varphi(x-y)$. Then, the distribution $f_t * \tilde{\varphi} := f_t \circ (\bullet * \varphi)$ is given by a Schwartz function, namely $g_t \colon y \mapsto \langle f_t, \varphi_y \rangle$. The Fourier transfom of g_t is given by:

$$\hat{g}_t(\xi) = \int_{\mathbb{R}^d} \langle f_t, \varphi_y \rangle \, \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot y} \, \mathrm{d}y$$

Claim:

$$-|2\pi\xi|^2 \hat{g}_t(\xi) = \int_{\mathbb{R}^d} \langle f_t, \Delta_x \varphi_y \rangle \, \mathrm{e}^{-2\pi\mathrm{i}\xi\cdot y} \, \mathrm{d}y \tag{$*$}$$

Then:

$$\begin{aligned} &\begin{cases} \partial_t \langle f_t, \varphi_y \rangle = \langle f_t, \Delta \varphi_y \rangle, & \forall y \in \mathbb{R}^d \\ f_0 = 0 \end{cases} \\ \implies &\begin{cases} \partial_t \hat{g}_t(\xi) \overset{(*)}{=} -|2\pi\xi|^2 \hat{g}_t(\xi), & \forall \xi \in \mathbb{R}^d \\ \hat{g}_0(\xi) = 0 \end{cases} \\ \iff &\hat{g}_t = 0 \iff g_t = 0 \end{aligned}$$

We conclude: $\langle f_t, \varphi_y \rangle = 0$, $\forall y \in \mathbb{R}^d$, so $\langle f_t, \varphi \rangle = 0$, forall $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Hence $f = 0$.
Proof of ($*$): TODO

(2) $\forall t > 0$:

$$\|f(t,\bullet)\|_{H^s}^2 = \int_{\mathbb{R}^d} (1+|2\pi k|^2)^s |\hat{f}(t,k)|^2 \, \mathrm{d}k = \int_{\mathbb{R}^d} (1+|2\pi k|^2)^s \mathrm{e}^{-2t|2\pi k|^2} |\hat{f}_0(k)|^2 \, \mathrm{d}k < \infty \,,$$

when $\int_{\mathbb{R}^d} (1+|2\pi k|^2)^{s_0} |\hat{f}_0(k)|^2 \, \mathrm{d}k < \infty$ for some $s_0 \in \mathbb{R}$.

(3) Dominated convergence: $f_0 \in L^2$:

$$\|f(t,\bullet) - f_0\|_{L^2}^2 = \int_{\mathbb{R}^d} |\hat{f}(t,k) - \hat{f}_0(k)|^2 \, \mathrm{d}k = \int_{\mathbb{R}^d} \underbrace{|\mathrm{e}^{-t|2\pi k|^2} - 1|^2}_{|\cdot|<2} |\hat{f}_0(k)|^2 \, \mathrm{d}k \xrightarrow{t\to 0} 0$$

□

Theorem 2.3 (Tychonoff). *The equation*

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & x \in \mathbb{R},\ t > 0 \\ u(0,x) = 0 \end{cases}$$

has a nontrivial solution

$$u(t,x) = \sum_{n=0}^{\infty} \frac{\partial_t^n g(t) x^{2n}}{(2n)!} \,,$$

where $g(t) = \mathrm{e}^{-\frac{1}{t^2}} \in C^\infty(\mathbb{R}_+)$.

Proof. Easy step: Assuming, that the series is absolutely convergent for $t > 0$:

$$\partial_t u = \sum_{n=0}^{\infty} \frac{\partial_t^{n+1} g(t) x^{2n}}{(2n)!}$$

$$\partial_x^2 u = \sum_{n=0}^{\infty} \frac{\partial_t^n g(t)(2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{\partial_t^n g(t) x^{2(n-1)}}{(2(n-1))!} \overset{\substack{m=n-1\\ \downarrow}}{=} \sum_{m=0}^{\infty} \frac{\partial_t^{m+1} g(t) x^{2m}}{(2m)!}$$

Nontrivial: Why is the series absolutely convergent and why $u(t,x) \to 0$ as $t \to 0$?
absolute convergence: Note that:

$$g(t) = \mathrm{e}^{-\frac{1}{t^2}},\ \partial_t g = g\left(-\frac{2}{t^3}\right),\ \partial_t^2 g = \left(\frac{2}{t^3}\right) + g\frac{2\cdot 3}{t^4},\ \partial_t^3 g = -g\left(\frac{2}{t^3}\right)^3 + \cdots$$

Lemma 2.4. *$\forall n \in \mathbb{N}: \ \partial_t^n g(t) = P_n(\frac{1}{t})g(t)$, where P_n is a polynomial of degree $3n$ and all coefficients of P_n are bounded by $n!3^n$.*

This concludes the (absolute) convergence of the series:

$$|u(t,x)| \leq \sum_{n=0}^{\infty} \frac{|\,\partial_t^n g(t)||x|^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{|g(t)|P_n(\frac{1}{t})||x|^{2n}}{(2n)!} \overset{\substack{\text{for some} \\ |\alpha_{n,k}|\leq n!3^n \\ \downarrow}}{=} \sum_{n=0}^{\infty} \frac{|g(t)|\sum_{k=0}^{3n}\alpha_{n,k}(\frac{1}{t})^k||x|^{2n}}{(2n)!}$$
$$\leq \sum_{n=0}^{\infty}\sum_{k=0}^{3n} \frac{n!3^n}{t^k} g(t) \frac{|x|^{2n}}{(2n)!}$$

If $t > 1$, then $\frac{1}{t^k}$ is bounded by 1 and else by $\frac{1}{t^{3n}}$. Hence:

$$|u(t,x)| \leq \begin{cases} \displaystyle\sum_{n=0}^{\infty}(3n+1)n!3^n g(t)\frac{|x|^{2n}}{(2n)!}, & t > 1 \\ \displaystyle\sum_{n=0}^{\infty}\frac{(3n+1)n!3^n}{t^{3n}} g(t)\frac{|x|^{2n}}{(2n)!}, & t \leq 1 \end{cases}$$

Note that:

$$\frac{(3n+1)n!3^n}{(2n)!} = \frac{\overbrace{(3n+1)}^{\leq 3^n}3^{n+1}}{\underbrace{(2n)\cdots(n+1)}_{\geq n!}} \leq \frac{9^n}{n!}$$

Hence, if $t > 1$, we have that

$$|u(x,t)| \leq g(t)\sum_{n=0}^{\infty}\frac{1}{n!}(9|x|^2)^n = \mathrm{e}^{-\frac{1}{t^2}+9|x|^2}$$

and if $t \leq 1$:

$$|u(x,t)| \leq g(t)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{9|x|^2}{t^3}\right)^n = \mathrm{e}^{-\frac{1}{t^2}+\frac{9|x|^2}{t^3}}$$

This proves the absolute convergence, but is however not good enough for the convergence $u(t,x) \overset{t\downarrow 0}{\longrightarrow} 0$.

Lemma 2.5. *Let $(P_n)_{n\in\mathbb{N}}$ be the polynomials from Lemma 2.4. Then, $\forall n \in \mathbb{N}$:*

$$\left|P_n\left(\frac{1}{t}\right)\right| \leq \max_{0\leq k\leq n}\frac{2^{n+k}(3n)^{n-k}}{t^{n+2k}}.$$

Note that:

$$g(t) = \mathrm{e}^{-\frac{1}{2t^2}}\cdot\mathrm{e}^{-\frac{1}{2t^2}} = \mathrm{e}^{-\frac{1}{2t^2}}\cdot\left(\sum_{k=0}^{\infty}\frac{1}{k!(2t^2)^k}\right)^{-1} \leq k!(2t^2)^k\mathrm{e}^{-\frac{1}{2t^2}} \qquad (*)$$

Now, using this Lemma 2.5 and $(*)$:

$$|u(x,t)| \leq \sum_{n=0}^{\infty}\max_{0\leq k\leq n}\frac{2^{n+k}(3n)^{n-k}}{t^{n+2k}}g(t)\frac{|x|^{2n}}{(2n)!} \leq \sum_{n=0}^{\infty}\max_{0\leq k\leq n}\frac{2^{n+k}(3n)^{n-k}}{t^{n+2k}}k!(2t^2)^k\mathrm{e}^{-\frac{1}{2t^2}}\frac{|x|^{2n}}{(2n)!}$$

Since

$$\max_{0\leq k\leq n}\frac{2^{n+k}(3n)^{n-k}}{t^{n+2k}}k!(2t^2)^k = \frac{1}{t^n}\max_{0\leq k\leq n}2^{n+2k}(3n)^{n-k}\overbrace{k!}^{\leq n^k} \leq \frac{1}{t^n}\max_{0\leq k\leq n}2^{n+2k}3^{n-k}n^n \leq \frac{24^n n^n}{t^n},$$

we get

$$|u(x,t)| \le \sum_{n=0}^{\infty} \frac{24^n}{t^n} \underbrace{\frac{n^n}{(2n)!}}_{\le \frac{1}{n!}} \cdot |x|^{2n} \mathrm{e}^{-\frac{1}{2t^2}} \le \exp\left(\frac{24|x|^2}{t} - \frac{1}{2t^2}\right) \xrightarrow{t\downarrow 0} 0 \quad \forall x$$

□

8th lecture

2.2 Schrödinger equation

$$\begin{cases} \mathrm{i}\,\partial_t\, u(t,x) = \Delta u(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^d \\ \quad u(0,x) = u_0(x) \end{cases}$$

This is similar to the heat equation $\partial_t\, u = \Delta u$, but for the imaginary time ($t \rightsquigarrow -\mathrm{i}t$). Therefore heuristically:

$$u(x,t) = \mathrm{e}^{-\mathrm{i}t\Delta} u_0(x)\,,$$

since

$$\mathrm{i}\,\partial_t\, u = \mathrm{i}\,\partial_t(\mathrm{e}^{-\mathrm{i}t\Delta}u_0) = \mathrm{i}(-\mathrm{i}\Delta \underbrace{\mathrm{e}^{-\mathrm{i}t\Delta}u_0}_{u}) = \Delta u\,.$$

Theorem 2.6. *For any $u_0 \in H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, the Schrödinger equation has a unique solution $u(x,t) = \mathrm{e}^{-\mathrm{i}t\Delta}u_0(x)$, which satisfies*

$$\hat{u}(t,k) = \mathrm{e}^{\mathrm{i}t|2\pi k|^2}\widehat{u_0}(k) \implies |\hat{u}| = |\widehat{u_0}|$$

In particular, if $u_0 \in L^1$, then

$$u(t,x) = \mathrm{e}^{-\mathrm{i}t\Delta}u_0(x) = \frac{1}{(-4\pi\mathrm{i}t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{|x-y|^2}{4\pi\mathrm{i}t}}\, u_0(y)\,\mathrm{d}y\,. \qquad (*)$$

Moreover, we have:

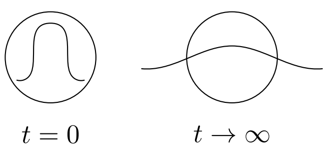
- $\|u(t,\bullet)\|_{H^\alpha} = \|u_0\|_{H^\alpha}$, $\forall \alpha \in \mathbb{R}$. *In particular:*

$$\alpha = 0: \quad \|u(t,\bullet)\|_{L^2} = \|u_0\|_{L^2} \qquad \text{(mass conservation)}$$
$$\alpha = 1: \quad \|u(t,\bullet)\|_{H^1} = \|u_0\|_{H^1} \qquad \text{(energy conservation)}$$

- $u(t,\bullet) \xrightarrow{t\to 0} u_0$ *in* $H^s(\mathbb{R}^d)$
- *(Dispersive estimate) If $u_0 \in L^2(\mathbb{R}^d)$, then for every ball $B_R \subset \mathbb{R}^d$:*

$$\int_{B_R} |u(t,x)|^2\,\mathrm{d}x \xrightarrow{t\to\infty} 0\,,$$

i.e. the mass "escapes to infinity":



Proof. If $u_0 \in L^1 \cap L^2$, then $(*)$ makes sense and

$$\|u(t,\bullet)\|_{L^2} = \|u_0\|_{L^2}\,.$$

Hence, by a density argument, we can define $\mathrm{e}^{-\mathrm{i}t\Delta}u_0$ for all $u_0 \in L^2$ and

$$|\hat{u}(t,k)| = |\widehat{u_0}(k)| \text{ for a.e. } k \in \mathbb{R}^d\,.$$

Then, using the Fourier transform, we can extend the solution $\mathrm{e}^{-\mathrm{i}t\Delta}u_0$ for all $u_0 \in H^t$ for any $t \in \mathbb{R}$. This solution is unique, because the Fourier transform is a bijective map on each Sobolev space. The conservation laws follow by

$$\|u(t,\bullet)\|_{H^\alpha}^2 = \int_{\mathbb{R}^d} (1+|2\pi k|^2)^\alpha \underbrace{|\hat{u}(t,k)|^2}_{|\widehat{u_0}(k)|^2} \mathrm{d}k = \|u_0\|_{H^\alpha}^2 .$$

The convergence $u(t,\bullet) \xrightarrow{t\to 0} u_0$ follows from the dominated convergence theorem.
Proof of the dispersive estimate ($t \to \infty$):

- $\|\mathrm{e}^{-\mathrm{i}t\Delta}u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$
- $\|\mathrm{e}^{-\mathrm{i}t\Delta}u_0\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}}\|u_0\|_{L^1(\mathbb{R}^d)}$, $\forall t > 0$

Lemma 2.7. *If $u_0 \in L^2(\mathbb{R}^d)$, then $\forall\, \varepsilon > 0$, we can decompose*

$$u_0 = a + b\,,$$

where $\|a\|_{L^2(\mathbb{R}^d)} \leq \varepsilon$ and $\|b\|_{L^1(\mathbb{R}^d)} \leq C_\varepsilon$.

Proof. exercise □

Conclusion of the dispersive bound: Fix $B_R \subset \mathbb{R}^d$, then

$$\begin{aligned}
\int_{B_R} |u(t,x)|^2\,\mathrm{d}x &= \int_{B_R} |\mathrm{e}^{-\mathrm{i}t\Delta}\underbrace{u_0(x)}_{a+b}|^2\,\mathrm{d}x = \int_{B_R} |\mathrm{e}^{-\mathrm{i}t\Delta}a(x) + \mathrm{e}^{-\mathrm{i}t\Delta}b(x)|^2\,\mathrm{d}x \\
&\leq 2\int_{B_R} |\mathrm{e}^{-\mathrm{i}t\Delta}a(x)|^2\,\mathrm{d}x + 2\int_{B_R} |\mathrm{e}^{-\mathrm{i}t\Delta}b(x)|^2\,\mathrm{d}x \\
&\leq 2\int_{\mathbb{R}^d} |\mathrm{e}^{-\mathrm{i}t\Delta}a(x)|^2\,\mathrm{d}x + 2|B_R|\|\mathrm{e}^{-\mathrm{i}t\Delta}b\|_{L^\infty(\mathbb{R}^d)}^2 \\
&\leq 2\|a\|_{L^2(\mathbb{R}^d)}^2 + 2|B_R|\frac{1}{(4\pi t)^d}\|b\|_{L^1(\mathbb{R}^d)}^2 \\
&\leq 2\,\varepsilon^2 + 2|B_R|\frac{C_\varepsilon}{(4\pi t)^d} \quad \forall\,\varepsilon > 0,\ t > 0
\end{aligned}$$

Hence,

$$\forall\,\varepsilon > 0: \ \limsup_{t\to\infty} \int_{B_R} |u(t,x)|^2\,\mathrm{d}x \leq 2\,\varepsilon^2 \implies \lim_{t\to\infty}\int_{B_R} |u(t,x)|^2\,\mathrm{d}x = 0$$

□

So far, we considered *homogeneous* equations, e.g. $\partial_t u = \Delta u$ or $\mathrm{i}\,\partial_t u = \Delta u$. The corresponding *inhomogeneous* equations, $\partial_t - \Delta u = g$ or $\mathrm{i}\,\partial_t u - \Delta u = g$ can be solved by Duhamel's formula. In ODE we have:

- If for some $a \in \mathbb{R}$
$$\begin{cases} u'(t) = au(t), & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}, \qquad \text{(homogeneous problem)}$$
then $u(t) = \mathrm{e}^{at}u_0$ ($\iff$ ($\mathrm{e}^{-at}u(t) = u_0$).
- The inhomogenous problem is
$$\begin{cases} u'(t) - au(t) = g(t), & t \in \mathbb{R} \\ u(0) = u_0 \end{cases}.$$

Define $v(t) = \mathrm{e}^{-at} u(t)$, then

$$\begin{aligned} \partial_t v = \partial_t(\mathrm{e}^{-at}u(t)) &= -a\mathrm{e}^{-at}u(t) + \mathrm{e}^{-at}u'(t) = \mathrm{e}^{-at}(u' - au) = \mathrm{e}^{-at}g(t) \\ \implies v(t) &= v(0) + \int_0^t \mathrm{e}^{-as} g(s)\,\mathrm{d}s \\ \implies u(t) &= \mathrm{e}^{at}v(t) = \mathrm{e}^{at}u(0) + \int_0^t \mathrm{e}^{a(t-s)}g(s)\,\mathrm{d}s \end{aligned}$$

In case of the Heat equation $\partial_t u(x,t) - \Delta u(x,t) = g(t,x)$, this means:

$$u(t,x) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{\Delta(t-s)} g(s,x)\,\mathrm{d}s$$

We will discuss transport equations:

$$\begin{cases} \partial_t f + v \cdot \nabla f + A \cdot f = g(t,x), & t \in \mathbb{R}^d,\ x \in \mathbb{R}^d \\ f(0,x) = f_0 \end{cases}$$

Here $f(x,t) \in \mathbb{R}^d$, $v(t,x) \in \mathbb{R}^d$, $A(t,x) \in \mathbb{R}^d \times \mathbb{R}^d$, $g(t,x) \in \mathbb{R}^d$.

2.3 General discussion on ODE

$$x(t) = x_0 + \int_0^t F(s, x(s))\,\mathrm{d}s \xleftarrow{\text{weak formulation}} \begin{cases} \partial_t x(t) = F(t, x(t)), & t > 0 \\ x(0) = x_0 \end{cases}$$

Theorem 2.8 (Cauchy-Lipschitz theorem). *If*

$$\exists L > 0\ \forall x, y: \quad |F(t,x) - F(t,y)| \le L|x - y|\,,$$

then there exists a unique solution. Moreover, the solution can be obtained by the Picard method:

$$x_{k+1} = x_0 + \int_0^t F(s, x_k(s))\,\mathrm{d}s$$

is a Cauchy sequence and as F is Lipschitz, $x_k \xrightarrow{k\to\infty} x(t)$.

More general condition:

Definition 2.9 (Osgood). Let $I \subseteq \mathbb{R}$ be an open interval. A measure $\mu\colon I \to [0,\infty]$ is an *Osgood modulus of continuity* iff

$$\int_I \frac{\mathrm{d}r}{\mu(r)} = \infty\,.$$

Idea: Lipschitz means

$$|F(t,x) - F(t,y)| \le \mu(|x - y|)$$

with $\mu(r) = Lr$ and $\int_I \frac{\mathrm{d}r}{\mu(r)} = \infty$, if $0 \in \overline{I}$. More generally, if $\mu(r) \le r|\log r|^\beta$ with $\beta \le 1$, then $\int_I \frac{\mathrm{d}r}{\mu(r)} = \infty$.

Theorem 2.10 (Generalized Cauchy-Lipschitz). *Let E be a Banach space and $I \subseteq \mathbb{R}$ an open interval with $0 \in I$. Consider*

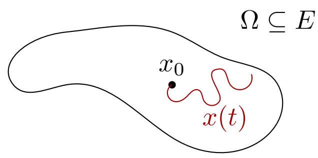
$$x(t) = x_0 + \int_0^t F(s, x(s))\,\mathrm{d}s\,,$$

where $x\colon I \times E \to E$. Assume $F \in L^1_{\mathrm{loc}}(I, C_\mu(\Omega, E))$ where $\Omega \subset E$ is open and

$$C_\mu(\Omega, E) = \left\{ u\colon \Omega \to E \;\middle|\; \|u\|_{C_\mu} = \|u\|_{L^\infty} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|u(x) - u(y)\|}{\mu(\|x-y\|)} < \infty \right\}$$

and μ is an Osgood modulus of continuity. Then:

$$\forall x_0 \in \Omega,\ \exists J^{open} \subset I,\ 0 \in J \text{ and } \exists!\ \text{solution } x(t) \text{ for } t \in J$$



Lemma 2.11 (Osgood). *Let* $\rho\colon [0,\tau] \to [0,a]$ *be measurable and* $\gamma \in L^1_{\mathrm{loc}}([0,\tau],\mathbb{R}_+)$*. Let* $\mu\colon [0,a] \to \mathbb{R}_+$ *be continuous, non-decreasing and an Osgood modulus of continuity. Assume for some c:*

$$\rho(t) \le c + \int_0^t \gamma(s)\mu(\rho(s))\,\mathrm{d}s\,, \quad \forall t \in [0,\tau]$$

Then

$$M(c) - M(\rho(t)) \le \int_0^t \gamma(s)\,\mathrm{d}s\,,$$

where

$$M(x) = \int_x^a \frac{\mathrm{d}r}{\mu(r)}\,.$$

Note: $M(0) = \infty$, and hence if $c = 0$, then $\rho(t) = 0$ if $|t|$ small enough. (Recall the Gronwall Lemma: $\rho(t) \le \int_0^t \rho(s)\,\mathrm{d}s \implies \rho = 0$).

Proof. (of Theorem 2.10 assuming Osgood's Lemma): **Uniqueness**: Assume $x_1(t), x_2(t)$ are two solutions. Define $\delta(t) = x_1(t) - x_2(t)$. Then

$$\delta(t) = \int_0^t F(s,x_1(s)) - F(s,x_2(s))\,\mathrm{d}s$$

and hence

$$\|\delta(t)\| \le \int_0^t \|F(s,x_1(s)) - F(s,x_2(s))\|\,\mathrm{d}s \le \int_0^t \gamma(s)\mu(\|x_1(s)-x_2(s)\|)\,\mathrm{d}s = \int_0^t \gamma(s)\mu(\|\delta(s)\|)\,\mathrm{d}s$$

Here we used $F \in L^1_{\mathrm{loc}}(I,C_\mu)$, so $\|F(s,x)-F(s,y)\| \le \overbrace{\gamma(s)}^{L^1_{\mathrm{loc}}(I)}\mu(\|x-y\|)$. Osgoods lemma yields $\|\delta(s)\| = 0$ for $|s|$ small enough. Thus

$$x_1(s) = x_2(s)\,, \quad \forall s \in J \text{ if } |J| \text{ small}, 0 \in J$$

Here

$$a = \max_{t\in J}\|x(t) - x_0\|$$

$$\|\delta(s)\| = \|x_1(s) - x_2(s)\| \le \|x_1(s) - x_0\| + \|x_2(s) - x_0\| \le a\,, \quad \text{if } s \in J\,.$$

Existence: Define $x_{k+1} := x_0 + \int_0^t F(s,x_k(s))\,\mathrm{d}s$. We prove that $\{x_k\}$ is a Cauchy-sequence, i.e. $\sup_{n\in\mathbb{N}}\|x_{k+n} - x_k\| \xrightarrow{k\to\infty} 0$. Define

$$\rho_{k,n}(t) := \sup_{0\le s\le t}\|x_{k+n}(s) - x_n(s)\|$$

From the equation:

$$
\begin{aligned}
& x_{k+n+1}(t) - x_{k+1} = \int_0^t (F(s, x_{k+m}(s)) - F(s, x_{k+1}(s)))\,\mathrm{d}s \\
\implies & 0 \le \rho_{k+1,n}(t) = \sup_{0\le s\le t} \|x_{k+n+1}(s) - x_{k+1}(s)\| \le \int_0^t \gamma(s)\mu(\|x_{k+1}(s) - x_k(s)\|)\,\mathrm{d}s \\
& \le \int_0^t \gamma(s)\mu(\rho_{k,n}(s))\,\mathrm{d}s \\
\implies & \rho_{k+1}(t) = \sup_{n\ge 1} \rho_{k+1,n}(t) \le \int_0^t \gamma(s) \sup_n \mu(\rho_{k,n}(s))\,\mathrm{d}s \le \int_0^t \gamma(s)\mu(\rho_k(s))\,\mathrm{d}s \\
\implies & \rho(t) = \limsup_{k\to\infty} \rho_k(t) \le \int_0^t \gamma(s)\mu(\rho(s))\,\mathrm{d}s \\
\implies & \rho(t) = 0\,, \quad \text{if } t \in J,\ |J| \text{ small, } 0 \in J
\end{aligned}
$$

□

9th lecture

Proof. (of the Osgood Lemma (Lemma 2.11)): Take $g(t) = c + \int_0^t \gamma(s)\mu(\rho(s))\,\mathrm{d}s \ge \rho(t)$, then

$$g'(t) = \gamma(t)\mu(\rho(t)) \le \gamma(t)\mu(g(t)) \qquad (\mu\uparrow)$$

Consider

$$
\begin{aligned}
& \partial_t(M(g(t)) = M'(g(t))g'(t) = -\frac{1}{\mu(g(t))}\cdot g'(t) \ge -\gamma(t) \\
\implies & M(c) - M(g(t)) = -\int_0^t \partial_s(M(g(s))\,\mathrm{d}s \le \int_0^t \gamma(s)\,\mathrm{d}s \\
\implies & M(c) - M(\rho(t)) \le \int_0^t \gamma(s)\,\mathrm{d}s
\end{aligned}
$$

□

Theorem 2.12 (Blow-up criterion). *Let $I \subseteq \mathbb{R}$ be an open interval, $0 \in I$ and $\mu\colon I \to \mathbb{R}$ an Osgood modulus of continuity. Assume $\|F(t,x) - F(t,y)\| \le \mu(\|x-y\|)$.*

- *(Local theory) $\forall x_0 \in \Omega\, \exists J^{open}$, $0 \in J \subset I$ such that there exists a unique solution $x(t)$, $t \in J$.*
- *(Blow-up criterion) Assume additionally*

  $$\|F(t,x)\| \le \beta(t)M(\|x\|)\,,$$

  where $\beta \in L^1_{\mathrm{loc}}$, $M \in L^\infty_{\mathrm{loc}}$. Then the local solution from (i) can be extended uniquely to the maximal interval $J^* = (T_*, T^*)$. Moreover, if $T^* \in I$ (i.e. T^* is not an endpoint of I), then

  $$\|x(t)\| \to \infty \quad \text{as } t \uparrow T^*\,.$$

  Similar statement holds for T_*.

Proof. (of the Blow-up criterion) Assume a solution $x(t)$ exists on $J = (T_*, T^*)$ and $T^* \in I$ (not an endpoint case). We prove that there exists a limit $x(t) \to x^*$ as $t \uparrow T^*$. We need to show that $\{x(t)\}_{t\uparrow T^*}$ is Cauchy:

$$\|x(t_2) - x(t_1)\| = \left\|\int_{t_1}^{t_2} F(s,x(s))\,\mathrm{d}s\right\| \le \int_{t_1}^{t_2} \beta(t)M(\underbrace{\|x(s)\|}_{\substack{\text{bounded in } S<T^* \\ \text{uniformly in } S}})\,\mathrm{d}s \le C\int_{t_1}^{t_2} \rho(t)\,\mathrm{d}t \xrightarrow{t_1,t_2\uparrow T^*} 0$$

Then using local theory (1), we can extend $x(t)$ with $x(T^*) = x^*$, so the solution exists in a larger domain. □

2.4 Cubic nonlinear Schrödinger equation (NLS)

$$\begin{cases} \mathrm{i}\,\partial_t u = \Delta u + |u|^2 u, \quad u = u(t,x),\, t \in \mathbb{R},\, x \in \mathbb{R}^d \\ u(0) = u_0 \in H^s(\mathbb{R}^d) \end{cases}$$

Theorem 2.13 (1D). *For all $u_0 \in H^1(\mathbb{R})$, there exists a local solution $u(t) \in H^1(\mathbb{R}^d)$, i.e $\exists T > 0$ such that $\forall t \in (-T, T)$, $u(t) \in H^1(\mathbb{R})$ is a solution. Furthermore, the blow-up criterion holds.*

Proof. Weak formulation: Duhamel formula:

$$u(t) = \mathrm{e}^{-\mathrm{i}t\Delta} u_0 + \int_0^t \mathrm{e}^{-\mathrm{i}(t-s)\Delta} F(u(s))\, \mathrm{d}s\,, \quad F(u) = |u|^2 u$$

Let us verify the Lipschitz property:

$$\begin{aligned} &\left\| \mathrm{e}^{-\mathrm{i}(t-s)\Delta} F(u(s)) - \mathrm{e}^{-\mathrm{i}(t-s)\Delta} F(v(s)) \right\|_{H^1(\mathbb{R})} \\ &= \|F(u(s)) - F(v(s))\|_{H^1(\mathbb{R})} \\ &\lesssim \|u - v\|_{H^1} (\|u\|^2_{L^\infty} + \|v\|^2_{L^\infty}) \\ &\lesssim \|u - v\|_{H^1} (\|u\|^2_{H^1} + \|v\|^2_{H^1}) \end{aligned}$$

$\implies$ existence locally and blow up criterion

□

10th lecture

<u>Easy case</u>: $s > \frac{d}{2}$, then there exists a global solution in $H^s(\mathbb{R}^d)$, $\forall u_0 \in H^s(\mathbb{R}^d)$. (Hint:

$$H^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \implies \||u|^2 u - |v|^2 v\|_{H^s} \lesssim \|u - v\|_{H^s} (\|u\|^2_{H^s} + \|v\|^2_{H^s})$$

exercise: $s = d = 1$)
<u>Difficult case</u>: $s \leq \frac{d}{2}$: Critical regularity (scaling argument):

$$u \text{ solution} \implies u_\lambda(t,x) = \lambda^{-1} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \text{ also a solution}$$

What is the critical s for the solution? $\|u\|_{H^{s_{cr}}}$ is invariant under that scaling!

$$\|u_\lambda\|^2_{H^s} \sim \|\sqrt{\Delta_x}^{\,s} u_\lambda\|^2_{L^2} \sim \|\frac{1}{\lambda^s} u_\lambda\|^2_{L^2} \sim \int_{\mathbb{R}^d} \frac{1}{\lambda^{2s+2}} |u(\bullet, y)^2|\, \mathrm{d}y \lambda^d = \lambda^{d-2s-2} \|u\|^2_{L_x}(\frac{t}{\lambda})$$

$$\implies d - 2s_{cr} - 2 = 0 \implies s_{cr} = \frac{d-2}{2} < \frac{d}{2}$$

Example:

$$\left[\begin{array}{ll} 1D: & s_{cr} = -\frac{1}{2} \\ 2D: & s_{cr} = 0 \\ 3D: & s_{cr} = \frac{1}{2} \end{array}\right.$$

Another way: $|u|^2 u$ is well-defined if $u \in H^{s_{cr}}$ (3D: $u \in H^{\frac{1}{2}}$, Sobolev $\|u\|_{L^3} \lesssim \|u\|_{H^{\frac{1}{2}}}$)
Expectation: If $s \geq s_{cr}$, there exists a global solution (we only discuss defocusing case $+|u|^2 u$). If $s < s_{cr}$, no!

Theorem 2.14 ($d = 3$, $s = 1$). *For all $u_0 \in H^1(\mathbb{R}^3)$, there exists a unique global solution $u(t) \in H^1(\mathbb{R}^3)$. $u(t)$ satisfies the scattering property*

$$\|u(t) - \mathrm{e}^{\mathrm{i}t\Delta} u_{\pm\infty}\|_{H^1} \xrightarrow{t \to \pm\infty} 0$$

<u>Strichatz estimate</u>:

- $\|\mathrm{e}^{\mathrm{i}t\Delta} u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$

- $\|e^{it\Delta}u_0\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{d}{2}}}\|u_0\|_{L^1(\mathbb{R}^d)}$

By interpolation (Riesz-Thorin/complex):

$$\|e^{it\Delta}u_0\|_{L^p} \lesssim t^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^{p'}(\mathbb{R}^d)}\,, \quad \forall 1 \le p \le \infty,\ \frac{1}{p}+\frac{1}{p'}=1$$

Definition 2.15.

$$\|u(t,x)\|_{L_t^q L_x^p} := \|\|u(x,t)\|_{L_x^p}\|_{L_t^q} = \left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}^d}|u(x,t)|^p\,dx\right|^{\frac{q}{p}}dt\right)^{\frac{1}{q}}$$

Example: $(L_t^q L_x^p)' = L_t^{q'}L_x^{p'}$ (exercise)

Theorem 2.16 (Strichartz). *Let $d \ge 1$, $(q,r) \in (2,\infty] \times [2,\infty]$ admissible pair: $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$. Then:*

(1) $\left\|e^{it\Delta}u_0\right\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}$

(2) $\left\|\int_{\mathbb{R}} e^{-it\Delta}F(t,\bullet)\,dt\right\|_{L_x^2} \lesssim \|F\|_{L_t^{q'}L_x^{r'}}$

(3) $\left\|\int_{\mathbb{R}} e^{i(t-s)\Delta}F(s,\bullet)\,ds\right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'}L_x^{r'}}$

(They are all equivalent!)

Proof. Step 1: (1) $\iff$ (2) $\iff$ (3): Duality argument!
Easiest duality: $|\int fg| \le \|f\|_{L^p}\|g\|_{L^{p'}}$, also $\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}\le 1}|\int fg|$
We use this for $L_t^{"}L_x^{"}$ as well.

$$\begin{aligned} & e^{it\Delta}u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2} && (1)\\ \iff & \left\langle F(t,x), e^{it\Delta}u_0\right\rangle_{t,x} \lesssim \|u_0\|_{L^2}\|F\|_{L_t^{q'}L_x^{r'}} \\ \iff & \left\langle \int_{\mathbb{R}} e^{-it\Delta}F(t,x)\,dt, u_0\right\rangle_x \lesssim \|u_0\|_{L^2}\|F\|_{L_t^{q'}L_x^{r'}} && (2)\end{aligned}$$

TODO
We will prove (3) using

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^d)} \lesssim t^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^{p'}(\mathbb{R}^d)}$$

plus HLS inequality. □

Hardy-Littlewood-Sobolev:
Sobolev: $0 < s < \frac{d}{2}$: $\|u\|_{L^p} \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^d)}$, $p = \frac{ds}{d-2s}$

$$\begin{aligned}&\iff \|u\|_{L^p} \lesssim \|(-\Delta)^{\frac{s}{2}}u\|_{L^2}\\ &\iff \textit{TODO}\end{aligned}$$

Theorem 2.17 (Hardy-Littlewood-Sobolev). *If $p,q \in (1,\infty]$, $0 < \lambda < d$, $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d}=2$:*

$$\left|\iint_{\mathbb{R}^{2d}} \frac{f(x)g(y)}{|x-y|^\lambda}\,dx\,dy\right| \lesssim \|f\|_{L^p}\|g\|_{L^q}$$

proof of (3):

$$LHS = \|\|\int_{\mathbb{R}}\|_{L_x^r}\|_{L_t^q} \le \|\int_{\mathbb{R}}\|\|_{L_x^p}\|_{L_t^q} \le \left\|\int_{\mathbb{R}}\underbrace{\|e^{i(t-s)}G(s,x)\|_{L_x^{r'}}}\,ds\right\|_{L^q(\mathbb{R})}$$

TODO

Back to the cubic nonlinear Schrödinger equation

$$\begin{cases} \mathrm{i}\,\partial_t u = -\Delta u + |u|^2 u, & t \in \mathbb{R},\, x \in \mathbb{R}^3 \\ \quad u_0 \in H^1(\mathbb{R}^3) \end{cases}$$

and the proof of of local existence:

Proof. (of Theorem 2.14) Duhamel:

$$u(t) = \underbrace{\mathrm{e}^{\mathrm{i}t\Delta} u_0 + \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta}(|u|^2 u)(s)\,\mathrm{d}s}_{=:B(u)(t)}$$

The right space to do the fixed point argument on: Define

$$\|u\|_{S^0} := \|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^4 L_x^3}$$

$$\|u\|_{S^1} = \|u\|_{S^0} + \|\nabla u\|_{S^0}$$

$$X = \left\{ u \,\middle|\, \|u\|_{S^1} \le 2\|u_0\|_{H_x^1} \right\}$$

We prove that $B\colon X \to X$ and then obtain a fixed point $B(u) = u$ for $|t| < T$ and T small. 11th lecture
Claim: If $T > 0$ is small enough, then $B\colon X \to X$ and $\|Bu - Bv\|_{S^1} \le (1-\varepsilon)\|u - v\|_{S^1}$, $\forall u, v \in X$, for some $\varepsilon > 0$. This implies the local existence $\exists T > 0$, $\exists u = Bu$ for time $t \in [-T, T]$.
Proof: By Strichartz

$$\|\mathrm{e}^{\mathrm{i}t\Delta} u_0\|_{S^1} \lesssim \|u_0\|_{H^1}$$

$$\begin{aligned} \left\| \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta}(|u|^2 u)(s)\,\mathrm{d}s \right\|_{S^0} &= \|\ldots\|_{L_t^\infty L_x^2} + \|\ldots\| \overset{(3)}{\lesssim} \||u|^2 u\|_{L_t^1 L_x^2} + \||u|^2 u\|_{L_t^{\frac{4}{3}} L_x^{\frac{3}{2}}} \\ &\lesssim T\|u\|^3_{L_t^\infty L_x^6} + T^{\frac{3}{4}}\|u\|^3_{L_t^\infty L_x^{\frac{9}{2}}} \lesssim T\|\nabla u\|^3_{L_t^\infty L_x^2} + T^{\frac{3}{4}}\|u\|_{L_t^\infty H_x^1} \\ &\le (T + T^{\frac{3}{4}})\|u\|_{S^1} \end{aligned}$$

Furthermore,

$$\begin{aligned} \left\| \nabla_x \int_0^t \mathrm{e}^{i(t-s)\Delta}(|u|^2 u)(s)\,\mathrm{d}s \right\|_{S^0} &\sim \left\| \int_0^t \mathrm{e}^{i(t-s)\Delta}(|u|^2 \nabla u)(s)\,\mathrm{d}s \right\|_{S^0} = \|\cdots\|_{L_t^\infty L_x^2} + \|\cdots\|_{L_t^4 L_x^3} \\ &\overset{(3)}{\lesssim} \|(\nabla u)|u|^2\|_{L_t^1 L_x^2} + \|(\nabla u)|u|^2\|_{L_t^{\frac{4}{3}} L_x^{\frac{3}{2}}} \end{aligned}$$

By a calculation using HLS (TODO):

$$\|(\nabla u)|u|^2\|_{L_t^1 L_x^2} \le TODO \le T^{\frac{1}{3}}\|u\|^3_{S^1}$$

and by another calculation (TODO):

$$\|(\nabla u)|u|^2\|_{L_t^{\frac{4}{3}} L_x^{\frac{3}{2}}} \lesssim TODO \le T^{\frac{1}{2}}\|u\|^3_{S^1}$$

<u>Scaling</u>: $\|u\|_{L^{12}(\mathbb{R}^3)} \lesssim \|\Delta u\|_{L^3(\mathbb{R}^3)} + \|u\|_{L^2(\mathbb{R}^3)} \iff \|u\|_{L^{12}} \lesssim \|\Delta u\|^\theta_{L^3}\|u\|^{1-\theta}_{L^2}$ In summary:

$$\begin{aligned} \|Bu\|_{S^1} &\le \|\mathrm{e}^{\mathrm{i}t\Delta} u_0\|_{S^1} + \left\| \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\Delta}(|u|^2 u)(s)\,\mathrm{d}s \right\|_{S^1} \le C_0\|u_0\|_{H_x^1} + \|C_1 T^\alpha\|u\|^3_{S^1} \\ &\overset{u\in X}{\le} C_0\|u_0\|_{H_1^x} + C_1 T^\alpha (C\|u_0\|_{H_x^1})^3 \end{aligned}$$

for universal constants $C_0, C_1, \alpha > 0$. We can take $C = C_0 + 1$ and $T = T(c_0, \|u_0\|_{H^1_x}) > 0$ small enough, such that $\|Bu\|_{S^1} \leq C\|u_0\|_{H^1_x}$, i.e. $B\colon X \to X$.
Contraction property:

$$\|B(u) - B(v)\|_{S^1} = \left\| \int_0^t e^{i(t-s)\Delta} \underbrace{|u|^2 u - |v|^2 v}_{\sim (u-v)(|u|^2+|v|^2)} (s)\, ds \right\|_{S^1} \lesssim T^\alpha \|u - v\|_{S^1} (\|u\|_{S^1}^2 + \|v\|_{S^1}^2)$$

by the same analysis. Hence, $\forall u, v \in X$:

$$\|Bu - Bv\|_{S^1} \lesssim T^\alpha \|u - v\|_{S^1} (C\|u_0\|_{H^1})^2 \leq \varepsilon \|u - v\|_{S^1}, \forall u, v \in X \text{ if } T > 0 \text{ small enough!}$$

Thus B is a contraction on the complete metric space X, so the fixed point theorem gives the existence of a fixed point.

□

Lemma 2.18. *If $u(t,x)$ is a local solution of the cubic nonlinear Schrödinger equation, $i\,\partial_t u = -\Delta u + |u|^2 u$ on $t \in (-T, T)$, then:*

- *Mass conservation: $\|u\|_{L^2_x} = \|u_0\|_{L^2_x}$, $\forall t \in (-T, T)$*
- *Energy conservation: $\mathscr{E}(u) = \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^3} |u|^4\, dx = \mathscr{E}(u_0)$*

In particular, these imply that

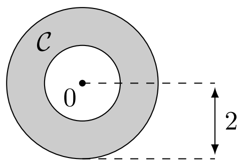
$$\|\nabla u\|_{L^2_x}^2 + \|u\|_{L^2_x}^2 = \|u\|_{H^1_x}^2$$

is bounded uniformly in time. This helps to get the global existence.

3 Littlewood-Paley theory

Motivation: Decompose $f\colon \mathbb{R}^d \to \mathbb{C}$ into $f = \sum_{n\in\mathbb{Z}} f_n$, where $\hat{f}_n(k) = \hat{f}(k) \mathbb{1}_{\{2^{n-1} \leq |k| \leq 2^n\}}$.

Lemma 3.1 (Bernstein). *Let $\mathcal{C} = \{1 \leq |k| \leq 2\}$ and supp $\hat{f} \subset \lambda\mathcal{C}$ ($\lambda\mathcal{C} = \{\lambda \leq |k| \leq 2\lambda\}$).*



(1) Let $p \in [1, \infty]$, then

$$\sup_{|\alpha|=n} \|D^\alpha f\|_{L^p(\mathbb{R}^d)} \sim \lambda^n \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) Let $1 \leq p \leq q \leq \infty$, then

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \lambda^{d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

Interpretation (heuristically): Since $|k| \sim \frac{1}{|x|}$, $|k| \sim \lambda \iff |x| \sim \lambda^{-1}$.

Proof. Since supp $\hat{f} \subset \lambda C$, we have that $\hat{f} = \hat{\chi}\hat{f} = \widehat{\chi * f}$, where $\hat{\chi}(k)$ is a smooth function, $\hat{\chi}(k) = 1$ if $k \in \lambda C$. Thus $f = \chi * f$.

(1) Since $D^\alpha f = D^\alpha(\chi * f)$, we have

$$\|D^\alpha f\|_{L^p} = \|D^\alpha \chi * f\|_{L^p} \leq \|D^\alpha \chi\|_{L^1} \|f\|_{L^p}. \qquad \text{(Young inequality)}$$

Here

$$\widehat{\chi}(k) = \widehat{\chi_\lambda}(k) = \widehat{\chi_1}\left(\frac{k}{\lambda}\right) \implies \chi(x) = \chi_1(\lambda x)\lambda^d.$$

Hence:

$$\|D^\alpha \chi\|_{L^1} = \|D^\alpha \chi_1(\lambda x)\lambda^d\|_{L^1} = \|\lambda^{|\alpha|}(D^\alpha\chi_1)(\lambda x)\lambda^d\|_{L^1(\mathrm{d}x)} = \lambda^{|\alpha|}\|D^\alpha \chi_1\|_{L^1}\,.$$

Reversely: exercise

(2) For $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{p}$:

$$\|f\|_{L^q} = \|\chi * f\|_{L^q} \overset{\substack{\text{Young}\\ \downarrow}}{\leq} \|\chi\|_{L^r}\|f\|_{L^p}$$

and

$$\|\chi\|_{L^r} = \left(\int_{\mathbb{R}^d}\left|\chi_1(\lambda x)\lambda^d\right|^r \mathrm{d}x\right)^{\frac{1}{r}} = \lambda^{\frac{dr-d}{r}}\|\chi_1\|_{L^r} = \lambda^{d(\frac{1}{p}-\frac{1}{q})}\|\chi_1\|_{L^r}$$

□

12th lecture

We want to extend this, i.e. replacing $D^\alpha \cong (2\pi\mathrm{i}k)^\alpha$ by a general function:

Lemma 3.2. *Assume, supp $\hat f \subset \lambda\mathcal{C}$. Take $\sigma\colon \mathbb{R}^d \to \mathbb{C}$ such that*

$$|D^\alpha\sigma(k)| \lesssim_\alpha |k|^{m-|\alpha|}\,, \quad \forall|\alpha| \leq n = 2\left(1+\left[\frac{d}{2}\right]\right)$$

for some $m\in\mathbb{R}$. Then

$$\|\sigma(k)f\|_{L^p} \lesssim \lambda^m\|f\|_{L^p}\,, \quad \forall 1\leq p\leq\infty$$

with the notation

$$\sigma(k)f = \mathcal{F}^{-1}\left(\sigma\hat f\right)\,,\ i.e.\ \ \widehat{\sigma(k)f} = \sigma(k)\hat f(k)\,.$$

Proof. Let $\chi\in C_c^\infty(\mathbb{R}^d)$, with $\chi\equiv 1$ on $\{1\leq|k|\leq 2\}$ (and $\chi\equiv 0$ outside say $\{\frac{1}{2}\leq|k|\leq 3\}$). Observe $\sigma(k)\hat f(k) = \sigma(k)\hat f(k)\chi(\frac{k}{\lambda})$, so

$$\sigma(k)f(x) = (g*f)(x)\,,$$

where $\hat g(k) = \sigma(k)\chi(\frac{k}{\lambda})$. By the Young inequality:

$$\|\sigma(k)f(x)\|_{L^p_x} = \|g*f\|_{L^p_x} \leq \|g\|_{L^1_x}\|f\|_{L^p_x}$$

We prove that $\|g\|_{L^1_x}\lesssim\lambda^m$. The decay of g is related to the smoothness of $\hat g$, and $\hat g$ is indeed smooth enough:

$$(1+|\lambda x|^2)^M g(x) = (1+|\lambda x|^2)^M\int_{\mathbb{R}^d}\mathrm{e}^{2\pi\mathrm{i}k\cdot x}\sigma(k)\chi\left(\frac{k}{\lambda}\right)\mathrm{d}k = \sum_{|\alpha|\leq 2M}\int_{\mathbb{R}^d}c_\alpha\underbrace{(2\pi\mathrm{i}\lambda x)^\alpha\mathrm{e}^{2\pi\mathrm{i}k\cdot x}}_{\lambda^{|\alpha|}D^\alpha(\mathrm{e}^{2\pi\mathrm{i}k\cdot x})}\sigma(k)\chi\left(\frac{k}{\lambda}\right)\mathrm{d}k$$

$$\overset{\substack{\text{partial integration}\\ \downarrow}}{=} \sum_{|\alpha|\leq 2M}(-1)^\alpha\lambda^{|\alpha|}\int_{\mathbb{R}^d}c_\alpha\mathrm{e}^{2\pi\mathrm{i}k\cdot x}D^\alpha\left(\sigma(k)\chi\left(\frac{k}{\lambda}\right)\right)\mathrm{d}k$$

By the Leibniz's formula:

$$D^\alpha\left(\sigma(k)\chi\left(\frac{k}{\lambda}\right)\right) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta\sigma(k)D^{\alpha-\beta}\left(\chi\left(\frac{k}{\lambda}\right)\right)$$

Using this:

$$\left|D^\alpha\left(\sigma(k)\chi\left(\frac{k}{\lambda}\right)\right)\right| \lesssim_\alpha \sup_{\beta\leq\alpha}\left|D^\beta\sigma(k)\right|\left|D^{\alpha-\beta}\left(\chi\left(\frac{k}{\lambda}\right)\right)\right| \lesssim_\alpha \underbrace{|k|^{m-|\beta|}}_{\sim\lambda^{m-|\beta|}}\left(\frac{1}{\lambda}\right)^{|\alpha-\beta|}\underbrace{|\,D^{\alpha-\beta}\chi\left(\frac{k}{\lambda}\right)|}_{\operatorname{supp}\subset\{3\lambda\geq|k|\geq\frac{\lambda}{2}\}}$$

$$\lesssim_\alpha \lambda^{m-|\alpha|}\mathbb{1}_{\{\frac{\lambda}{2}\leq|k|\leq 3\lambda\}}$$

Thus:

$$|(1+|\lambda x|^2)^M g(x)| \lesssim \sum_{|\alpha|\le 2M} \int_{\mathbb{R}^d} \lambda^{|\alpha|}\lambda^{m-|\alpha|} \mathbb{1}_{\{\lambda\le|k|\le 2\lambda\}}\,\mathrm{d}k = \lambda^{m+d}\,,$$

i.e.

$$|g(x)| \lesssim \frac{\lambda^{m+d}}{(1+|\lambda x|^2)^M}\,.$$

Then

$$\|g\|_{L^1} \le \int_{\mathbb{R}^d} \frac{\lambda^{m+d}}{(1+|\lambda x|^2)^M}\,\mathrm{d}x = \lambda^m \int_{\mathbb{R}^d} \frac{1}{(1+|y|^2)^M}\,\mathrm{d}y \lesssim \lambda^m\,,$$

if $2M = n = 2(1+[\frac{d}{2}]) > d$. □

Lemma 3.3 (Smoothing effect of the heat equation). *Assume supp* $\hat{f} \subset \lambda\mathcal{C}$*. Then*

$$\|\mathrm{e}^{t\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim C\mathrm{e}^{-ct\lambda^2}\|f\|_{L^p(\mathbb{R}^d)}\,, \quad \forall t>0$$

for some $C, c > 0$.

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ on $\mathcal{C}$ and $\operatorname{supp}\chi \subseteq \{\frac{1}{2} \le |k| \le 3\}$. Then

$$\widehat{\mathrm{e}^{-t\Delta} f}(k) = \mathrm{e}^{t|2\pi k|^2}\hat{f}(k) = \mathrm{e}^{-t|2\pi k|^2}\chi\left(\frac{k}{\lambda}\right)\hat{f}(k)\,,$$

so

$$(\mathrm{e}^{t\Delta} f)(x) = (g * f)(x), \quad \hat{g}(k) = \mathrm{e}^{-t|2\pi k|^2}\chi\left(\frac{k}{\lambda}\right)$$

where $\hat{g}(k) = \mathrm{e}^{-t|2\pi k|^2}\chi\left(\frac{k}{\lambda}\right)$. Thus, by the Young inequality, we obtain

$$\|\mathrm{e}^{t\Delta} f\|_{L^p_x} \le \|g\|_{L^1_x}\|f\|_{L^p_x}\,.$$

We have:

$$\begin{aligned}(1+|\lambda x|^2)^M g(x) &= \sum_{|\alpha|\le 2M}\int_{\mathbb{R}^d} c_\alpha \underbrace{(2\pi\mathrm{i}\lambda x)^\alpha \mathrm{e}^{2\pi\mathrm{i}k\cdot x}}_{D_k^\alpha(\mathrm{e}^{2\pi\mathrm{i}k\cdot x})}\underbrace{\mathrm{e}^{-t|2\pi k|^2}\chi\left(\frac{k}{\lambda}\right)}_{\hat{g}(k)}\mathrm{d}k \\ &= \sum_{\alpha\le 2M}(-1)^\alpha c_\alpha \lambda^{|\alpha|}\int_{\mathbb{R}^d}\mathrm{e}^{2\pi\mathrm{i}k\cdot x}D_k^\alpha \hat{g}(k)\,\mathrm{d}k\end{aligned}$$

By the Leibniz formula:

$$\begin{aligned}|D^\alpha g(k)| &\lesssim \sup_{\beta\le\alpha}\left|D^\beta\left(\mathrm{e}^{-t|2\pi k|^2}\right)D^{\alpha-\beta}\left(\chi\left(\frac{k}{\lambda}\right)\right)\right| \\ &\overset{\substack{\text{inductively}\\\text{by product-}\\\text{and chain rule}\\\downarrow}}{\lesssim} \sup_{\beta\le\alpha}\frac{(1+t|k|^2)^{|\beta|}}{|k|^{|\beta|}}\mathrm{e}^{-t|2\pi k|^2}\left(\frac{1}{\lambda}\right)^{|\alpha|-|\beta|}\underbrace{\left|D^{\alpha-\beta}\chi\left(\frac{k}{\lambda}\right)\right|}_{\operatorname{supp}\subseteq\frac{\lambda}{2}\le|k|\le 3\lambda} \\ &\lesssim \sup_{\beta\le\alpha}\frac{1+t\lambda^2}{\lambda^{|\beta|}}\lambda^{|\beta|-|\alpha|}\mathrm{e}^{-ct\lambda^2}\mathbb{1}_{\{\lambda\le|k|\le 3\lambda\}}\end{aligned}$$

so

$$(1+|\lambda x|^2)^M|g(x)| \lesssim \sup_{\substack{\beta\le\alpha\\|\alpha|\le 2M}}\frac{\lambda^{|\alpha|}}{\lambda^{|\beta|}}\lambda^{|\beta|-|\alpha|}(1+t\lambda^2)^{|\beta|}\mathrm{e}^{-ct\lambda^2}\lambda^d \lesssim \sup_{\substack{\beta\le\alpha\\|\alpha|\le 2M}}\lambda^d(1+t\lambda^2)^{|\beta|}\mathrm{e}^{ct\lambda^2}\,.$$

Thus:

$$\|g\|_{L^1_x} \lesssim \underbrace{\left(\int_{\mathbb{R}^d} \frac{1}{(1+|\lambda x|^2)^M}\right)}_{\lambda^{-d}} \sup_{\substack{\beta\leq\alpha \\ |\alpha|\leq 2M}} \lambda^d (1+\lambda^2 t)^{|\beta|} \mathrm{e}^{-ct\lambda^2} \lesssim \sup_{\substack{\beta\leq\alpha \\ |\alpha|\leq 2M}} (1+\lambda^2 t)^{|\beta|} \mathrm{e}^{-ct\lambda^2}$$

This completes the prove, since

$$(1+\lambda^2 t)^{|\beta|} \mathrm{e}^{-\frac{c}{2}t\lambda^2} \lesssim_\beta 1$$

uniformly in t and λ. □

Corollary 3.4. *If*

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = 0, & t>0 \\ u(x,t) = u_0(x), & t=0 \end{cases}$$

and $\operatorname{supp}\widehat{u_0} \subseteq \lambda\mathcal{C}$, *then*

$$\|u\|_{L^q_t L^b_x} \lesssim \lambda^{-\frac{2}{q}} \|u_0\|_{L^b} \quad \forall b,q \in [1,\infty]\,.$$

Proof. By Lemma 3.3

$$\|u\|_{L^b_x} = \|\mathrm{e}^{t\Delta} u_0\|_{L^b_x} \lesssim \mathrm{e}^{-ct\lambda^2} \|u_0\|_{L^b_x}\,,$$

so

$$\|u\|_{L^q_t L^b_x} \lesssim \|u_0\|_{L^b_x} \|\mathrm{e}^{-ct\lambda^2}\|_{L^q_t} = \|u_0\|_{L^b_x} \underbrace{\left(\int_{\mathbb{R}_+} (\mathrm{e}^{-ct\lambda^2})^q\right)^{\frac{1}{q}}}_{=\left(\int_{\mathbb{R}_+} \mathrm{e}^{-cs} \frac{\mathrm{d}s}{\lambda^2}\right)^{\frac{1}{q}} \sim \lambda^{-\frac{2}{q}}}$$

More generally, if $1 \leq a \leq b \leq \infty$:

$$\|u\|_{L^q_t L^b_x} \lesssim \lambda^{-\frac{2}{q} + d(\frac{1}{a}-\frac{1}{b})} \|u_0\|_{L^q_x}$$

by the Bernstein Lemma. □ 13th lecture

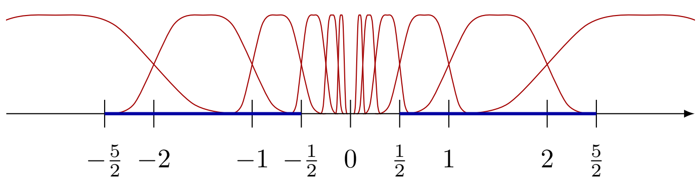
Motivation: Functions in L^p, $H^s = W^{s,2}$, $W^{s,p}$ have some regularity. The homogenous functions $\frac{1}{|x|^\lambda}$, $\lambda \in (0,1)$ are in no L^p space.

$$L^p \rightsquigarrow L^{p,q} \text{ (Lorentz space)} : \quad \frac{1}{|x|^d} \in L^{1,\infty} \quad \text{weak } L^1$$

$$H^s \rightsquigarrow B^s_{p,q} \text{ (Besov space)}$$

Definition 3.5 (Dyadic decomposition of unity). Let $\mathcal{C}$ be the annulus $\{\frac{1}{2} < |k| < \frac{5}{2}\}$, then $\exists\, \varphi \in C^\infty_c(\mathcal{C})$, such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}k) = 1 \quad \forall k \in \mathbb{R}^d \setminus \{0\}\,.$$



This is a smooth version of $\mathbb{1} = \sum_{j\in\mathbb{Z}} \mathbb{1}\{2^{j-1} \leq |k| < 2^j\}$, $\forall k \neq 0$.

Proof. Let $\tilde{\varphi} \in C^\infty_c(\mathcal{C})$ such that $\tilde{\varphi} \equiv 1$ on $\{1 \leq |k| \leq 2\}$. Define:

$$S(k) = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}k)\,.$$

Then:

- $S(k)$ is well-defined. It is easy, since $\forall k \neq 0$, the rhs is just a finite sum.
- $S(k) > 0$, $\forall k$, since $\tilde{\varphi} \equiv 1$ on $\{1 \leq |k| \leq 2\}$.
- $S(2^{-N}k) = S(k)$, $\forall N \in \mathbb{Z}$

Then we define $\varphi = \frac{\tilde{\varphi}}{S}$. □

Remark 3.6. Littlewood-Paley decomposition:

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \|\hat{f}\|^2_{L^2(\mathbb{R}^d)} = \sum_j \left\| \sqrt{\varphi(2^{-j}k)}\hat{f}(k) \right\|^2_{L^2(\mathbb{R}^d)}$$

This can be generalized to L^p-spaces! We will study the decomposition

$$f = \sum_j \varphi(2^j k) f$$

in the general situation ($f \in \mathscr{S}'$).

Lemma 3.7. *Let $\{u_j\}_{j\in\mathbb{N}}$ such that supp $\widehat{u_j} \subset 2^j\mathcal{C}$, $\forall j$ and $\exists N$ such that $\{2^{-jN}\|u_j\|_{L^\infty}\}_{j\in\mathbb{N}}$ is bounded. Then $\sum_{j\in\mathbb{N}} u_j$ converges in $\mathscr{S}'(\mathbb{R}^d)$.*

Proof. Since supp $\widehat{u_j} \subseteq 2^j\mathcal{C}$:

$$\begin{aligned} &\widehat{u_j} = \varphi(2^{-j}k) = \widehat{u_j} \\ \Longrightarrow\ &u_j(x) = \check{\varphi}(2^{-j}k) * u_j(x) = 2^{-jn} \sum_{|\alpha|=n} \widetilde{g_\alpha}(2^{-j}k) * D^\alpha_x u_j(x)\,, \quad \forall n \in \mathbb{N} \end{aligned}$$

We want to define the limit $\sum_{j\in\mathbb{N}} \langle u_j, \varphi\rangle$ for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$. We can write

$$\langle u_j, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = 2^{-jn} \sum_{|\alpha|=n} \left\langle \widetilde{g_\alpha}(2^{-j}k) * D^\alpha_x u_j, \varphi\right\rangle_{\mathscr{S}',\mathscr{S}} = 2^{-jn} \sum_{|\alpha|=n} \left\langle \widetilde{g_\alpha}(2^{-j}k) * u_j, (-1)^{|\alpha|} D^\alpha \varphi\right\rangle_{\mathscr{S}',\mathscr{S}}\,,$$

so

$$|\langle u_j, \varphi\rangle_{\mathscr{S}',\mathscr{S}}| \leq 2^{-jn} \sum_{|\alpha|=n} \underbrace{\|\widetilde{g_\alpha}(2^{-j}k)\|_{L^1_x}}_{\leq C_n} \underbrace{\|u_j\|_{L^\infty_x}}_{\lesssim 2^{jN}} \|D^\alpha \varphi\|_{L^1_x} \leq C_n 2^{-j(n-N)} \sup_{|\alpha|=n} \|D^\alpha \varphi\|_{L^1}\,.$$

Taking $n > N$, gives that $\{\langle u_j, \varphi\rangle_{\mathscr{S}',\mathscr{S}}\}$ decays L^1 exponentially in j. Hence $\sum_j \langle u_j, \varphi\rangle$ converges to $\langle T, \varphi\rangle$, $\forall \varphi \in \mathscr{S}(\mathbb{R}^d)$. From the proof:

$$|T(\varphi)| = \left| \sum_j \langle u_j, \varphi\rangle \right| \leq C_N 2^{-j} \sup_{|\alpha|=n=N+1} \|D^\alpha \varphi\|_{L^1}\,, \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^d)$$

so $T \in \mathscr{S}'(\mathbb{R}^d)$ with order $N+1$. □

Definition 3.8. $\mathscr{S}'_h(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)$ is the subset containing $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\lim_{\lambda\to\infty} \|\theta(\lambda k) f\|_{L^\infty_x(\mathbb{R}^d)} = 0\,,\ \text{where } \widehat{\theta(\lambda k)} f(k) = \theta(\lambda k)\hat{f}(k)\,,$$

holds for some $\theta \in C^\infty_c$, such that $\theta(0) \neq 0$, i.e.

$$\mathscr{S}'_h(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) \,\middle|\, \exists \theta \in C^\infty_c(\mathbb{R}^d)\,, \theta(0) \neq 0 :\ \lim_{\lambda\to\infty} \|\theta(\lambda k) f\|_{L^\infty_x(\mathbb{R}^d)} = 0 \right\}.$$

Heuristically, this allows us to avoid the situation, where $\hat{f}(k) = \delta_k$.

Example 3.9. (1) If f is a (nonzero) polynomial, then $f \notin \mathscr{S}'_h(\mathbb{R}^d)$.

(2) If $\hat{f} \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $f \in \mathscr{S}'_h(\mathbb{R}^d)$.

(3) If $0 < \lambda < d$, then $\frac{1}{|x|^\lambda} \in \mathscr{S}'_h(\mathbb{R}^d)$, since $\widehat{\frac{1}{|x|^\lambda}} \sim \frac{1}{|k|^{d-\lambda}} \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Definition 3.10 (Homogeneous Besov space). Let $s \in \mathbb{R}$, $p, r \in [1, \infty]$. Define $\dot{B}^s_{p,r} \subset \mathscr{S}'_h$ with

$$\|f\|_{\dot{B}^s_{p,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \| \varphi(2^{-j}k) f \|^r_{L^p_x} \right)^{\frac{1}{r}} , & r \in [1, \infty) \\ \sup_{j\in\mathbb{Z}} 2^{js} \| \varphi(2^{-j}k) f \|_{L^p}, & r = \infty \end{cases} = \left(2^{js} \| \varphi(2^{-j}k) f \|_{L^p} \right)_{\ell^r(\mathbb{Z})}$$

Remark 3.11. $\forall j \in \mathbb{Z}$, $\widehat{\varphi(2^{-j}k) f}$ is supported in $2^j\mathcal{C}$. Hence, by the Bernstein lemma:

$$2^{js} \| \varphi(2^{-j}k) f \|_{L^p} \sim \sup_{|\alpha|=s} \| D^\alpha (\varphi(2^{-j}k) f) \|_{L^p} \quad \text{if } s \in \mathbb{N}$$

Hence, we can think of $\dot{B}^s_{p,r}$ as a variant of $\dot{W}^{s,p}(\mathbb{R}^d) = \{\sup_{|\alpha|=s} \|D^\alpha f\|_{L^p} < \infty\}$. Rigorously, the norm $\dot{B}^s_{2,2}$ is equivalent to the seminorm of $\dot{H}^s(\mathbb{R}^d)$. If $|s| < \frac{d}{2}$, then $\dot{B}^s_{2,2} = \dot{H}^s(\mathbb{R}^d)$ is a Hilbert space.

Lemma 3.12. *The space $\dot{B}^s_{p,r}$ is a normed space.*

Proof. It is easy to see that $\|f\|_{\dot{B}^s_{p,r}}$ is a seminorm! Assume $\|f\|_{\dot{B}^s_{p,r}} = 0$, then

$$\begin{aligned} &\varphi(2^{-j}k) f = 0 \, , \quad \forall j \in \mathbb{Z} \\ \Longrightarrow\ &\varphi(2^{-j}k) \hat{f}(k) = 0 \, , \quad \text{for a.e. } k, \ \forall j \in \mathbb{Z} \\ \Longrightarrow\ &\hat{f}(k) = 0 \text{ for a.e. } k \, . \end{aligned}$$

The only dangerous possibility is $\hat{f}(k) \sim \delta_k$, but it is ruled out by $f \in \mathscr{S}'_h$. Thus $f = 0$. Hence $\dot{B}^s_{p,r}$ is a normed space. □

Lemma 3.13. *Take $N \in \mathbb{N}$, $f \in \mathscr{S}'_h$. Then $\|u\|_{\dot{B}^s_{p,r}} = 2^{N(s-\frac{d}{p})} \|u_N\|_{\dot{B}^s_{p,r}}$, where $u_N(x) = u(2^N x)$.*

Proof. $u_N(x) = u(2^N x)$, so $\widehat{u_N}(k) = 2^{-dN} \hat{u}(2^{-N}k)$. Hence:

$$\begin{aligned} \|u_n\|_{B^s_{2,r}} &= \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \| \varphi(2^{-j}k) u_N \|^r_{L^2_x} \right)^{\frac{1}{r}} = \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \left(\int_{\mathbb{R}^d} \left| \varphi(2^{-j}k) 2^{-dN} \hat{u}(2^{-N}k) \right|^2 \mathrm{d}k \right)^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \left(\int_{\mathbb{R}^d} \left| \varphi(2^{-j}2^N\xi) 2^{-dN} \hat{u}(\xi) \right|^2 2^{dN} \mathrm{d}\xi \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \\ &\overset{l:=j-N}{\downarrow}{=} 2^{-\frac{dN}{2}} \left(\sum_{l\in\mathbb{Z}} \underbrace{2^{r(l+N)s}}_{2^{rNs}2^{rls}} \left(\int_{\mathbb{R}^d} \left| \varphi(2^{-l}\xi) \hat{u}(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \\ &= 2^{N(s-\frac{d}{2})} \|u\|_{\dot{B}^s_{2,r}} = \left(2^{N(s-\frac{d}{p})} \|u\|_{\dot{B}^s_{p,r}} \right) \Big|_{p=2} \end{aligned}$$

Exercise: general case $p \in [1, \infty]$ □

Lemma 3.14 (Sobolev type). *Let $1 \le p_1 \le p_2 \le \infty$, $1 \le r_1 \le r_2 \le \infty$. Then*

$$\dot{B}^s_{p_1,r_1} \subset \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$$

with continuous embedding.

This is a variant of the Sobolev embedding theorem $\dot{W}^{s,2}(\mathbb{R}^d) = \dot{H}^s(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) = \dot{W}^{0,p}(\mathbb{R}^d)$.

Proof. Take $r_1 = r_2 =: r$ for simplicity. Then:

$$\| \underbrace{\varphi(2^{-j}k)}_{\mathrm{supp}\subseteq 2^j\mathcal{C}} f\|_{L^{p_2}} \overset{\text{Bernstein}}{\overset{\downarrow}{\lesssim}} 2^{-jd(\frac{1}{p_1}-\frac{1}{p_2})}\| \varphi(2^{-jk})f\|_{L^{p_1}}$$

Thus:

$$\begin{aligned}\|f\|^r_{\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r}} &= \sum_{j\in\mathbb{Z}} 2^{rj(s-d(\frac{1}{p_1}-\frac{1}{p_2}))}\| \varphi(2^{-j}k)f\|^r_{L^{p_2}} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{rjs}2^{-rjd(\frac{1}{p_1}-\frac{1}{p_2})}2^{rjd(\frac{1}{p_1}-\frac{1}{p_2})}\| \varphi(2^{-j}k)f\|^r_{L^{p_1}} \\ &= \|f\|_{\dot{B}^s_{p_1,r}}\end{aligned}$$

If $r_1 < r_2$, we also need to use that $\ell^{r_1}(\mathbb{Z}) \subset \ell^{r_2}(\mathbb{Z})$ with continuous embedding. □

Lemma 3.15. *Let $0 < \lambda < d$. Then*

$$f(x) = \frac{1}{|x|^\lambda} \in \dot{B}^{\frac{d}{p}-\lambda}_{p,\infty}, \quad \forall p \in [1,\infty]\,.$$

Proof. By Sobolev (Lemma 3.14), $\dot{B}^{\frac{d}{p}-\lambda}_{p,\infty} \supset \dot{B}^{d-\lambda}_{1,\infty}$, $\forall p \in [1,\infty]$, so it suffices to prove this for $p = 1$. Consider

$$\|f\|_{\dot{B}^{d-\lambda}_{1,\infty}} = \sup_{j\in\mathbb{Z}} 2^{j(d-\lambda)}\| \varphi(2^{-j}k)f\|_{L^1(\mathbb{R}^d)}$$

Key point: $x \mapsto f(x)$ is homogeneous:

$$f(tx) = t^{-\lambda}f(x) \implies \hat{f}(tk) = t^{-d}t^{\lambda}\hat{f}(k)$$

And hence

$$\begin{aligned}\| \varphi(2^{-j}k)f\|_{L^1(\mathbb{R}^d_x)} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \varphi(2^{-j}k)\hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}k\cdot x}\,\mathrm{d}k\right|\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} \varphi(k)\underbrace{\hat{f}(2^jk)}_{2^{-j(d-\lambda)}\hat{f}(k)}\mathrm{e}^{2\pi\mathrm{i}2^jk\cdot x}2^{jd}\,\mathrm{d}k\right|\mathrm{d}x \\ &= \int_{\mathbb{R}^d} 2^{j\lambda}\left|\int_{\mathbb{R}^d} \varphi(k)\hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}2^jk\cdot x}\,\mathrm{d}k\right|\mathrm{d}x \\ &= \int_{\mathbb{R}^d} 2^{j\lambda}2^{-jd}\left|\varphi(k)\hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}k\cdot x}\right|\mathrm{d}x = 2^{-j(d-\lambda)}\| \varphi(k)f\|_{L^1_x}\end{aligned}$$

$$\implies \|f\|_{\dot{B}^{d-\lambda}_{1,\infty}} = \| \varphi(k)f\|_{L^1_x} < \infty\,.$$

□

5th Tutorial

Theorem (Hardy-Littlewood-Sobolev).

$$\left|\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda}\,\mathrm{d}x\,\mathrm{d}y\right| \le C\|f\|_{L^p}\|g\|_{L^q}\,,$$

where $C = C(p,q,\lambda,d)$, $\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{d} = 2$ and $p,q \in (1,\infty)$, $\lambda \in (0,d)$.

This is equivalent to

$$\sup_{\|g\|_{L^q}} \left| \int_{\mathbb{R}^d} \left(f * \frac{1}{|x|^\lambda} \right) g \, \mathrm{d}x \right| \le C \|f\|_{L^p} \iff \left\| f * \frac{1}{|x|^\lambda} \right\|_{L^{q'}} \le C \|f\|_{L^p}$$

For $1 + \frac{1}{q'} = \frac{1}{p} + \frac{1}{r}$ (hence $r \leftrightarrow \frac{d}{\lambda}$):

$$\|f * h\|_{L^{q'}} \le \|f\|_{L^p} \|h\|_{L^r} \qquad \text{(Young's inequality)}$$

Actually, $\frac{1}{|x|^\lambda} \in L_w^{\frac{d}{\lambda}}(\mathbb{R}^d)$ (weak $L^{\frac{d}{\lambda}}$ space).

$$w \in L_w^r(\mathbb{R}^d) := \left\{ w \mid \sup_{s>0} s |\{|w| > s\}|^{\frac{1}{r}} < \infty \right\}$$

$$w \in L^r(\mathbb{R}^d) \implies \int_{\mathbb{R}^d} |w|^r \, \mathrm{d}x < \infty \implies r \int_0^\infty \lambda^{r-1} \left(\int 1_{\{|w|>\lambda\}} \, \mathrm{d}k \right) \mathrm{d}\lambda < \infty$$

Also for $q', p, r \in (1, \infty)$:

$$\|f * h\|_{L^{q'}} \le C \|f\|_{L^p} \|h\|_{L_w^r} \qquad \text{(Weak Young's inequality)}$$

Proof. (of HLS) By the Layer-Cake representation for $f, g \ge 0$:

$$f(x) = \int_0^\infty \mathbb{1}_{f(x)>a} \, \mathrm{d}a$$

$$g(y) = \int_0^\infty \mathbb{1}_{g(y)>b} \, \mathrm{d}b$$

$$\frac{1}{|z|^\lambda} = \int_0^\infty \mathbb{1}_{\frac{1}{|z|^\lambda}<c} \, \mathrm{d}c = \lambda \int_0^\infty \frac{1}{c^{\lambda+1}} \, \mathrm{d}c$$

Thus (set $z = x - y$):

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)g(x)}{|x-y|^\lambda} \, \mathrm{d}x \, \mathrm{d}y = \lambda \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{d}y \frac{\mathbb{1}_{f(x)>a} \mathbb{1}_{g(y)>b} \mathbb{1}_{|x-y|>c}}{c^{\lambda+1}}}_{=:I(a,b,c)}$$

Main estimate:

$$\begin{aligned} &\mathbb{1}_{f(x)>a} \mathbb{1}_{g(y)>b} \mathbb{1}_{|x-y|>c} \\ &\implies I(a,b,c) \le \frac{1}{c^{\lambda+1}} u(a) u(b) \end{aligned}$$

where $u(a) = \int_{\mathbb{R}^d} \mathbb{1}_{f(x)>a} \, \mathrm{d}x$, $v(b) = \int_{\mathbb{R}^d} \mathbb{1}_{g(y)>b} \, \mathrm{d}y$, $w(c) = \int_{\mathbb{R}^d} \mathbb{1}_{|z|<c} \, \mathrm{d}z \sim c^d$. Similarly:

$$I(a,b,c) \le \frac{1}{c^{\lambda+1}} \min(u(a)v(b), u(a)w(c), v(b)w(c)) = \frac{1}{c^{\lambda+1}} \frac{u(a)v(b)w(c)}{\max(u(a), u(b), u(c))}$$

We need to bound

$$\int_0^\infty \int_0^\infty \int_0^\infty c^{d-\lambda-1} \frac{u(a)v(b)}{\max(u(a), v(b), w(c))} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, .$$

Assume

$$1 = \|f\|_{L^p}^p = p \int_0^\infty a^{p-1} |\{f > a\}| \, \mathrm{d}a = p \int_0^\infty a^{p-1} u(a) \, \mathrm{d}a$$

$$1 = \|g\|_{L^q}^q = \cdots = q \int_0^\infty b^{q-1} v(b) \, \mathrm{d}b$$

It suffices to assume $u(a) \geq v(b)$. Then on that domain:

$$T_1 \leq \iiint c^{d-\lambda-1} \frac{u(a)v(b)}{\max(u(a), c^d)} \,\mathrm{d}a \,\mathrm{d}b \,\mathrm{d}c$$

Let's do the c-integration:

$$\int_0^\infty c^{d-\lambda-1} \frac{1}{\max(u(a), c^d)} \,\mathrm{d}c = \int_0^{u(a)^{\frac{1}{d}}} c^{d-\lambda-1} \frac{1}{u(a)} \,\mathrm{d}c \sim (c^{d-\lambda})\Big|_{c=u(a)^{\frac{1}{d}}} \frac{1}{u(a)}$$
$$= u(a)^{-\frac{\lambda}{d}} + \int_{u(a)^{\frac{1}{d}}}^\infty c^{d-\lambda-1} \frac{1}{c^d} \,\mathrm{d}c \sim (c^{-\lambda})\Big|_{c=u(a)^{\frac{1}{d}}} = u(a)^{\frac{-\lambda}{d}}$$

Hence:

$$T_1 \leq \int_0^\infty \int_0^\infty u(a)^{1-\frac{\lambda}{d}} v(b) \,\mathrm{d}a \,\mathrm{d}b$$

Try Hölder:

$$\int_0^{b^\alpha} u(a)^{1-\frac{\lambda}{d}} \,\mathrm{d}a \leq \left(\int a^{p-1} u(a) \,\mathrm{d}a \right)^{1-\frac{\lambda}{d}} \left(\int_0^{b^\alpha} a^\beta \right)^{\frac{\lambda}{d}} \lesssim \left((a^{\beta+1})\Big|_{a=b^\alpha} \right)^{\frac{\lambda}{d}} = b^{\alpha(\beta+1)\frac{\lambda}{d}}$$

$$\implies \iint_{a<b^\alpha} u(a)^{1-\frac{\lambda}{d}} v(b) \,\mathrm{d}a \,\mathrm{d}b \lesssim \int b^{q-1} v(b) \,\mathrm{d}b \lesssim 1$$

$$\iint_{a>b^\alpha} u(a)^{1-\frac{\lambda}{d}} v(b) \,\mathrm{d}a \,\mathrm{d}b = \iint_{b<a^{\frac{1}{\alpha}}} u(a)^{1-\frac{\lambda}{d}} v(b) \,\mathrm{d}a \,\mathrm{d}b \leq \iint_{b<a^{\frac{1}{\alpha}}} u(a) v(b)^{1-\frac{\lambda}{d}} \,\mathrm{d}a \,\mathrm{d}b$$

Hölder:

$$\int_{b<a^{\frac{1}{\alpha}}} v(b)^{1-\frac{\lambda}{d}} \,\mathrm{d}b \leq \left(\int b^{q-1} v(b) \,\mathrm{d}b \right)^{1-\frac{\lambda}{d}} \left(\int_0^{a^{\frac{1}{\alpha}}} b^\gamma \right)^{\frac{\lambda}{d}} \lesssim a^{\frac{1}{\alpha}(\gamma+1)\frac{\lambda}{d}} a^{p-1}$$

$$\implies \iint_{a>b^\alpha} u(a)^{1-\frac{\lambda}{d}} v(b) \,\mathrm{d}a \,\mathrm{d}b \lesssim \int a^{p-1} u(a) \,\mathrm{d}a \lesssim 1$$

□

14th lecture

Lemma 3.16 (Interpolation). *If $s_1 < s_2 \in \mathbb{R}$, $\theta \in (0,1)$, $p, r \in [1, \infty]$,*

$$\|u\|_{\dot{B}^{s_\theta}_{p,r}} \leq \|u\|^\theta_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}, \quad s_\theta = \theta s_1 + (1-\theta) s_2$$

Proof. This is equivalent to

$$\left(2^{js_\theta} \| \varphi(2^{-j}k) u \|_{L^p} \right)_{\ell^r(\mathbb{Z})} \leq \left(2^{js_1} \| \varphi(2^{-j}k) u \|_{L^p} \right)^\theta_{\ell^r(\mathbb{Z})} \left(2^{js_2} \| \varphi(2^{-j}k) u \|_{L^p} \right)^{1-\theta}_{\ell^r(\mathbb{Z})}.$$

Note that

$$2^{js_\theta} = (2^{js_1})^\theta \cdot (2^{js_2})^{1-\theta} = 2^{j(\theta s_1 + (1-\theta)s_2)},$$

so

$$(\mathrm{LHS})^r = \sum_{j\in\mathbb{Z}} 2^{js_\theta} \| \varphi(2^{-j}k) u \|^r_{L^p} = \sum_{j\in\mathbb{Z}} \left(2^{js_1} \| \varphi(2^{-j}k) u \|^r_{L^p} \right)^\theta \left(2^{js_2} \| \varphi(2^{-j}k) u \|^r_{L^p} \right)^{1-\theta}$$
$$\overset{\text{Hölder}}{\leq} \left(\sum_{j\in\mathbb{Z}} \cdots \right)^\theta \left(\sum_{j\in\mathbb{Z}} \cdots \right)^{1-\theta} = (\mathrm{RHS})^r.$$

□

Remark 3.17. (i) $\dot{B}^s_{p,r}$ is a normed space, but not necessarily a Banach space. Actually, if $s > \frac{d}{p}$ (or $s = \frac{d}{p}$ and $r > 1$), then $\dot{B}^s_{p,r}$ is not complete.

(ii) By the Bernstein Lemma 3.1:

$$\operatorname{supp} \hat{f} \subset \lambda \mathcal{C} \implies \|D^\alpha f\|_{L^q} \lesssim \lambda^{|\alpha|+d(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p} ,$$

in particular for $\alpha = 0$, $q = \infty$:

$$\|\varphi(2^{-j}k)f\|_{L^\infty} \lesssim 2^{j\frac{d}{p}} \|\varphi(2^{-j}k)f\|_{L^p} \tag{$*$}$$

If $s < \frac{d}{p}$ and $f \in \dot{B}^s_{p,r}$, then

$$\begin{aligned}\infty > \sum_{j\in\mathbb{Z}} 2^{rjs} \|\varphi(2^{-j}k)f\|^r_{L^p} &\geq \sum_{j\in\mathbb{Z}} 2^{js} \|\varphi(2^{-j}k)f\|_{L^p} \geq \sum_{j<0} 2^{js} \|\varphi(2^{-j}k)f\|_{L^p} \\ &\geq \sum_{j<0} 2^{j\frac{d}{p}} \|\varphi(2^{-j}k)f\|_{L^p} \overset{(*)}{\gtrsim} \sum_{j<0} \|\varphi(2^{-j}k)f\|_{L^\infty} ,\end{aligned}$$

so $\sum_{j<0} \varphi(2^{-j}k)f$ is convergent in L^∞. Hence $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}k)f \in \mathscr{S}'_h(\mathbb{R}^d)$: Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\|\sum_{j\leq -N} \varphi(2^{-j}k)f\|_{L^\infty} < \varepsilon$. Then

$$\left\|\theta(\lambda k) \sum_{j\in\mathbb{Z}} \varphi(2^{-j}k)f\right\|_{L^\infty} \leq \left\|\theta(\lambda k) \sum_{-N<j} \varphi(2^{-j}k)f\right\|_{L^\infty} + \left\|\theta(\lambda k) \sum_{j\leq -N} \varphi(2^{-j}k)f\right\|_{L^\infty} ,$$

where

$$\left\|\theta(\lambda k) \sum_{j\leq -N} \varphi(2^{-j}k)f\right\|_{L^\infty} \overset{\text{Young}}{\leq} \underbrace{\|\mathcal{F}^{-1}(\theta(\lambda k))\|_{L^1}}_{\text{ind. of } \lambda} \left\|\sum_{j\leq -N} \varphi(2^{-j}k)f\right\|_{L^\infty} \lesssim \varepsilon .$$

For the low-frequency part, we may choose λ large enough, such that

$$\operatorname{supp}(\theta(\lambda \bullet)) \cap \operatorname{supp}\left(\sum_{-N<j} \varphi(2^{-j}\bullet)\right) = \emptyset .$$

Similarly, if $s = \frac{d}{p}$ and $r = 1$, the same holds.

Theorem 3.18. *Let $s_1, s_2 \in \mathbb{R}$, $p_1, p_2, r_1, r_2 \in [1, \infty]$. Assume*

$$s_1 < \frac{d}{p_1} \quad \text{or} \quad \left(s_1 = \frac{d}{p_1} \text{ and } r_1 = 1\right)$$

then

- $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ *is a Banach space with* $\|u\|_{B^{s_1}_{p_1,r_1}} + \|u\|_{\dot{B}^{s_2}_{p_2,r_2}}$.
- *Fatou property: If $\{u_n\}_n$ is bounded in $\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$, then up to a subsequence, $u_n \to u$ in $\mathscr{S}'$ with $u \in \mathscr{S}'_h$ and $u \in \dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ and*

$$\|u\|_{\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}} \leq C \liminf_{n\to\infty} \|u_n\|_{\dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}} .$$

Lemma 3.19. *Assume supp $\hat{u}_j \subset 2^j\mathcal{C}$, $\mathcal{C}$ an annulus, $(2^{js}\|u_j\|_{L^p})_{\ell^r(\mathbb{Z})}$ is finite and $\sum_{j\in\mathbb{Z}} u_j = u$ in $\mathscr{S}'$ and $u \in \mathscr{S}'_h$. Then $u \in \dot{B}^s_{p,r}$ and $\|u\|_{\dot{B}^s_{p,r}} \leq C(2^{js}\|u_j\|_{L^p})_{\ell^r(\mathbb{Z})}$*

Proof. Observe that $\varphi(2^{-j'}k)u_j = 0$, if $|j'-j| \geq C$. Thus:

$$\left\|\varphi(2^{-j'}k)u\right\|_{L^p} = \left\|\sum_{|j'-j|\leq C} \varphi(2^{-j'}k)u_j\right\|_{L^p} \leq \sum_{|j'-j|\leq C} \left\|\varphi(2^{-j'}k)u_j\right\|_{L^p} \lesssim \sum_{|j'-j|\leq C} \|u_j\|_{L^p}$$
$$\implies 2^{sj'}\|\varphi(2^{-j'}k)u\|_{L^p} \lesssim \sum_{|j'-j|\leq C} \underbrace{2^{sj'}}_{\lesssim 2^{sj}} \|u_j\|_{L^p}$$
$$\implies \sum_{j'\in\mathbb{Z}} 2^{sj'r}\|\varphi(2^{-j'}k)u\|_{L^p}^r \lesssim \sum_{j'\in\mathbb{Z}} \Big(\sum_{|j-j'|\leq C} 2^{sj}\|u_j\|_{L^p}\Big)^r \lesssim \left\|2^{sj}\|u_j\|_{L^p}\right\|_{\ell^r(\mathbb{Z})}^r$$

□

Proof of Theorem 3.18. **Fatou property**: Let $\{u_n\}_n \subseteq \dot{B}^{s_1}_{p_1,r_1} \cap \dot{B}^{s_2}_{p_2,r_2}$ be bounded.
$\forall j$: $\{\varphi(2^{-j}k)u_n\}_n \xrightarrow[\text{up to subsequence of } u_n]{\text{Cantor diagonal}} \tilde{u}_j$, $\operatorname{supp}\widehat{\tilde{u}_j} \subset 2^j\mathcal{C}$, where

$$\|\tilde{u}_j\|_{L^p} \leq \liminf_{n\to\infty} \|\varphi(2^{-j}k)u_n\|_{L^p}\,.$$

Then we define $u = \sum_{j\in\mathbb{Z}} \tilde{u}_j$. The condition $s_1 < \frac{d}{p_1}$ (or $s_1 = \frac{d}{p_1}$ and $r_1 = 1$) gives $\sum_{j<0} \tilde{u}_j$ is convergent in L^∞, so $u \in \mathscr{S}_h'$ (see Remark 3.17(ii)). Applying Lemma 3.19:

$$\varepsilon \in \{1,2\}: \quad \|u\|_{\dot{B}^{s_\varepsilon}_{p_2,r_2}} \leq C\|2^{js_\varepsilon}\|\tilde{u}_j\|_{L^{p_\varepsilon}}\|_{\ell^{r_\varepsilon}(\mathbb{Z})} \leq C\liminf_{n\to\infty} \|2^{js_\varepsilon}\|\varphi(2^{-j}k)u_n\|_{L^p}\|_{\ell^{r_\varepsilon}(\mathbb{Z})}$$

Completeness follows from the Fatou property.

□ 15th lecture

Remark 3.20. This implies, that $\dot{B}^s_{p,r}$ is a Banach space if $s < \frac{d}{p}$ or $s = \frac{d}{p}$ and $r = 1$.

Theorem 3.21. *$\forall p, q \in [1,\infty]$, $p \leq q$, then*

$$\dot{B}^{d(\frac{1}{p}-\frac{1}{q})}_{p,1}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)\,.$$

Moreover, if $p < \infty = q$:

$$\dot{B}^{\frac{d}{p}}_{p,1} \subset C_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} \;\; \textit{continuos and vanishing at } \pm\infty\}$$

Remark 3.22. We proved Sobolev $\dot{B}^s_{p,r_1} \subset \dot{B}^{s-d(\frac{1}{p}-\frac{1}{q})}_{q,r_2}$ if $p \leq q$, $r_1 \leq r_2$. But $\dot{B}^0$ is quite complicated.

Proof. Take $u \in \dot{B}^{d(\frac{1}{p}-\frac{1}{q})}_{p,1}(\mathbb{R}^d)$, then $u = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}k)u$ and

$$\|u\|_{L^q(\mathbb{R}^d)} = \|\sum_{j\in\mathbb{Z}} \varphi(2^{-j}k)u\|_{L^q(\mathbb{R}^d)} \leq \sum_{j\in\mathbb{Z}} \|\varphi(2^{-j}k)u\|_{L^q(\mathbb{R}^d)} \overset{\text{Bernstein}}{\leq} \sum_{j\in\mathbb{Z}} 2^{jd(\frac{1}{p}-\frac{1}{q})}\|\varphi(2^{-j}k)u\|_{L^p(\mathbb{R}^d)}$$
$$= \|u\|_{\dot{B}^s_{p,1}}$$

with $s = d(\frac{1}{p}-\frac{1}{q})$. □

Theorem 3.23. *$\forall p \in [1,\infty]$, $L^p(\mathbb{R}^d) \subset \dot{B}^0_{p,\infty}(\mathbb{R}^d)$. When $p = 1$, we even have $\mathcal{M}(\mathbb{R}^d) \subset \dot{B}^0_{1,\infty}(\mathbb{R}^d)$, where $\mathcal{M}(\mathbb{R}^d)$ is the set of bounded measures on $\mathbb{R}^d$.*

Proof. Take $u \in \dot{B}^0_{p,\infty}(\mathbb{R}^d)$ then

$$\|u\|_{\dot{B}^0_{p,\infty}} = \sup_{j\in\mathbb{Z}} \|\varphi(2^{-j}k)u\|_{L^p(\mathbb{R}^d)}$$

Note $\varphi(2^{-j}k)u(x) = G_j * u(x)$, where $\widehat{G_j}(k) = \varphi(2^{-j}k)$, hence $G_j(x) = 2^{jd}G(2^j x)$, $\|G_j\|_{L^1} = \|G\|_{L^1}$. By Young's inequality:

$$\|\,\varphi(2^{-j}k)u\|_{L^p} = \|G_j * u\|_{L^p} \le \|G_j\|_{L^1}\|u\|_{L^p} \lesssim \|u\|_{L^p}$$

uniformly in j.

$$\implies \|u\|_{\dot{B}^0_{p,\infty}} \lesssim \|u\|_{L^p}$$

When $p = 1$:

$$\|\,\varphi(2^{-j}k)u\|_{\dot{B}^0_{1,\infty}} = \sup_j \|G_j * u\|_{L^1} \le \|G_j\|_{L^1}\|u\|_{\mathcal{M}} \lesssim \|u\|_{\mathcal{M}}$$

□

Theorem 3.24. *Let $s > 0$, $p, r \in [1,\infty]$. Then*

$$\|u\|_{\dot{B}^{-2s}_{p,r}} \sim \left\|\|t^s \mathrm{e}^{t\Delta}u\|_{L^p_x}\right\|_{L^r(\mathbb{R}_+,\frac{\mathrm{d}t}{t})}\,.$$

Proof. **Case 1** ($r = \infty$): In this case,

$$\|u\|_{\dot{B}^{-2s}_{p,r}} = \sup_{j\in\mathbb{Z}} 4^{-j}\|\,\varphi(2^{-j}k)u\|_{L^p}\,.$$

Consider

$$t^s\|\mathrm{e}^{t\Delta}u\|_{L^p_x} = t^s\|\mathrm{e}^{t\Delta}\sum_{j\in\mathbb{Z}}\varphi(2^{-j}k)u\|_{L^p_x} \le \sum_{j\in\mathbb{Z}} t^s\|\mathrm{e}^{t\Delta}\underbrace{\varphi(2^{-j}k)u}_{\substack{=:u_j,\\ \operatorname{supp}\widehat{u_j}\subset 2^j\mathcal{C}}}\|_{L^p_x} \overset{\text{Lemma 3.3}}{\lesssim} \sum_{j\in\mathbb{Z}} t^s \mathrm{e}^{-ct4^j}\|\,\varphi(2^{-j}k)u\|_{L^p}$$

$$\le \underbrace{\left(\sum_{j\in\mathbb{Z}} t^s 4^{js}\mathrm{e}^{-ct4^j}\right)}_{=:\,(*)}\underbrace{\sup_{j\in\mathbb{Z}} 4^{-js}\|\,\varphi(2^{-j}k)u\|_{L^p}}_{=\|u\|_{\dot{B}^{-2j}_{p,\infty}}}$$

Note that $(*) \lesssim_s 1$ (c universal), since ($j \rightsquigarrow x \in \mathbb{R}$):

$$\int_{\mathbb{R}} t^s 4^{xs}\mathrm{e}^{-ct4^x}\,\mathrm{d}x \sim \int_{\mathbb{R}} t^s \mathrm{e}^{xs}\mathrm{e}^{-ct\mathrm{e}^x}\,\mathrm{d}x = \int_{\mathbb{R}_+} t^s y^s \mathrm{e}^{-cty}\frac{\mathrm{d}y}{y} = \int_{\mathbb{R}_+} t^s\left(\frac{z}{ct}\right)^s \mathrm{e}^{-z}\frac{\mathrm{d}z}{z}$$
$$= \frac{1}{c^s}\int_{\mathbb{R}_+} z^{s-1}\mathrm{e}^{-z}\,\mathrm{d}z = \frac{1}{c^s}\Gamma(s)$$

Let us consider the reverse inequality. Note that:

$$\Gamma(\gamma+1) = \int_0^\infty t^\gamma \mathrm{e}^{-t}\,\mathrm{d}t = \int_0^\infty t^\gamma a^{\gamma+1}\mathrm{e}^{-ta}\,\mathrm{d}t\,,\quad \forall a > 0\,.$$

Hence, the functional calculus yields:

$$\Gamma(\gamma+1) = \int_0^\infty t^\gamma(-\Delta)^{\gamma+1}\mathrm{e}^{t\Delta}\,\mathrm{d}t$$

Thus

$$\varphi(2^{-j}k)u = \frac{1}{\Gamma(s+1)}\int_0^\infty t^s(-\Delta)^{s+1}\mathrm{e}^{t\Delta}\,\varphi(2^{-j}k)u\,\mathrm{d}t\,,$$

so

$$\begin{aligned}\sup_{j\in\mathbb{Z}} 4^{-js}\|\,\varphi(2^{-j}k)u\|_{L^p} &= \sup_{j\in\mathbb{Z}} 4^{-js}\frac{1}{\Gamma(s+1)}\left\|\int_0^\infty t^s(-\Delta)^{s+1}\mathrm{e}^{t\Delta}\,\varphi(2^{-j}k)u\,\mathrm{d}t\right\|_{L^p}\\ &\lesssim \sup_{j\in\mathbb{Z}} 4^{-js}\int_0^\infty t^s 4^{j(s+1)}\mathrm{e}^{-c+4^j}\|\mathrm{e}^{\frac{t\Delta}{2}}u\|_{L^p}\,\mathrm{d}t\\ &\lesssim \sup_{j\in\mathbb{Z}}\underbrace{\left(\int_0^\infty 4^j\mathrm{e}^{-ct4^j}\,\mathrm{d}t\right)}_{=\int_0^\infty z\mathrm{e}^{-z}}\sup_{t>0} t^s\|\mathrm{e}^{\frac{t\Delta}{2}}u\|_{L^p}\\ &\lesssim \sup_{t>0} t^s\|\mathrm{e}^{\frac{t\Delta}{2}}u\|_{L^p}\lesssim_s \sup_{t>0} t^s\|\mathrm{e}^{t\Delta}u\|_{L^p}\,.\end{aligned}$$

Case 2 ($r<\infty$): We have

$$t^s\|\mathrm{e}^{t\Delta}u\|_{L^p} = t^s\left\|\mathrm{e}^{t\Delta}\sum_{j\in\mathbb{Z}}\varphi(2^{-j}k)u\right\|_{L^p}\lesssim \sum_{j\in\mathbb{Z}} t^s\mathrm{e}^{-ct4^j}4^{js}\underbrace{4^{-js}\|\,\varphi(2^{-j}k)u\|_{L^p}}_{=:c_j},$$

where $\left(c_j^r\right)^{\frac{1}{r}} = \|u\|_{\dot{B}^{-2s}_{p,r}}$. Now,

$$\begin{aligned}\int_0^\infty \left(t^s\|\mathrm{e}^{t\Delta}u\|_{L^p}\right)^r\frac{\mathrm{d}t}{t} &\lesssim \int_0^\infty\left(\sum_{j\in\mathbb{Z}} t^s\mathrm{e}^{-ct4^j}4^{js}c_j\right)^r\frac{\mathrm{d}t}{t}\\ &\lesssim \int_0^\infty\underbrace{\left(\sum_{j\in\mathbb{Z}} t^s\mathrm{e}^{-ct4^j}4^{js}\right)^{r-1}}_{\lesssim 1}\left(\sum_{j\in\mathbb{Z}} t^s\mathrm{e}^{-ct4^j}4^{js}c_j^r\right)\frac{\mathrm{d}t}{t}\\ &\lesssim \int_0^\infty\sum_{j\in\mathbb{Z}} t^s\mathrm{e}^{ct4^j}4^jc_j^r\frac{\mathrm{d}t}{t} = \int_0^\infty\sum_{j\in\mathbb{Z}}\left(y^s\mathrm{e}^{-cy}\frac{\mathrm{d}y}{y}\right)c_j^r\lesssim \|u\|^r_{\dot{B}^{-2s}_{p,r}}\end{aligned}$$

Other direction: ? □

6th tutorial

Definition 3.25. Consider $\sum_{j\geq 0}\varphi(2^{-j}k)+\chi(k)=1$, $\forall k\in\mathbb{R}^d$. The *nonhomogeneous Besov norm* is

$$\|u\|_{B^s_{p,r}} = \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j u\|^r_{L^p}\right)^{\frac{1}{r}},$$

where

$$\Delta_j u = \begin{cases}\varphi(2^{-j}k)u, & \text{if } j\geq 0\\ \chi(k)u, & \text{if } j=-1\\ 0, & \text{if } j\leq 2\end{cases}.$$

Note that $\operatorname{supp}\chi\subset\{|k|\leq\frac{3}{2}\}$.

Remark 3.26. $B^s_{2,2}=H^s$, $\forall s\in\mathbb{R}$ (nonhomogeneous Sobolev space). In general, $B^s_{p,r}$ contains the Sobolev space.

Theorem 3.27. *$\forall s\in\mathbb{R}$, $p,r\in[1,\infty]$, the nonhomogenoues Besov space $B^s_{p,r}(\mathbb{R}^d)$ is a Banach space. Moreover, it satisfies the Fatou property: If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $B^s_{p,r}$, then there exists a subsequence and $u\in B^s_{p,r}$ such that $u_n\xrightarrow{n\to\infty}u$ in $\mathscr{S}'(\mathbb{R}^d)$ and $\|u\|_{B^s_{p,r}}\lesssim\liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}$.*

Proof. We use the decomposition $u_n = \chi(k)u_n + \sum_{j\geq 0}\varphi(2^{-j}k)u_n = \sum_{j\in\mathbb{Z}}(\Delta_j u_n)$. For any $j \in \mathbb{Z}$, the sequence $(\Delta_j u_n)_{n\in\mathbb{N}}$ satisfies supp $\widehat{\Delta_j u_n}$ is uniformly bounded in n, so up to a subsequence: $\Delta_j \xrightarrow{n\to\infty} \tilde{u}_j$, in $\mathscr{S}'(\mathbb{R}^d)$. We define $u = \sum_{j\in\mathbb{Z}}\tilde{u}_j$. Then $u \in B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}} \sim \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\tilde{u}_j\|^r_{L^p}\right)^{\frac{1}{r}} \lesssim \liminf_{n\to\infty}\left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\Delta_j u_n\|^r_{L^p}\right)^{\frac{1}{r}} = \liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}$$

□

16th lecture

Theorem 3.28 ($s > 0$)**.** *Let $s > 0$, then*

(1) $B^s_{p,r} = \dot{B}^s_{p,r} \cap L^p(\mathbb{R}^d)$, i.e.

$$\|u\|_{B^s_{p,r}} \sim \|u\|_{\dot{B}^s_{p,r}} + \|u\|_{L^p}\,.$$

(2) If Ω is a compact subset of $\mathbb{R}^d$, and $B^s_{p,r}(\Omega) = \{u \in B^s_{p,r}(\mathbb{R}^d) \mid \operatorname{supp} u \subset \Omega\}$ (similarly $\dot{B}^s_{p,r}(\Omega) = \{u \in \dot{B}^s_{p,r}(\mathbb{R}^d) \mid \operatorname{supp} u \subseteq \Omega\}$), then

$$B^s_{p,r}(\Omega) = \dot{B}^s_{p,r}(\Omega)\,.$$

Actually, we have $\|u\|_{L^p} \lesssim |\Omega|^{\frac{s}{d}}\|u\|_{\dot{B}^s_{p,r}}$. Consequently,

$$\|u\|_{B^s_{p,r}} \lesssim (1 + |\Omega|^{\frac{s}{d}})\|u\|_{\dot{B}^s_{p,r}}$$

Proof. (1) exercise

(2) Take $j_0 \in \mathbb{Z}$ and decompose $u = \chi(2^{-j_0}k)u + (1-\chi(2^{-j_0}k))u$, so

$$\|u\|_{L^p(\Omega)} \leq \|\chi(2^{-j_0}k)u\|_{L^p(\Omega)} + \|(1+\chi(2^{-2j_0}k))u\|_{L^p(\Omega)}$$

Easy part:

$$\|\underbrace{(1-\chi(2^{-j_0}k))u}_{\operatorname{supp}\widehat{\ldots}\subset\{|k|\gtrsim 2^{j_0}\}}\|_{L^p} \lesssim \left(\sum_{j\geq j_0}\|\varphi(2^{-j}k)u\|^r_{L^p}\right)^{\frac{1}{r}} \leq \Big(\sum_{j\geq j_0}\underbrace{2^{-j_0 sr}2^{jsr}}_{\geq 1}\|\varphi(2^{-j}k)u\|^r_{L^p}\Big)^{\frac{1}{r}}$$

$$\leq 2^{-j_0 s}\|u\|_{\dot{B}^s_{p,r}}$$

More tricky part:

$$\begin{aligned}
\|\underbrace{\chi(2^{-j_0}k)u}\|_{L^p(\Omega)} &\leq |\Omega|^{\frac{1}{p}}\|\chi(2^{-j_0}k)u\|_{L^\infty(\mathbb{R}^d)} && \text{(Hölder)}\\
&\leq |\Omega|^{\frac{1}{p}}|\Omega|^{\frac{1}{p}}2^{j_0 d(\frac{1}{1}-\frac{1}{\infty})}\|\chi(2^{-j_0}k)u\|_{L^1(\mathbb{R}^d)} && \text{(Bernstein)}\\
&\leq |\Omega|^{\frac{1}{p}}2^{j_0 d}\underbrace{\|u\|_{L^1(\mathbb{R}^d)}}_{\|u\|_{L^1(\Omega)}} && \text{(Young)}\\
&\leq |\Omega|^{\frac{1}{p}}2^{j_0 d}|\Omega|^{1-\frac{1}{p}}\|u\|_{L^p(\Omega)} \leq 2^{j_0 d}|\Omega|\|u\|_{L^p}
\end{aligned}$$

Conclusion:

$$\|u\|_{L^p(\Omega)} \leq 2^{j_0 d}|\Omega|\|u\|_{L^p} + c2^{-j_0 s}\|u\|_{\dot{B}^s_{p,r}}\,, \quad \forall j_0 \in \mathbb{Z}$$

We can choose $j_0 \in \mathbb{Z}$ such that $2^{j_0 d}|\Omega| \in \left[\frac{1}{2^{2d}}, \frac{1}{2^d}\right]$, so

$$\|u\|_{L^p(\Omega)} \leq \frac{1}{2^d}\|u\|_{L^p} + c|\Omega|^{\frac{s}{d}}\|u\|_{\dot{B}^s_{p,r}} \implies \|u\|_{L^p} \lesssim_d |\Omega|^{\frac{s}{d}}\|u\|_{B^s_{p,r}}$$

□

Theorem 3.29. *Take $s' < s \in \mathbb{R}$ and $\phi \in \mathscr{S}(\mathbb{R}^d)$. Then the mulitplication operator $\phi\colon B^s_{p,\infty} \to B^{s'}_{p,1}$ is compact.*

Recall: $\phi\colon H^s \to H^{s'}$ is compact.

Proof. Take $(u_n)_{n\in\mathbb{N}}$ a bounded sequence in $B^s_{p,\infty}$. Up to a subsequence, $u_n \to u$ in $\mathscr{S}'(\mathbb{R}^d)$ and $u \in B^s_{p,\infty}$ ($u = \sum \tilde{u}_j$, $\tilde{u}_j = \lim_{n\to\infty} \Delta_j u_n$). We need to prove that $\phi u_n \to \phi u$ strongly in $B^{s'}_{p,1}$. It suffices to consider the case where $u = 0$.

$$\|\phi u_n\|_{B^{s'}_{p,1}} = \sum_{j\in\mathbb{Z}} 2^{js'} \|\Delta_j(\phi u_n)\|_{L^p} = \sum_{j\le j_0} + \sum_{j>j_0}$$

For $j > j_0$:

$$\sum_{j>j_0} 2^{js'}\|\Delta_j(\phi u_n)\|_{L^p} = \sum_{j>j_0} 2^{-j(s-s')}2^{js}\|\Delta_j(\phi u_n)\|_{L^p} \le \underbrace{\sum_{j>j_0} 2^{-j(s-s')}}_{\sim_{ss'} 2^{-j_0(s-s')}} \cdot \underbrace{\sup_{j>j_0} 2^{js}\|\Delta_j(\phi u_n)\|_{L^p}}_{\le \|\phi u_n\|_{B^s_{p,\infty}} \lesssim \|u_n\|_{B^s_{p,\infty}} \le C<\infty}$$

Here,

$$\Delta_j(\phi u_n) = \varphi(2^{-j}k)(\phi u_n) = 2^j h(2^j\bullet) * (\phi u_n)\,,$$

so by Young:

$$\|\Delta_j(\phi u_n)\|_{L^p} \le \|\phi u_n\|_{L^p} \le \|u_n\|_{L^p}\,.$$

We need to work a bit to get $\|\Delta_j u_n\|_{L^p}$ instead of $\|u_n\|_{L^p}$ (exercise). Thus:

$$\sum_{j>j_0} 2^{js'}\|\Delta_j(\phi u_n)\|_{L^p} \le C2^{-j_0}(s-s') \xrightarrow{j_0\to\infty} 0 \text{ uniformly in } n$$

Part $j \le j_0$: We can consider each j separately. We prove that

$$\|\Delta_j(\phi u_n)\|_{L^p} \xrightarrow{n\to\infty} 0, \quad \forall j \ge -1\,.$$

For all $j \ge -1$, $\Delta_j(\phi u_n) = g * (\phi u_n)$, where $\operatorname{supp}\hat{g}$ is bounded. Consider $g*(\phi u_n)(x) = \int_{\mathbb{R}^d} g(x-y)\phi(y)u_n(y)\,\mathrm{d}y \xrightarrow{n\to\infty} 0$. For fixed x,

$$g \in \mathscr{S}(\mathbb{R}^d) \implies y \mapsto g(x-y)\pi(y) \in \mathscr{S}(\mathbb{R}^d)$$

Moreover,

$$\| \underbrace{\Delta_j(\phi u_n)}_{\operatorname{supp}\widehat{\cdots}\text{ bounded}} \|_{L^\infty} \overset{\text{Bernstein}}{\lesssim_j} \|\Delta_j(\phi u_n)\|_{L^p} \lesssim_j \|\phi u_n\|_{B^s_{p,\infty}} \lesssim \|u_n\|_{B^s_{p,\infty}} \le C < \infty$$

Consequently, $\forall R > 0$,

$$\|\Delta_j(\phi u_n)\|_{L^p(B_R)} \xrightarrow{n\to\infty} 0$$

by dominated convergence (obivious if $p < \infty$, $p = \infty$ needs a modification). To get the convergence in $L^p(\mathbb{R}^d)$, we need some decay for $x \mapsto g*(\phi u_n)$. For all $n \in \mathbb{N}$:

$$\begin{aligned}
|x|^{2m}|g*(\phi u_n)(x)| &= \left|\int_{\mathbb{R}^d} |x|^{2m} g(x-y)\phi(x)u_n(y)\,\mathrm{d}y\right| \\
&\lesssim_m \int_{\mathbb{R}^d} ((x-y)^{2m} + y^{2m})|g(x-y)||\phi(y)||u_n(y)|\,\mathrm{d}y \\
&\le \int_{\mathbb{R}^d} \underbrace{(x-y)^{2m}g(x-y)}_{z^{2n}g(z)\in\mathscr{S}} |\phi(y)u_n(y)|\,\mathrm{d}y + \int_{\mathbb{R}^d} |g(x-y)|\underbrace{|y^{2m}\phi(y)|}_{y^{2n}\phi(y)\in\mathscr{S}} u_n(y)\,\mathrm{d}y \\
&\le \|u_n\|_{B^s_{p,\infty}} \le C < \infty
\end{aligned}$$

by the same argument as before. Thus:

$$|g*(\phi u_n)(x)| \lesssim_m \frac{1}{1+|x|^{2m}} \in L^p(\mathbb{R}^d)$$

if m large enough ($m \ge d$)! $\rightsquigarrow g*(\phi u_n) \to 0$ in L^p by dominated convergence. $\square$

4 Incompressible Navier-Stokes equation

For a vector field: $u = (u_j)_{j=1}^d$, $u_j \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $u = u(x,t)$, the *Navier-Stokes equation* is

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla P \\ \operatorname{div}(u) = 0 \\ u|_{t=0} = u_0 \end{cases},$$

where $u_0 = u_0(x)$ is the initial data and $P \colon \mathbb{R}^d \to \mathbb{R}$ the pressure. The equation is considered component-wise with

$$u \cdot \nabla u = \sum_{j=1}^d u_j\, \partial_j\, u = \left(\sum_{j=1}^d u_j\, \partial_j\, u_k \right)_{k=1}^d , \quad \operatorname{div}(u) = \nabla \cdot u = \sum_{j=1}^d \partial_j\, u_j\,.$$

(1) Leray weak solution

(2) (Weak) solution in $u_0 \in \dot{H}^{\frac{d}{2}-1}$, critical space

(3) Uniqueness of solution (2D ✓, 3D ?)

(4) Solution $u_0 \in B^{-1}_{\infty,\infty}$

4.1 Energy approach

Theorem 4.1 (Leray, 1934). *There exists a weak solution in $L^\infty(\mathbb{R}_+, L^2_x) \cap L^2(\mathbb{R}_+, H^1_x)$ such that*

$$\|u(t)\|^2_{L^2_x} + 2\int_0^t \|\nabla u(s)\|^2_{L^2_x}\, \mathrm{d}s \le \|u_0\|^2_{L^2_x}$$

Remark 4.2. Energy space $L^\infty(\mathbb{R}_+, L^2_x) \cap L^2(\mathbb{R}_+, \dot{H}^1_x)$ is invariant under the scaling

$$u_\lambda(x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{in } 2D\,,$$

so we get a good theory in 2D.

Formal derivation of the energy kernel: Do inner product in L^2_x with $u(t,x)$:

$$\frac{1}{2}\,\partial_t\, \|u\|^2_{L^2} + \langle u \cdot \nabla u, u\rangle_{L^2_x} - \int_{\mathbb{R}^d} |\nabla u|^2 = -\,\langle \nabla P, u\rangle_{L^2_x}$$

$$-\int_{\mathbb{R}^d} \nabla P \cdot u\, \mathrm{d}x = \sum_{j=1}^d -\int_{\mathbb{R}^d} \partial_j\, P \cdot u_j = \sum_{j=1}^d \int P\, \partial_j\, u_j = \int P \underbrace{\operatorname{div}(u)}_{=0} = 0$$

$$\langle u \cdot \nabla u, u\rangle_{L^2} = \sum_{j,k} \int u_j(\partial_j\, u_k) u_k = \sum_{j,k} \int u_j \frac{1}{2}\, \partial_j (u_k)^2 = -\sum_{j,k} \int \partial_j\, u_j \cdot t u_k^2 = -\int (\operatorname{div} u) \cdot |u|^2 = 0$$

Integrating over time:

$$\|u(t)\|_{L^2_x} + 2\int_0^t \|\nabla u(s)\|^2_{L^2_x}\, \mathrm{d}s = \|u_0\|^2_{L^2_x}$$

7th tutorial

The Navier-Stokes equation can be written as

$$\begin{cases} \partial_t\, u - \Delta u = Q(u,u) \\ u(t=0) = u_0 \\ \operatorname{div} u = 0 \end{cases},$$

where $Q(u,v)$ is a bilinear form (cf. an inner product). We will try to use a fixed point argument to prove existence of a solution (at least local existence).

Lemma 4.3. *If Q is a bilinear form on a Banach space and*

$$\|Q\| = \sup_{\substack{\|u\|\leq 1 \\ \|v\|\leq 1}} \|Q(u,v)\| < \infty\,.$$

Then $\forall a$ such that $\|a\| \leq \alpha < \frac{1}{4\|Q\|}$, there exists a unique solution for

$$x = a + Q(x,x)\,, \quad x \in B(0,2\alpha)\,.$$

Proof. Define $x_0 = a$, $x_1 = a + Q(x_0, x_0)$, $x_{n+1} = a + Q(x_n, x_n)$, $\forall n$. We prove that $x_n \in B(0, 2\alpha)$. In fact:

- $x_0 = a \implies \|x_0\| < \alpha \implies x_0 \in B(0, 2\alpha)$
- Assume $\|x_n\| < 2\alpha$. We need to show that $\|x_{n+1}\| < 2\alpha$.

$$\|x_{n+1}\| = \|a + Q(x_n,x_n)\| \leq \|a\| + \|Q(x_n,x_n)\| < \alpha + \|Q\|\|x_n\|^2 < \alpha + \underbrace{\|Q\|(2\alpha)^2}_{<\alpha} < 2\alpha\,.$$

We want to prove that $\{x_n\}$ is a Cauchy sequence. We have

$$\begin{aligned}\|x_{n+1} - x_n\| =& \|Q(x_n,x_n) - Q(x_{n-1},x_{n-1})\| = \|Q(x_n - x_n, x_n) + Q(x_{n-1}, x_n - x_{n-1})\| \\ \leq& \|Q(x_n - x_{n-1}, x_n\| + \|Q(x_{n-1}, x_n - x_{n-1})\| \leq \|Q\|\|x_n - x_{n-1}\|(\|x_n\| + \|x_{n-1}\|) \\ \leq& \|Q\|\|x_n - x_{n-1}\|(\|x_n\| + \|x_{n-1}\|) \leq \underbrace{4\|Q\|\alpha}_{=:\beta<1}\|x_n - x_{n-1}\|\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence since

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \leq C(\beta^m + \beta^{m-1} + \cdots + \beta^n) \xrightarrow{m,n\to\infty} 0$$

$$\|x_{n+1}\| \leq \beta\|x_{n-1}\| \leq \beta^2\|x_{n-1} - x_{n-2}\| \leq \cdots \leq \beta^n C$$

Hence there exists $x = \lim_{n\to\infty} x_n$. From $x_{n+1} = a + Q(x_n,x_n) \xrightarrow{n\to\infty} x = a + Q(x,x)$. Finally, we check, that the solution in $B(0,2\alpha)$ is unique. Assume $\exists \tilde{x} = a + Q(\tilde{x},\tilde{x})$, then

$$\|x - \tilde{x}\| = \|Q(x,x) - Q(\tilde{x},\tilde{x})\| \leq \underbrace{4\|Q\|\alpha}_{<1}\|x - \tilde{x}\| \implies \|x - \tilde{x}\| \leq 0 \implies x = \tilde{x}\,.$$

□

Navier-Stokes reads:

$$\begin{cases} \partial_t\, u + \operatorname{div}(u \otimes u) - \Delta u = -\nabla P \\ u(t=0) = u_0 \\ \operatorname{div} u = 0 \end{cases},$$

where

$$\operatorname{div}(u \otimes u) = u \cdot \nabla u = \left(\sum_k \partial_k(u_j u_k)\right)_{j=1}^d = (\operatorname{div}(u_j u))_{j=1}^d\,.$$

Definition 4.4 (Weak solution of the Navier-Stokes equation). We want that u satisfies:

$$\int_0^t \int_{\mathbb{R}^d} u(s,x)\psi(s,x)\,\mathrm{d}s\,\mathrm{d}x = \int_0^t \int_{\mathbb{R}^d} (u \cdot \Delta\psi + \langle u \otimes u, \nabla\psi\rangle + u \cdot \partial_t\,\psi)\,(s,x)\,\mathrm{d}s\,\mathrm{d}x + \int_{\mathbb{R}^d} u_0(x)\psi(0,x)\,\mathrm{d}x\,,$$

$\forall t \in [0,T]$, $\forall \psi \in C([0,T], H^2(\mathbb{R}^d))$.

Leray condition: If u is a weak to solution to the Navier-Stokes equation, then

$$\|u(t)\|_{L^2_x}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2_x}^2 \, \mathrm{d}s = \|u_0\|_{L^2_x}^2$$

This implies the global existence in $d = 2$. The Leray condition makes sense, if $u \in L^\infty(\mathbb{R}_+, L^2_x) \cap L^2(\mathbb{R}_+, \dot{H}^1_x)$, but this energy space is only scaling compatible in $d = 2$.

Scaling argument: Let $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$, then

$$\partial_t\, u_\lambda(t,x) = \lambda^3\, \partial_t\, u(\lambda^2 t, \lambda x)$$

$$\Delta_x u_\lambda(t,x) = \lambda^3 \Delta_x u(\lambda^2 t, \lambda x$$

$$\underbrace{Q(u_\lambda)}_{\equiv \operatorname{div}(u\otimes u)} \sim \lambda^3 \quad \text{(this requires the factor } \lambda)$$

We want to find an energy space such that

$$\|u_\lambda\|_E = \|u\|_E\,.$$

For $d = 2$, $E = L^\infty(\mathbb{R}_+, L^2_x)$:

$$\begin{aligned}\|u_\lambda\|_{L^\infty(\mathbb{R}_+,L^2_x)} &= \sup_t \|u_\lambda\|_{L^2_x} = \sup_t \left(\int_{\mathbb{R}^2} \lambda^2 |u(\lambda^2 t, \lambda x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \sup_t \left(\int_{\mathbb{R}^d} |u(\lambda^2 t, x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \sup_t \left(\int_{\mathbb{R}^d} |u(t,x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \|u\|_{L^\infty_t L^2_x}\end{aligned}$$

Similarly:

$$\begin{aligned}\|u_\lambda\|_{L^2(\mathbb{R}_+,\dot{H}^1_x)} &= \left(\int_{\mathbb{R}_+} \|u_\lambda\|_{\dot{H}^1_x}^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^2} |\nabla_x u_\lambda(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}^2} \lambda^2\lambda^2 |\nabla u(\lambda^2 t, \lambda x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} = \|u\|_{L^2_t \dot{H}^1_x}\end{aligned}$$

In general, the following energy space is scaling invariant in $d = 2, 3$:

$$L^\infty_t L^d_x,\ L^2_t \dot{H}_x^{\frac{d}{2}},\ L^\infty_t \dot{H}_x^{\frac{d}{2}-1},\ L^4_t \dot{H}_x^{\frac{d-1}{2}} \qquad \text{(exercise)}$$

Theorem 4.5 (Local well-posedness). *If $d = 2, 3$ and $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, then there exists $T > 0$ such that the Navier-Stokes equality has a unique weak solution*

$$u \in \underbrace{L^4_t([0,T])\dot{H}_x^{\frac{d-1}{2}}}_{\substack{\uparrow \\ \text{the Banach space to do} \\ \text{the fixed point argument}}} \cap L^\infty_t([0,T])\dot{H}_x^{\frac{d}{2}-1} \cap L^2_t([0,T])\dot{H}_x^{\frac{d}{2}}\,.$$

17th lecture

We will use the fixed point argumentation on the Banach space $L^4([0,T])\dot{H}_x^{\frac{d-1}{2}}$. By the Duhamel formula, we can rewrite the equation:

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} \operatorname{div}(u\otimes u)(s)\,\mathrm{d}s$$

($\rightsquigarrow x = a + Q(x,x) \rightsquigarrow$ need $\|a\| < \frac{1}{4\|Q\|}$). Denote

$$Q(u,v) = -\int_0^t \mathrm{e}^{(t-s)\Delta} \operatorname{div}(u\otimes v)(s)\,\mathrm{d}s\,.$$

Lemma 4.6.
$$\|Q(u,v)\|_{L^4_t \dot{H}_x^{\frac{d-1}{2}}} \le C\|u\|_{L^4_T \dot{H}_x^{\frac{d-1}{2}}} \|v\|_{L^4_T \dot{H}_x^{\frac{d-1}{2}}}$$

Lemma 4.7. *If $T>0$ small and $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, then $\|\mathrm{e}^{t\Delta}u_0\|_{L^4_T H_x^{\frac{d-1}{2}}}$ is small.*

Then the conclusion that there exists a unique solution $u \in L^4_t H_x^{\frac{d-1}{2}}$ follows from the general fixed point result.
Proof of Lemma 4.6: **Step 1**: We show:

$$\|\mathrm{div}(u\otimes v)\|_{\dot{H}^{\frac{d}{2}-2}} \le C\|u\|_{\dot{H}^{\frac{d-1}{2}}} \|v\|_{\dot{H}^{\frac{d-1}{2}}}$$

For $d=2$:

$$\|\mathrm{div}(u\otimes v)\|_{\dot{H}^{-1}(\mathbb{R}^2)} \lesssim \|uv\|_{L^2(\mathbb{R}^2)} \le \|u\|_{L^4(\mathbb{R}^2)}\|v\|_{L^4(\mathbb{R}^2)} \lesssim \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}\|v\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)},$$

by the Sobolev inequality $L^q(\mathbb{R}^d) \supset \dot{H}^s(\mathbb{R}^d)$ with $q=\frac{2d}{d-2s}$ ($d=2$, $s=\frac{1}{2}$, $q=4$).
For $d=3$:
$$\|\mathrm{div}(u\otimes v)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|\nabla u\cdot v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le \|\nabla u\|_{L^2(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)}$$

Sobolev:
$$L^3(\mathbb{R}^3) \supset H^{\frac{1}{2}}(\mathbb{R}^3) \implies H^{-\frac{1}{2}} \supset L^{\frac{3}{2}}(\mathbb{R}^3)$$

NR:
$$\|\nabla u\cdot v\|^{\frac{3}{2}}_{L^{\frac{3}{2}}} = \int |\nabla u|^{\frac{3}{2}}|v|^{\frac{3}{2}} \le \left(|\nabla u|^2\right)^{\frac{3}{4}}\left(|v|^6\right)^{\frac{1}{4}} = \|\nabla u\|^2_{L^2}\|v\|^{\frac{6}{4}}_{L^6}$$

Step 2: Consider
$$\begin{cases} \partial_t v - \Delta v = f \in L^2([0,T],\dot{H}^{s-2}) \\ v(t,0) = v_0 \in \dot{H}^s(\mathbb{R}^d) \end{cases}$$

($f \sim \mathrm{div}(u\otimes v)$ in our application). Then, we claim that

$$v \in L^\infty([0,T],\dot{H}^s(\mathbb{R}^d)) \cap L^4([0,T],\dot{H}^{s+\frac{1}{2}}) \cap L^2([0,T],\dot{H}^{s+1})$$

and we get quantivative information

$$\|v(t)\|^2_{\dot{H}^s} + 2\int_0^t \|\nabla v\|^2_{\dot{H}^s}\,\mathrm{d}s = \|v_0\|_{\dot{H}^s} + \int_0^t \langle f,v\rangle_s\,\mathrm{d}s\,.$$

Duhamel gives:

$$\begin{aligned} & v = \mathrm{e}^{t\Delta}v_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}f(s)\,\mathrm{d}s \\ \iff & \hat{v}(k) = \mathrm{e}^{-t|4\pi k|^2}\widehat{v_0}(k) + \int_0^t \mathrm{e}^{-(t-s)|4\pi k|^2}\hat{f}(s,k)\,\mathrm{d}s \\ \implies & \|v\|_{H^s} \le \|\mathrm{e}^{t\Delta}v_0\|_{\dot{H}^s} + \left\|\int_0^t \mathrm{e}^{(t-s)\Delta}f(s)\,\mathrm{d}s\right\|_{\dot{H}^s} \le \|v_0\|_{\dot{H}^s} + \left\|\int_0^t (-\Delta)^{\frac{s}{2}}\mathrm{e}^{(t-s)\Delta}f(s)\,\mathrm{d}s\right\|_{L^2} \end{aligned}$$

The second term equals:

$$\begin{aligned} & \left(\int_{\mathbb{R}^d}\left|\int_0^t |2\pi k^s|\mathrm{e}^{-(t-s)|2\pi k|^2}\hat{f}(s,k)\,\mathrm{d}s\right|\mathrm{d}k\right)^{\frac{1}{2}} = c\left(\left|\int_0^t \mathrm{e}^{-(t-s)|2\pi k|^2}|k||k|^{s-1}\hat{f}(s,k)\,\mathrm{d}s\right|^2\mathrm{d}k\right)^{\frac{1}{2}} \\ \lesssim & \left(\int_{\mathbb{R}^d}\left(\int_0^t \mathrm{e}^{-(t-s)|2\pi k|^2}|k|^2\,\mathrm{d}s\right)\left(\int_0^t |\hat{f}(s,k)|^2|k|^{2(s-1)}\,\mathrm{d}s\right)\right)^{\frac{1}{2}} \le \|f\|_{L^2_t\dot{H}_x^{s-1}} \end{aligned}$$

Let us multiply the equation with $\hat{v}(k)|2\pi k|^{2s}$ and integrate over $k \in \mathbb{R}^d$:

$$\begin{aligned} &\partial_t \hat{v}(t,k) + |2\pi k|^2 \hat{v}(k,t) = \hat{f}(t,k) \\ \implies &\frac{1}{2}\partial_t \int_{\mathbb{R}^d} |2\pi k|^2 |\hat{v}|^2 + \int_{\mathbb{R}^d} |2\pi k|^{2s+2} |\hat{v}|^2 = \int_{\mathbb{R}^d} \hat{f}(t,k)\overline{\hat{v}(k)}|2\pi k|^{2s}\, \mathrm{d}k\,, \end{aligned}$$

i.e. $\frac{1}{2}\partial_t \|v\|^2_{\dot{H}^s_x} + \|v\|_{\dot{H}^{s+1}_x} = \langle v, f\rangle_{\dot{H}^s_x}$. Integration by parts yields:

$$\frac{1}{2}\left(\|v(t)\|^2_{\dot{H}^s} - \|v_0\|_{\dot{H}^s}\right) + \int_0^t \|v\|^2_{H^{s+1}_x} = \int_0^t \langle v, f\rangle_{H^s_x}\, \mathrm{d}t'$$

Consequently:

$$\int_0^T \|v(t)\|^2_{\dot{H}^{s+1}_x} \le \frac{1}{2}\|v_0\|_{\dot{H}^s} + \int_0^T \underbrace{\langle v, f\rangle_{\dot{H}^{\frac{d}{x}}}}_{\substack{\sim \int |k|^{2s}\overline{\hat{v}(k)}\hat{f}(k)\,\mathrm{d}k \\ \le \int |k|^{s+1}\hat{v}(k)|k|^{s-1}\hat{f}(k)\,\mathrm{d}k \\ \le \|v\|_{\dot{H}^{s+1}}\|f\|_{\dot{H}^{s-1}}}} \mathrm{d}t$$

$$\implies \int_0^T \langle v, f\rangle_{\dot{H}^s_x}\, \mathrm{d}t \le \int_0^T \|v(t)\|_{\dot{H}^{s+1}}\|f(t)\|_{\dot{H}^{s+1}}\, \mathrm{d}t \le \frac{1}{2}\int_0^T \|v(t)\|^2_{\dot{H}^{s+1}_x} + \frac{1}{2}\int_0^T \|f(t)\|_{\dot{H}^{s-1}_x}\, \mathrm{d}t$$

$$\implies \int_0^T \|v(t)\|_{\dot{H}^{s+1}_x} \le \|v_0\|^2_{\dot{H}^s} + \int_0^T \|f(t)\|^2_{\dot{H}^{s+1}_x}\, \mathrm{d}t$$

The fact $v \in L^4$ follows by interpolation (exercise). Let us conclude Lemma 4.6 from this: We already proved

$$\|\mathrm{div}(u\otimes v)\|_{\dot{H}^{\frac{d}{2}-2}} \lesssim \|u\|_{\dot{H}^{\frac{d-1}{2}}}\|v\|_{\dot{H}^{\frac{d-1}{2}}}$$

We now apply the previous claim with $s = \frac{d}{2}-1$, $f = \pm\mathrm{div}(u\otimes v)$ to get:

$$\begin{aligned} \|f\|_{L^2([0,T],\dot{H}^{s-1})} &= \int_0^T \|\mathrm{div}(u\otimes v)\|_{\dot{H}^{\frac{d}{2}-2}}\, \mathrm{d}t \lesssim \int_0^T \|u\|^2_{\dot{H}^{\frac{d-1}{2}}}\|v\|^2_{\dot{H}^{\frac{d-1}{2}}}\, \mathrm{d}t \\ &\le \left(\int_0^T \|u\|^4_{\dot{H}^{\frac{d-1}{2}}}\right)^{\frac{1}{2}}\left(\int_0^T \|v\|^4_{\dot{H}^{\frac{d-1}{2}}}\, \mathrm{d}t\right)^{\frac{1}{2}} = \|u\|_{L^4_t\dot{H}_x^{\frac{d-1}{2}}}\|v\|_{L^4_t\dot{H}_x^{\frac{d-1}{2}}}{}^2 \end{aligned}$$

Thus

$$\|Q(u,v)\|_{L^4_t\dot{H}^{\frac{d-1}{2}}} \lesssim \|\mathrm{div}(u\otimes v)\|_{L^2_T\dot{H}_x^{\frac{d}{2}-2}} \lesssim \|u\|_{L^4_t\dot{H}_x^{\frac{d-1}{2}}}\|v\|_{L^4_t\dot{H}^{\frac{d-1}{2}}}\,,$$

i.e. the bilinear form $Q(u,v)$ is bounded in $L^4_t\dot{H}_x^{\frac{d-1}{2}}$ (and $\|Q\|$ is independent of T).

□

Proof of Lemma 4.7:

- Easy step: Assume $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$ is small. Then by the previous proof:

$$\|\mathrm{e}^{t\Delta}u_0\|_{L^4_t\dot{H}_x^{\frac{d-1}{2}}} \lesssim \|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$$

  small.

- Additional step: for any $u_0 \in \dot{H}^{\frac{d}{2}-1}$, we decompose

$$u_0 = u_0^< + u_0^>\,, \quad \widehat{u_0^<} = \widehat{u_0}(k)\mathbb{1}_{\{|k|\le L\}}\,, \quad L \text{ large}$$

  By taking $L \to \infty$, we can make $\|u_0^>\|$ as small as we want. Easy step applies.

$$u_0^< \in \dot{H}^{\frac{d-1}{2}} \implies \|\mathrm{e}^{t\Delta}u_0\|_{\dot{H}^{\frac{d-1}{2}}} \le \|u_0\|_{\dot{H}^{\frac{d-1}{2}}} \implies \|\mathrm{e}^{t\Delta}u_0\|_{L^4_T\dot{H}^{\frac{d-1}{2}}} \text{ small, if } T>0 \text{ small}$$

□ 18th lecture

Remark 4.8. • To be safer, we should define

$$Q(u,v) = \int_0^t \mathrm{e}^{(t-s)\Delta} P\left(-\frac{1}{2}\mathrm{div}(u\otimes v) - \frac{1}{2}\mathrm{div}(v\otimes u)\right)(s)\,\mathrm{d}s\,,$$

where P is the projection onto the divergent-free function.

$$\rightsquigarrow \text{ Duhamel : } \quad u(t) = \mathrm{e}^{t\Delta}u_0 + Q(u,u)$$

$$\Longrightarrow \text{ automatically div}u = 0$$

$$P(\mathrm{div}(u\otimes u)) = \mathrm{div}(u\otimes u) + \left(\sum_{k,l=1}^d \partial_j(-\Delta)^{-1}\,\partial_k\,\partial_l(u_k u_l)\right)_{j=1}^d$$

$$\begin{aligned}\mathrm{div}(P(\mathrm{div}(u\otimes u))) &= \sum_j \partial_j \sum_k \partial_k(u_j u_k) + \sum_j \partial_j \left(\sum_{k,l=1}^d \partial_j(-\Delta)^{-1}\,\partial_k\,\partial_l(u_k u_l)\right)_{j=1}^d \\ &= \sum_{j,k} \partial_j\,\partial_k(u_j,u_k) + \underbrace{\sum_j \partial_j^2(-\Delta)^{-1}}_{=-1} \sum_{k,l} \partial_k\,\partial_l(u_k u_l) = 0\end{aligned}$$

• The fact that u is divergent-free allows us to get

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \underbrace{\|u\|_{\dot H^1}^2}_{\|\nabla u\|_{L^2}^2}\,\mathrm{d}t' = \|u_0\|_{L^2}^2$$

Multiply the equation with u:

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2\,\mathrm{d}t' = \|u_0\|_{L^2}^2 + \underbrace{2\int_0^t \langle -\mathrm{div}(u\otimes u), u\rangle_{L^2}\,\mathrm{d}t'}_{2\int_0^t \langle u\otimes u, \mathrm{div}u\rangle}$$

Two key estimates that we used:

- $\|\mathrm{div}(u\otimes v)\|_{\dot H^{\frac{d}{2}-2}(\mathbb{R}^d)} \lesssim \|u\|_{\dot H^{\frac{d-1}{2}}}\|v\|_{\dot H^{\frac{d-1}{2}}}$
- If $f\in L_t^2\dot H_x^{s-1}$, then

$$\int_0^t \mathrm{e}^{(t-s)\Delta}f(s)\,\mathrm{d}s \in L_T^\infty \dot H_x^s \cap L_T^2\dot H_x^{s+1}\cap L_T^4\dot H_x^{d+\frac{1}{2}} \qquad \text{(exercise)}$$

(use $\int_0^t \mathrm{e}^{-sk^2}k^2\,\mathrm{d}s \le \int_0^\infty \mathrm{e}^{-sk^2}k^2\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}s = 1$)

$\Longrightarrow f = -\mathrm{div}(u\otimes v)$

$\Longrightarrow \|f\|_{L_t^2\dot H} \lesssim \|u\|_{\dot H^{\frac{d-1}{2}}}\|v\|_{\dot H^{\frac{d-1}{2}}}$

$\Longrightarrow \|f\|_{L_T^2\dot H^{\frac{d}{2}-2}} = \int_0^T \|f\|_{\dot H^{\frac{d}{2}-2}}^2\,\mathrm{d}t \lesssim \int_0^T \|u\|_{\dot H^{\frac{d-1}{2}}}^2\,\mathrm{d}t \le \left(\|u\|_{\dot H^{\frac{d-1}{2}}}^4\right)^{\frac{1}{2}}\int v = \|u\|_{L^4\dot H_x^{\frac{d-1}{2}}}^2\|v\|_{L^4}^2$

$\Longrightarrow \|Q(u,v)\|_{L^4\dot H^{\frac{d-1}{2}}} \lesssim \|u\|_{L^4}\|v\|_{L^4}\|$

Why $u \in L^\infty_T \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$?

$$u(t) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}(-\mathrm{div}(u\otimes u)\,\mathrm{d}s$$

$$u_0 \in \dot{H}^{\frac{d}{1}-1} \implies \mathrm{e}^{t\Delta}u_0 \in \dot{H}^{\frac{d}{2}-1}$$

$$u \in L^4_T\dot{H}^{\frac{d-1}{2}} \implies \mathrm{div}(u\otimes u) \in L^2_T\dot{H}_x^{\frac{d}{2}-2} \implies \int_0^t \mathrm{e}^{(t-s)\Delta}\mathrm{div}(u\otimes u)\,\mathrm{d}s \in L^\infty_T\dot{H}_x^{\frac{d}{2}-1} \cap L^2_T\dot{H}_x^{\frac{d}{2}}$$

$$\begin{aligned}\|\mathrm{e}^{t\Delta}u_0\|^2_{L^2_t\dot{H}_x^{\frac{d}{2}}} &= \int_0^T \|\mathrm{e}^{t\Delta}u_0\|^2_{\dot{H}^{\frac{d}{2}}}\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{-t|2\pi k|^2}|2\pi k|^d|\widehat{u_0}(k)|^2\,\mathrm{d}k\,\mathrm{d}t \\ &= \int_0^T\int_{\mathbb{R}^d} \mathrm{e}^{-t|2\pi k|^2}|2\pi k|^2|2\pi k|^{d-2}|\hat{u}(k)|^2\,\mathrm{d}k\,\mathrm{d}t\end{aligned}$$

Theorem 4.9 (2D). *If $u_0 \in L^2(\mathbb{R}^d))$, $\mathrm{div}u_0 = 0$, then the Navier-Stokes equation has a unique global solution in $L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$. Moreover, the solution belongs to $L^\infty(\mathbb{R}_+, L^2) \cap L^2(\mathbb{R}_+, \dot{H}^1)$ and*

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(t')\|^2_{L^2}\,\mathrm{d}t' = \|u_0\|^2_{L^2}\,, \quad \forall t > 0\,.$$

Proof. Usual unique continuation! Here the local theory with $u_0 \in \dot{H}^{\frac{d}{2}-1}$ applies to give us a local solution. Then by the Leray energy identity, $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$, thus we can extend the solution. From this identity, the global solution satisfies

$$\int_0^\infty \|\nabla u\|^2_{L^2}\,\mathrm{d}t \le \|u_0\|^2_{L^2} \implies u \in L^2(\mathbb{R}_+, \dot{H}^1)\,.$$

This also implies $u \in L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^2))$. ☐

Theorem 4.10 (3D). *If $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\mathrm{div}(u_0) = 0$, and $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough, then the Navier-Stokes equation has a unique global solution.*

Open problem: Does there exists a global solution for large data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$?

Lemma 4.11. *If $\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \le \varepsilon$, then the solution to the Navier-Stokes equation satisfies:*

$$t \mapsto \|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \textit{ is decreasing}\,.$$

Proof. $\partial_t\, u - \Delta u = -\mathrm{div}(u\otimes u)$, multiply with $\sqrt{-\Delta}u$ and integrate over x and t:

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} + 2\int_0^t \|u\|^2_{\dot{H}^{\frac{3}{2}}}\,\mathrm{d}s = \|u_0\|^2_{\dot{H}^{\frac{1}{2}}} + \int_0^t \langle -\mathrm{div}(u\otimes u), u\rangle_{\dot{H}^{\frac{1}{2}}}\,\mathrm{d}s$$

By Cauchy-Schwarz inequality:

$$\begin{aligned}|\langle \mathrm{div}u\otimes u, u\rangle_{\dot{H}^{\frac{1}{2}}}| &\sim \left|\int_{\mathbb{R}^d} |k|\widehat{\mathrm{div}(u\otimes u)}(k)\hat{u}(k)\,\mathrm{d}k\right| \\ &\le \left(\int_{\mathbb{R}^d}|k|^3|\hat{u}(k)|^2\,\mathrm{d}k\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|k|^{-1}|\widehat{\mathrm{div}(u\otimes u)}(k)|^2\,\mathrm{d}k\right)^{\frac{1}{2}} \le \|u\|_{\dot{H}^{\frac{3}{2}}}\|\mathrm{div}(u\otimes u)\|_{\dot{H}^{-\frac{1}{2}}} \\ &\lesssim \|u\|_{\dot{H}^{\frac{3}{2}}}\underbrace{\|u\|^2_{\dot{H}^1}}_{\le \|u\|_{\dot{H}^{\frac{3}{2}}}\|u\|_{\dot{H}^{\frac{1}{2}}}} \le \|u\|^2_{\dot{H}^{\frac{3}{2}}}\|u\|_{\dot{H}^{\frac{1}{2}}}\end{aligned}$$

Conclusion:

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} + 2\int_0^t \|u(s)\|^2_{\dot{H}^{\frac{3}{2}}}\,\mathrm{d}s \le \|u_0\|^2_{\dot{H}^{\frac{1}{2}}} + c\int_0^t \|u(s)\|^2_{\dot{H}^{\frac{3}{2}}}\|u(s)\|_{\dot{H}^{\frac{1}{2}}}\,\mathrm{d}s$$

We have claimed that our local solution belongs to $L^\infty_T \dot{H}^{\frac{1}{2}}_x$. This can be improved to $C([0,T], H^{\frac{1}{2}}(\mathbb{R}^3))$. Consequently, $t \mapsto \|u(t)\|_{\dot{H}^{\frac{1}{2}}}$ is continuous. So if $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \varepsilon$ gives that for $T > 0$ small, $\|u(t)\|_{\dot{H}^{\frac{1}{2}}} \leq 2\,\varepsilon$, $\forall t \in [0,T]$.

$$C \int_0^t \|u(s)\|^2_{\dot{H}^{\frac{3}{2}}} \|u(s)\|_{\dot{H}^{\frac{1}{2}}} \,\mathrm{d}s \leq 2C\,\varepsilon \int_0^t \|u(s)\|^2_{\dot{H}^{\frac{3}{2}}} \,\mathrm{d}s$$

Thus if $2C\,\varepsilon < 1 \implies \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq \|u_0\|^2_{\dot{H}^{\frac{1}{2}}}$. □

19th lecture

Lemma 4.12 (3D). *Take* $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, $\mathrm{div}(u_0) = 0$. *If there exists a global solution of the Navier-stokes equation wrt. the initial data* u_0*, then:*

- $\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \xrightarrow{t\to\infty} 0$.
- $\int_0^\infty \|\nabla u(t)\|^4_{L^2} \,\mathrm{d}t < \infty$.

Remark 4.13. If $u_0 \in H^{\frac{1}{2}}$ (i.e. in $\dot{H}^{\frac{1}{2}} \cap L^2$), then the proof is easy (exercise).

Proof. Let $u(t)$ be a global solution wrt. initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. From

$$\partial_t\, u - \Delta u + \mathrm{div}(u \otimes u) = 0$$

take the inner product with u in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$:

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\int_0^t \|\nabla u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} - \int_0^t \langle \mathrm{div}(u \otimes u), u\rangle_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \,\mathrm{d}t'$$

If $\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$ is small, then the nonlinear term from the RHS can be controlled by the LHS (this is the way we prove global existence for small data). For general data, we decompose $u_0 = u_0^- + u_0+$, where

$$\begin{cases} \widehat{u_0^-}(k) = \widehat{u_0}\mathbb{1}_{|k|\leq E} \\ \widehat{u_0^+}(k) = \widehat{u_0}\mathbb{1}_{|k|>E} \end{cases}$$

By taking $E > 0$ small enough, we find that $\|u_0^-\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}$ is small:

$$\|u_0^-\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \int_{|k|\leq E} |2\pi k| |\widehat{u_0}(k)|^2 \,\mathrm{d}k \xrightarrow{E\to 0} 0\,, \quad \text{by dominated convergence}$$

and $\|u_0^+\|_{L^2}$ is finite:

$$\|u_0^+\|_{L^2} = \int_{|k|>E} |\widehat{u_0}(k)|^2 \,\mathrm{d}k \leq \int \frac{|k|}{E} |\widehat{u_0}(k)|^2 \,\mathrm{d}k < \infty$$

By the global existence for small data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, there exists a global solution $u^-(t)$ of the Navier-Stokes equation with $u^-(t=0) = u_0^-$. Thus

$$\|u^-(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \|u_0^-\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}\,.$$

Define $v(t) = u(t) - u^-(t)$. Then $v(0) = u_0^+ \in H^{\frac{1}{2}}(\mathbb{R}^3) = \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and v solves the modified equation:

$$\partial_t\, v - \Delta v = -\mathrm{div}(v \otimes v) - \mathrm{div}(v \otimes u^-) - \mathrm{div}(u^- \otimes v)$$

By mimicing Leray's identity, we get:

$$\|v(t)\|_{L^2} + 2\int_0^t \|\nabla v\|^2_{L^2} = \|v_0\|^2_{L^2} - 2\int_0^t \left\langle \mathrm{div}(v \otimes v) + \mathrm{div}(v \otimes u^-) + \mathrm{div}(u^- \otimes v), v \right\rangle_{L^2}$$

We can bound:

$$|\left\langle \operatorname{div}(u\otimes u^-), v\right\rangle|_{L^2} \le \|vu^-\|_{L^2}\|\nabla v\|_{L^2} \le \|v\|_{L^6}\|u^-\|_{L^3}\|\nabla v\|_{L^2} \lesssim \|u^-\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla v\|_{L^2}^2$$

Conclusion: From Leray's identity for v:

$$\|v\|_{L^2}^2 + 2\int_0^t \|\nabla v\|_{L^2}^2 \le \|v_0\|_{L^2}^2 + C\int_0^t \underbrace{\|u^-\|_{\dot H^{\frac12}(\mathbb{R}^3)}}_{\le \|u_0^-\|_{\dot H^{\frac12}(\mathbb{R}^3)} \text{ small}} \|\nabla v\|^2$$

$$\implies \|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 \le \|v_0\|_{L^2}^2 = \|u_0^+\|_{L^2}^2$$

Consequently, $v(t)$ is uniformly bounded in L^2 and $\|\nabla v\|_{L^2}$ is small in an average sense, i.e. $\int_0^\infty \|\nabla v(t)\|_{L^2}^2 \,\mathrm{d}t < \infty$. Thus there exists $t_0 > 0$ such that $\|\nabla v(t_0)\|_{L^2}$ is small. Therefore:

$$\|v(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \le \|v(t_0)\|_{L^2}^{\frac12}\|v(t_0)\|_{\dot H^1}^{\frac12}$$

is small. By the triangle inequality:

$$\|u(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)} = \|u^-(t_0)+v(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \le \underbrace{\|u^-\|_{\dot H^{\frac12}(\mathbb{R}^3)}}_{\le \|u_0^-\|_{\dot H^{\frac12}(\mathbb{R}^3)} \text{ small}} + \|v(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)}$$

is small.

$$\implies t\mapsto \|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \text{ is decreasing, when } t\ge t_0\,.$$

$$\implies \limsup_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \le \|u(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)}$$

Since we can make $\|u(t_0)\|_{\dot H^{\frac12}(\mathbb{R}^3)}$ as small as we want, we conclude that $\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \xrightarrow{t\to\infty} 0$. By Leray's identity for $\dot H^{\frac12}(\mathbb{R}^3)$:

$$\|v(t)\|_{\dot H^{\frac12}}^2 + 2\int_0^t \|\nabla u\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2 = \|u_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2 + 2\int_0^t \langle -\operatorname{div}(u\otimes u), u\rangle_{\dot H^{\frac12}(\mathbb{R}^3)}$$

$$\implies \int_0^\infty \|u(t)\|_{\dot H^1}^4 \,\mathrm{d}t < \infty \qquad \text{(exercise)}$$

□

Theorem 4.14 (3D). *The subset of $\mathcal{H} = \{u_0 \in \dot H^{\frac12}(\mathbb{R}^3)\,|\,\operatorname{div}(u_0) = 0\}$ such that there exists a global solution for the Navier-Stokes equation is an open subset of $\mathcal{H}$.*

Proof. Assume $u_0 \in \dot H^{\frac12}(\mathbb{R}^3)$, $\operatorname{div}(u_0) = 0$ gives a solution. Take $\varepsilon_0 \in \dot H^{\frac12}(\mathbb{R}^3)$, $\operatorname{div}\varepsilon_0 = 0$, $\|\,\varepsilon_0\,\|_{\dot H^{\frac12}(\mathbb{R}^3)}$ is small. We need to prove that the initial data $u_0+\varepsilon_0$ gives us a global solution. Let $v(t)$ be the local solution with $v(0) = u_0+\varepsilon_0$. Define

$$\varepsilon(t) := \underbrace{v(t)}_{\text{local}} - \underbrace{u(t)}_{\text{global}}\,.$$

Then $\varepsilon(0) = \varepsilon_0$ and

$$\partial_t\,\varepsilon(t) - \Delta\,\varepsilon(t) = -\operatorname{div}(v\otimes v) + \operatorname{div}(u\otimes v) = -\operatorname{div}(\varepsilon\otimes\varepsilon) - \operatorname{div}(u\otimes\varepsilon) - \operatorname{div}(\varepsilon\otimes u)$$

$$\implies \|\,\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2 + 2\int_0^t \|\nabla\,\varepsilon(t')\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2 \,\mathrm{d}t' = \|\,\varepsilon_0\,\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2 - 2\int_0^t \langle \operatorname{div}(\dots) - \operatorname{div}\dots, \varepsilon\rangle_{\dot H^{\frac12}(\mathbb{R}^3)}$$

We have

$$|\,\langle \operatorname{div}(a\otimes b), c\rangle_{\dot H^{\frac12}(\mathbb{R}^3)}\,| \le \|\operatorname{div}(a\otimes b)\|_{\dot H^{-\frac12}(\mathbb{R}^3)}\|c\|_{\dot H^{\frac32}(\mathbb{R}^3)}$$

In particular:

$$|\langle \operatorname{div}(\varepsilon\otimes\varepsilon),\varepsilon\rangle_{\dot H^{\frac12}(\mathbb{R}^3)}| \lesssim \|\varepsilon\|_{\dot H^1}^2\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)} \le \|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2$$

$$\begin{aligned}\langle \operatorname{div}(\varepsilon\otimes u),\varepsilon\rangle_{H^{\frac12}(\mathbb{R}^3)} &\lesssim \|u\|_{\dot H^1}\|\varepsilon\|_{\dot H^1}\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)} \lesssim \|u\|_{\dot H^1}\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^{\frac12}\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^{\frac32}\\ &\le C_\delta\|u\|_{\dot H^1}^4\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+\delta\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\end{aligned}$$

Conclusion:

$$\begin{aligned}&\|\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+2\int_0^t\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\\ &\le \|\varepsilon_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+C\int_0^t\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+C\int_0^t(C_\delta\|u\|_{\dot H^1}\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+\delta\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\end{aligned}$$

We take $\delta>0$ small such that $C\delta<\frac12$. If we assume $\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}\le\delta$ for all $t\in[0,T]$, then

$$\|\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+\int_0^1\|\nabla\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\le\|\varepsilon_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2+C_\delta\int_0^t\|u\|_{\dot H^1}^4\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2$$

Since $\int_0^\infty\|u\|_{\dot H^1}^4\,\mathrm{d}t<\infty$, we can apply the Gronwall lemma (Osgood):

$$\|\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\le\|\varepsilon_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}^2\exp\left(C_\delta\int_0^\infty\|u(s)\|_{\dot H^1}^4\,\mathrm{d}s\right),\quad\forall t\in[0,T].$$

This holds as long as

$$\|\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\le\delta,\quad\forall t\in[0,T].$$

We can take $\|\varepsilon_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}$ small, so $\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}$ is uniformly bounded small, i.e.

$$\|\varepsilon\|_{\dot H^{\frac12}(\mathbb{R}^3)}\le\underbrace{C\|\varepsilon_0\|_{\dot H^{\frac12}(\mathbb{R}^3)}}_{\le\delta},\quad\forall t\in[0,T].$$

Consequently,

$$\|v(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\le\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}+\|\varepsilon(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}\le C$$

uniformly in time $\implies v(t)$ is global. □

4.2 L^p-approach

20th lecture

Theorem 4.15 (3D). *If $u_0\in L^3(\mathbb{R}^3)$, $\operatorname{div}(u_0)=0$, then there exists $T>0$ and a unique local solution $u(t)\in C([0,T],L^3(\mathbb{R}^3))$ of the Navier-Stokes equation. Moreover, if $\|u_0\|_{L^3}$ is small, then there exists a global solution.*

Remark 4.16. We will use the abstract fixed point argument for the equation

$$u(t)=\mathrm{e}^{t\Delta}u_0+\int_0^t\mathrm{e}^{(t-s)\Delta}(-\operatorname{div}(u\otimes u)(s))\,\mathrm{d}s.$$

The point here is that the fixed-point method does not work on $L^\infty L^3$, but we need to use the Besov space $\dot B_{p,\infty}^{\frac3p-1}(\mathbb{R}^3)$ for some $p>3$.

Definition 4.17 (Kato spaces). For $p\in[3,\infty]$ define

$$K_p(T)=\left\{u\in C((0,T],L^p(\mathbb{R}^3))\,\middle|\,\sup_{t\in(0,T]}t^{\frac12(1-\frac3p)}\|u(t)\|_{L^p(\mathbb{R}^3)}<\infty\right\}.$$

If $p\in[1,3]$, we can also define $K_p(T)$ similarly except $(0,T]\rightsquigarrow[0,T]$.

Let us consider the $e^{t\Delta}u_0$ with $u_0 \in L^3(\mathbb{R}^3)$. We claim:

Lemma 4.18. • *$\|e^{t\Delta}u_0\|_{K_p(T)} \lesssim \|u_0\|_{L^3}$ uniformly in T.*

• *Moreover, $\|e^{t\Delta}u_0\|_{K_p(T)} \xrightarrow{T\to 0} 0$, $\forall p > 3$.*

Proof. By definition:

$$e^{t\Delta}u_0(x) = \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4t}} * u_0(x)$$
$$\implies \|e^{t\Delta}u_0\|_{L^p_x} \lesssim \frac{1}{t^{\frac{3}{2}}} \|e^{-\frac{|x|^2}{4t}} * u_0\|_{L^p_x} \leq \frac{1}{t^{\frac{3}{2}}} \|e^{-\frac{|x|^2}{4t}}\|_{L^s} \|u_0\|_{L^3}$$

Here

$$\|e^{-\frac{|x|^2}{4t}}\|_{L^s} = \left(\int_{\mathbb{R}^3} e^{-\frac{s|x|^2}{4t}}\, dx\right)^{\frac{1}{s}} = \left(\int_{\mathbb{R}^3} e^{-\frac{s|y|^2}{4}}\, dx t^{\frac{3}{2}}\right)^{\frac{1}{s}} \sim t^{\frac{3}{2s}} = t^{1+\frac{3}{2p}},$$

so

$$\|e^{t\Delta}u_0\|_{L^p_x} \lesssim t^{-\frac{3}{2}} t^{1+\frac{3}{2p}} \|u_0\|_{L^3} = t^{\frac{1}{2}(\frac{3}{p}-1)} \|u_0\|_{L^3}, \quad \forall t > 0.$$

To get $\|e^{t\Delta}u_0\|_{K_p(T)} \xrightarrow{T\to\infty} 0$, $p > 3$, we first assume $u_0 \in C_c^\infty$. Then we can apply the previous proof to show that

$$\|e^{t\Delta}u_0\|_{K_p(T)} \xrightarrow{T\to\infty} 0. \qquad \text{(exercise)}$$

For general $u_0 \in L^3$, there exists $(u_n)_n \subseteq C_c^\infty$ such that $u_n \to u_0$ in L^3. Then

$$\|e^{t\Delta}u_0\|_{K_p(T)} \leq \|e^{t\Delta}(u_0 - u_n)\|_{K_p(T)} + \|e^{T\Delta}u_n\|_{K_p(T)} \leq C\|u_0 - u_n)\|_{L^3} + \|e^{t\Delta}u_n\|_{K_p(T)}$$
$$\implies \limsup_{T\to 0} \|e^{t\Delta}u_0\|_{K_p(T)} \leq C\|u_0 - u_n\|_{L^3}$$

Taking $n \to \infty$, we conclude that $\|e^{t\Delta}u_0\|_{K_p(T)} \xrightarrow{T\to 0} 0$. □

Lemma 4.19. *Denote*

$$B(f) = \int_0^t e^{(t-s)\Delta} \mathrm{div}(f(s))\, ds \quad (f = u \otimes v \text{ in application}).$$

Then

$$B(f)(t,x) = \int_0^t \Gamma(t-s,\bullet) *_x f(s,\bullet)\, ds = \int_0^t \int_{\mathbb{R}^3} \Gamma(t-s,y) f(s,x-y)\, dy\, ds,$$

where

$$|\Gamma(t,x)| \leq \frac{C}{t^2 + x^4}.$$

Proof. Using the Fourier transform:

$$\widehat{B(f)} = \int_0^t e^{-(t-s)|2\pi k|^2} k \cdot \hat{f}(s,k)\, ds$$

Define

$$\hat{\Gamma}(t,k) = e^{-t|2\pi k|^2} k,$$

then

$$\widehat{B(f)}(t,k) = \int_0^t \hat{\Gamma}(t-s,k) \cdot \hat{f}(s,k)\, ds = \int_0^t \widehat{\Gamma(t-s,\bullet) *_x f(s,\bullet)}\, ds.$$

To get the bound for $\Gamma(t,x)$, we use

$$\Gamma(t,x) = \int_{\mathbb{R}^3} e^{-t|2\pi k|^2} k e^{2\pi i k\cdot x}\, dk = \frac{1}{2\pi i} \partial_x \int_{\mathbb{R}^3} e^{-t|2\pi k|^2} e^{2\pi i k\cdot x}\, dk$$

From the heat kernel:

$$\widehat{\frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{3}{2}}}} = \mathrm{e}^{-t|2\pi k|^2}$$

$$\implies \Gamma(t,x) = \frac{\partial_x}{2\pi \mathrm{i}} \frac{\mathrm{e}^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{3}{2}}}$$

$$\implies |\Gamma(t,x)| \lesssim \frac{|x|}{t^{\frac{5}{2}}} \mathrm{e}^{-\frac{|x|^2}{4t}}$$

Using that $\mathrm{e}^{-\xi} \leq \xi^{-\alpha}$ for $\xi \geq 0$, $\alpha \geq 1$, we get

$$\text{RHS} \leq \frac{|x|}{t^{\frac{5}{2}}} \cdot \left(\frac{|x|^2}{t}\right)^{-\frac{1}{2}} = \frac{1}{t^2}$$

Also

$$\text{RHS} \leq \frac{|x|}{t^{\frac{5}{2}}} \cdot \left(\frac{|x|^2}{t}\right)^{-\frac{5}{2}} = \frac{1}{|x|^4},$$

so

$$|\Gamma(t,x)| \lesssim \min(\frac{1}{t^2}, \frac{1}{|x|^4}) \leq \frac{C}{t^2 + |x|^4}.$$

□

Lemma 4.20. *Define*

$$Q(u,v) = -\int_0^t \mathrm{e}^{(t-s)\Delta} \mathrm{div}(u \otimes v) \, \mathrm{d}s.$$

Take $1 \geq \frac{1}{p} + \frac{1}{q} > 0$, $\frac{1}{r} + \frac{1}{3} > \frac{1}{p} + \frac{1}{q} \geq \frac{1}{r}$. *Then,* $\forall T > 0$*:*

$$\|Q(u,v)\|_{K_r(T)} \leq C \|u\|_{K_p(T)} \|v\|_{K_q(T)}$$

Proof.

$$Q(u,v) = \int_0^t \int_{\mathbb{R}^3} \Gamma(t-s,y)(u \otimes v)(x-y) \, \mathrm{d}y \, \mathrm{d}s$$

for some

$$|\Gamma(t,x)| \lesssim \min\left(\frac{1}{t^2}, \frac{1}{|x|^4}\right)$$

$$\implies \|Q(u,v)\|_{L^r(\mathbb{R}^3)} \leq \int_0^t \|\Gamma(t-s,\bullet) * (u \otimes v)\|_{L^r} \, \mathrm{d}s \int_0^t \|\Gamma(t-s,\bullet)\|_{L^\eta_x} \|u\|_{L^p_x} \|v\|_{L^q_x} \, \mathrm{d}s$$

Using

$$|\Gamma(t,x)| \leq \frac{C}{t^2 + |x|^4} \implies \|\Gamma(t,x)\|^\eta_{L_x} \leq \left(\frac{C}{(t^2+|x|^4)^\eta} \, \mathrm{d}x\right)^{\frac{1}{\eta}} \sim \left(\frac{t^{\frac{d}{2}} \, \mathrm{d}y}{t^{2\eta}(1+|y|^4)}\right)^{\frac{1}{\eta}} = t^{\frac{3}{2q}-2}$$

TODO □

Proof of Theorem 4.15: Let $u_0 \in L^3(\mathbb{R}^3)$, consider $u(t) = \mathrm{e}^{t\Delta} u_0 + Q(u,u)$. We use the fixed point argument on the Banach space $K_p(T)$, $p > 3$. For $T > 0$ small, $\|\mathrm{e}^{t\Delta} u_0\|_{K_p(T)}$ is small. Moreover

$$\|Q(u,u)\|_{K_r(T)} \lesssim \|u\|_{K_p(T)}, \text{ if } r \geq \frac{p}{2}, \; p > 3$$

We can take $r = p > 3$, then $\|Q\| \leq C$ in $K_p(T)$ and $T > 0$ small: $\|\mathrm{e}^{t\Delta} u_0\|_{K_p(T)} \leq \frac{1}{4\|Q\|}$. Then Lemma 4.3 gives the existence and uniqueness of a solution in $K_p(T)$ with $\|u(t)\|_{K_p(T)} \leq 2\|\mathrm{e}^{t\Delta} u_0\|_{K_p(T)}$. Trick:

$$u(t) - \underbrace{\mathrm{e}^{t\Delta} u_0}_{\text{b.d. in } L^3} = Q(u,u)$$

Hence

$$\|u(t)-\mathrm{e}^{t\Delta}u_0\|_{L^\infty_T L^3_x} \le \|Q(u,u)\|_{K_3(T)} \le C\|u\|^2_{K_6(T)} \le C\|\mathrm{e}^{t\Delta}u_0\|^2_{K_6(T)}\,,$$

which is small if $T>0$ is small. Thus $\|u(t)\|_{L^3_x}$ is bounded small for all $t<T$, if $T>0$ and $\|u_0\|_{L^3}$ are small.

□ 8th tutorial

In general, the Navier-Stokes equation reads

$$\begin{cases} \partial_t\, u - \Delta u + \operatorname{div}(u\otimes u) = \nabla F \\ \operatorname{div} u = 0 \\ u(t=0)=u_0 \end{cases}.$$

So far, we only considered the case $F=0$. For $F\neq 0$, the same result applies. Duhamel gives

$$u(t) = \mathrm{e}^{t\Delta}u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}(-\operatorname{div}(u\otimes u)(s) + \nabla F(s))\,\mathrm{d}s = g + \int_0^t \mathrm{e}^{(t-s)\Delta}(-\operatorname{div}(u\otimes u)(s))\,\mathrm{d}s\,.$$

We are looking for a fixed point $x = a + Q(x,x)$. Therefore, we need $\|a\|$ small, then $\|\mathrm{e}^{t\Delta}u_0\| + \|\int_0^t \mathrm{e}^{(t-s)\Delta}\nabla F(s)\,\mathrm{d}s\|$ is small $\rightsquigarrow$ easy! For the global existence: Leray equality:

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(s)\|^2_{L^2}\,\mathrm{d}s = \|u_0\|^2_{L^2} + 2\int_0^t \underbrace{\langle \nabla F, u\rangle_{L^2}}_{-\langle F,\operatorname{div} u\rangle = 0}\,\mathrm{d}s$$

The same applies if we consider $\|u(t)\|_{H^s_x}$:

$$\langle \nabla F, u\rangle_{H^s} = \int_{\mathbb{R}^d} (2\pi \mathrm{i}k)\hat F(k)\cdot \hat u(k)|2\pi k|^{2s}\,\mathrm{d}k \sim \int_{\mathbb{R}^d} \overline{\hat F(k)}\underbrace{k\cdot \hat u(k)}_{=0}\,\mathrm{d}k$$

By Theorem 4.15: If $u_0\in L^3(\mathbb{R}^3)$, there exists $T>0$ and a local solution in $K_p(T)$. We claim that for $T=\infty$, $p>3$:

$$\|\mathrm{e}^{t\Delta}u_0\|_{K_p(\infty)} \sim \|u_0\|_{\dot B^{\frac{3}{p}-1}_{p,\infty}}$$

Remember the characterization of the Besov spaces (Theorem 3.24: For $p,r\in[1,\infty]$, $s>0$:

$$\|u\|_{\dot B^{-s}_{p,r}} \sim \|t^{\frac{s}{2}}\|\mathrm{e}^{t\Delta}u\|_{L^p_x}\|_{L^r(\mathbb{R}_+,\frac{\mathrm{d}x}{t})}$$

In particular, for $r=\infty$:

$$\|u\|^{-s}_{\dot B_{p,\infty}} \sim \sup_{t>0} t^{\frac{s}{2}}\|\mathrm{e}^{t\Delta}u\|_{L^p_x}$$

and for $s = 1-\frac{3}{p}>0$, $p>3$:

$$\|u_0\|_{\dot B^{\frac{3}{p}-1}_{p,\infty}} \sim \sup_{t>0} t^{\frac{1}{2}(1-\frac{3}{p})}\|\mathrm{e}^{t\Delta}u_0\|_{L^p} = \|\mathrm{e}^{t\Delta}u_0\|_{K_p(\infty)}$$

Theorem 4.21. *If $u_0\in\mathscr{S}'_h$ and $T>0$ such that $\|\mathrm{e}^{t\Delta}u_0\|_{K_p(T)}$ is small enough, then there exists a local solution to the Navier-Stokes equation.*

21st lecture

The best we can hope for is a solution with initial data $u_0\in\dot B^{-1}_{\infty,\infty}$. If we want to solve the Navier-Stokes equation by our fixed point argument, we cannot go beyond this space. This is because the Navier-Stokes equation is invariant under the scaling $u\mapsto u_\lambda(t,x) = u(\lambda^2 t,\lambda x)$. If we do the fixed point argument on a space $B\subseteq\mathscr{S}'(\mathbb{R}^3)$, its norm should be invariant under that scaling. If also $\|Q(u,v)\|_B\le C\|u\|_B\|v\|_B$, then $B\subseteq\dot B^{-1}_{\infty,\infty}$. Here

$$Q(u,v) = \int_0^t \mathrm{e}^{(t-s)\Delta}(-\operatorname{div}(u\otimes v)(s))\,\mathrm{d}s\,,$$

then Navier-Stokes reads:

$$u(t) = \mathrm{e}^{t\Delta}u_0 + Q(u,u)$$

We need to make sense for $Q(\mathrm{e}^{t\Delta}u_0,\mathrm{e}^{t\Delta}u_0)\rightsquigarrow \mathrm{e}^{t\Delta}u_0\in L^2_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}^3)$.

Definition 4.22. Define $X_0 \subseteq \mathscr{S}'(\mathbb{R}^d)$.

$$\|u_0(x)\|_{X_0} = \underbrace{\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}}}_{\sim \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} u_0\|_{L^\infty_x}} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} \left(R^{-3} \int_{P_{x,R}} |e^{t\Delta} u_0(y)|^2 \,dy\,dt \right)^{\frac{1}{2}} < \infty \,,$$

with $P_{x,R} = (0, R^2) \times B(x,R) \subseteq \mathbb{R}_+ \times \mathbb{R}^3$. Also

$$\|u(x,t)\|_X = \sup_{t>0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty_x} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} \left(R^{-3} \int_{P_{x,R}} |u(t,y)|^2 \,dy\,dt \right)^{\frac{1}{2}} < \infty$$

X_0 is called BMO^{-1} space (dual of BMO).

Remark 4.23. $\|e^{t\Delta} u_0\|_X \sim \|u_0\|_{X_0}$

Theorem 4.24. *If $u_0 \in X_0$ and $\|u_0\|_{X_0} \sim \|e^{t\Delta} u_0\|$ is small, then there exists a global solution $u \in X$ of the Navier-Stokes equation and $\|u\|_X \leq 2\|u_0\|_{X_0}$.*

Remark 4.25. $X_0 \supset \dot{B}^{\frac{3}{p}-1}_{p,\infty}$, $\forall 3 < p < \infty$.

Proof. Let $u_0 \in X_0$. We need to show if $u_0 \in \dot{B}^{\frac{3}{p}-1}_{p,\infty}$, i.e. $e^{t\Delta} u_0 \in K_p(\infty)$, then $u_0 \in X_0$. Assume $x = 0$. Then

$$\begin{aligned} R^{-3} \int_{0,R} |e^{t\Delta} u_0(y)|^2 \,dy\,dt &\leq R^{-3} \int_0^{R^2} dt \left(\int_{B_R} |e^{t\Delta} u_0(t)|^p \,dy \right)^{\frac{2}{p}} \left(\int_{B_R} 1 \,dy \right)^{1-\frac{2}{p}} \\ &\leq R^{-3} \int_0^{R^2} dt \|e^{t\Delta} u_0\|^2_{L^p} R^{3(1-\frac{2}{p})} \leq R^{-\frac{6}{T}} \left(\int_0^{R^2} dt t^{\frac{3}{p}-1} \right) \sup_{t>0} \left(t^{1-\frac{3}{p}} \|e^{t\Delta} u_0\|^2_{L^p} \right) \\ &\sim_p \underbrace{R^{-\frac{6}{p}} \cdot t^{\frac{2}{p}} \Big|_{t=R^2}}_{=1} \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}} \end{aligned}$$

□

We will prove the theorem by a fixed point argument on X.

Lemma 4.26.

$$\|Q(u,v)\|_X \leq C \|u\|_X \|v\|_X \,,$$

where

$$Q(u,v) = \int_0^t e^{(t-s)\Delta} (-\mathrm{div}(u \otimes v)(s)) \,ds \,.$$

Definition 4.27 (Technical space).

$$\|f\|_Y = \sup_{t>0} t \|f(t)\|_{L^\infty_x} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} R^{-3} \int_{P_{x,R}} |f(t,y)| \,dy\,dt$$

Lemma 4.28 (Key Lemma).

$$\mathcal{L}f = \int_0^t e^{(t-s)\Delta} \mathrm{div}(f(s,\bullet)) \,ds \implies \|Lf\|_X \leq \|f\|_Y$$

Remark 4.29. (1) The space Y is stable with scaling in Fourier: If $\theta \in \mathscr{S}(\mathbb{R}^3)$, then

$$\|\theta(t^{\frac{1}{2}} k) f(t,x)\|_Y \lesssim \|f\|_Y$$

(2) $\|ab\|_Y \lesssim \|a\|_X \|b\|_X$

Proof of Remark 4.29:

(2)

$$\|ab\|_Y = \sup_{t>0} t\|ab\|_{L^\infty_x} + \sup_{\substack{x\in\mathbb{R}^3\\R>0}} R^{-3}\int_{P_{x,R}} |(ab)(t,y)|\,\mathrm{d}y\,\mathrm{d}t$$

$$\leq \sup_{t>0}(t^{\frac12}\|a\|_{L^\infty})(t^{\frac12}\|b\|_{L^\infty}) + \sup_{\substack{x\in\mathbb{R}^3\\R>0}} \left(R^{-3}\int_{P_{x,R}}|a|^2\right)^{\frac12}\left(R^{-3}\int_{P_{x,R}}|b|^2\right)^{\frac12} \leq \|a\|_X\|b\|_X$$

(1) $\theta \in \mathscr{S}(\mathbb{R}^3)$

$$\theta(t^{\frac12}k)f(t,x) \sim t-\frac32\int_{\mathbb{R}^3}\check\theta(t^{-\frac12}(x-y))f(t,y)\,\mathrm{d}y$$

$$\Longrightarrow \|\theta(t^{\frac12}k)f(t,x)\|_{L^\infty_x} \overset{\substack{\text{Young}\\\downarrow}}{\leq} \|t^{-\frac32}\check\theta(t^{-\frac12}\bullet)\|_{L^1_x}\|f(t,\bullet)\|_{L^\infty_x} \lesssim \|f(t)\|_{L^\infty_x}$$

Moreover, for $|x|\leq R$:

$$\|\theta(t^{\frac12}k)f(t,x)\|_{L^1(P_{x,R})} \leq \|\theta(t^{\frac12}k)\mathbb{1}_{B_{2R}}f\|_{L^1(P_{x,R})} + \|\theta(t^{\frac12}k)\mathbb{1}_{B_{2R}^C}f\|_{L^1(P_{x,R})}$$

For $\mathbb{1}_{B_R}$

$$\|\check\theta(t^{\frac12}k)\mathbb{1}_{B_{2R}}f\|_{L^1(P_{x,R}} \leq \int_{P_{x,R}}\int_{\mathbb{R}^3}\underbrace{t^{-\frac32}|\check\theta(t^{-\frac12}(zy))|}_{\int \mathrm{d}d\leq C}\mathbb{1}_{B_{2R}}(y)|f(t,y)|\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

$$\leq \int_0^{R^3}\int \mathbb{1}_{B_{2R}}(y)|f(t,y)|\,\mathrm{d}y\,\mathrm{d}t \leq \int_{P_{0,3R}}|f(t,y)|\,\mathrm{d}y\,\mathrm{d}t = \|f\|_{L^1(P_{0,2R})}$$

For $\mathbb{1}_{B_R^C}$:

$$\|\theta(t^{\frac12}k)\mathbb{1}_{B_{2R}}f\|_{L^1(P_{x,R})} \leq \int_{P_{x,R}}\int_{\mathbb{R}^3} t^{-\frac32}|\check\theta(t^{-\frac12}(z-y))|\mathbb{1}_{B_{3R}^C}(y)\underbrace{|f(t,y)|}_{\leq\|f(t)\|_{L^\infty}}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

$$\lesssim \int_{P_{x,R}}\int_{\mathbb{R}^3} t^{-\frac32}\frac{\|f(t)\|_{L^\infty}}{1+t^{-2}|z-y|^4}\mathbb{1}_{|1-z|>R}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t$$

$$\lesssim \int_{P_{x,R}}\left(\int_{\mathbb{R}^3}\frac{t^{-\frac32}\|f(t)\|_{L^\infty}}{(1+t^{-2}|y|^4)^2}\mathbb{1}_{|y|>R}\,\mathrm{d}y\right)\mathrm{d}z\,\mathrm{d}t$$

$$\lesssim \int_0^{R^2}\mathrm{d}tR^3\int_{|\tilde y|>t^{\frac12}R}\frac{\|f(t)\|_{L^\infty}}{1+|y|^4}\,\mathrm{d}\tilde y\,\mathrm{d}t$$

$$\leq (\sup_{t>0} t\|f(t)\|_{L^\infty})\int_0^{R^2}\mathrm{d}t\frac1t\int_{|\tilde y|>t^{\frac12}R}\frac{1}{1+|\tilde y|^4}\,\mathrm{d}y$$

□ 22nd lecture

Proof of Theorem 4.24: In the proof, the key point is to use the fixed point argument in X, with

$$Q(u,v) = \int_0^t \mathrm{e}^{(t-s)\Delta}(-\mathrm{div}(u\otimes v))(s)\,\mathrm{d}s\,.$$

Recall:

$$\mathcal{L}f = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathrm{div}(f(s,\bullet))\,\mathrm{d}s(x) = \int_0^t\int_{\mathbb{R}^3}\Gamma(t-s,x-y)f(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

$$|\Gamma(t,x)| \lesssim \frac{1}{t^2+x^4} \rightsquigarrow Q(u,v) = \mathcal{L}(-u\otimes v))$$

Technical space:

$$\|f\|_Y = \sup_{t>0} t\|f(t)\|_{L^\infty_x} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} R^{-3}\int_{P_{x,R}} |f(t,y)|\,\mathrm{d}y\,\mathrm{d}t$$

Conclusion of the theorem:

$$\|Q(u,v)\|_X = \|\mathcal{L}(u\otimes v)\|_X \overset{\substack{\text{Key Lemma}\\ \downarrow}}{\lesssim} \|u\otimes v\|_Y \lesssim \|u\|_X\|v\|_X$$

$\rightsquigarrow Q$ is a bounded bilinear map $\rightsquigarrow x = a + Q(x,x)$ is solvable if $\|a\|_X < \frac{1}{4\|Q\|}$ and $\|x\|_X \leq 2\|a\|$.

Lemma 4.30. *Define* $\Gamma^{(1)}(t,x) = \Gamma(t,x)\mathbb{1}_{B_R^C}(x)$, $\Gamma^{(2)} = \Gamma(t,x)\mathbb{1}_{B_R}(x)$, *then*

$$\mathcal{L}f(\Gamma * f)(t,x) = \int_0^t\int_{\mathbb{R}^3} \Gamma(t-s,x-y)f(s,y)\,\mathrm{d}y\,\mathrm{d}s = (\Gamma^{(1)}*f + \Gamma^{(2)}*f)(t,x)\,.$$

Then:

$$\|\Gamma^{(1)}*f\|_{L^\infty((0,R^2)\times\mathbb{R}^3)} \lesssim \frac{t}{R^3}\|f\|_Y$$

and if $t \geq R^2$*:*

$$\|\Gamma^{(2)}*f\|_{L^\infty((R^2,\infty)\times R^2)} \lesssim \frac{1}{R}\|f\|_Y$$

Consequently, if $R = \sqrt{t}$*, then*

$$\|\Gamma^{(1)/(2)}*f\|_{L^\infty_x} \lesssim \frac{1}{\sqrt{t}}\|f\|_Y$$

$$\implies \sup_{t>0}\sqrt{t}\|\mathcal{L}f(t)\|_{L^\infty_x} \leq \sqrt{t}\left(\|\Gamma^{(1)}*f\|_{L^\infty_x} + \|\Gamma^{(2)}*f\|_{L^\infty_x}\right)$$

$$\implies \sup_{t>0} t^{\frac{1}{2}}\|\mathcal{L}f(t)\|_{L^\infty_x} \lesssim \|f\|_Y$$

Proof.

$$\begin{aligned}
|(\Gamma^{(1)}*f)(t,x)| &\leq \int_0^t\int_{\mathbb{R}^3} |\Gamma^{(1)}(t-s,x-y)||f(s,y)|\,\mathrm{d}y\,\mathrm{d}s \leq \int_0^t\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_R^C}(x-y)}{|t-s|^2+|x-y|^4}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s\\
&\leq t\int_{|x-y|>R}\frac{1}{|x-y|^4}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s \leq \int_0^t\sum_{l=1}^\infty\int_{2^{l+1}R\geq|x-y|>2^lR}\frac{1}{|x-y|^4}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s\\
&\lesssim \int_0^{R^2}\sum_{l=1}^\infty\frac{1}{(2^lR)^4}\int_{|x-y|<2^{l+1}R}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s\\
&\leq \sum_{l=1}^\infty\frac{1}{2^lR}\underbrace{\left(\frac{1}{(2^lR)^3}\int_0^{R^2}\int_{B_{x,2^{l+1}R}}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s\right)}_{\leq\|f\|_Y} \lesssim \frac{1}{R}\|f\|_Y
\end{aligned}$$

On the other hand:

$$|(\Gamma^{(2)}*f)(t,x)| \lesssim \int_0^t\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_R}(x-y)}{|t-s|^2+|x-y|^4}|f(s,y)|\,\mathrm{d}y\,\mathrm{d}s \lesssim \int_0^t\frac{\mathbb{1}_{B_R}(x-y)}{|t-s|^2+|x-y|^4}\frac{1}{s}\,\mathrm{d}y\,\mathrm{d}s$$

Let us divide the time integration over s into $s\in[0,\frac{R^2}{2}]$ and $\frac{R^2}{2},t]$. The second part is good:

$$\int_{\frac{R^2}{2}}^t\frac{\mathbb{1}_{B_R}(x-y)}{|t-s|^2+|x-y|^4}\frac{1}{s}\,\mathrm{d}y\,\mathrm{d}s \lesssim R^{-2}\int_0^\infty\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_R}(x-y)}{s^2+|x-y|^4}\,\mathrm{d}s\,\mathrm{d}y \lesssim \frac{1}{R^2}\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_R(x-y)}}{|x-y|^2}\,\mathrm{d}y \lesssim \frac{1}{R}$$

For the other one:

$$R^{-3}\int_{P_{x,R}} |\Gamma * f(t,y)|^2 \,\mathrm{d}t\,\mathrm{d}y \lesssim \|f\|_Y^2$$

For $\Gamma^{(1)}$ easy:

$$R^{-3}\int_0^{R^2}\int_{B_{x,R}} |\Gamma^{(1)} * f(t,y)|^2 \,\mathrm{d}t\,\mathrm{d}y \le R^{-3}\int_0^{R^2}\int_{B_{x,R}} \frac{1}{R^2}\|f\|_Y^2 \,\mathrm{d}y\,\mathrm{d}s \sim \|f\|_Y^2$$

The case $\Gamma^{(2)}$ is difficult: The previous bound does not apply! Problem:

$$R^{-3}\int_0^{R^2}\int_{B_{x,R}} \left|\int_0^t\int_{\mathbb{R}^3} \frac{\mathbb{1}_{B_R}(y-z}{|t-s|^2+|y-z|^4}|f(s,z)|\,\mathrm{d}s\,\mathrm{d}z\right|^2 \mathrm{d}y\,\mathrm{d}t \lesssim \|f\|_Y^2$$

By translation and rescaling, we may assume $x=0$ and $R=1$. Decompose 23rd lecture

$$f = f^+ + f^-\,,$$

where

$$\widehat{f^+}(t,\xi) = \theta(t^{\frac12}\xi)\hat{f}(t,\xi)$$

for $\theta \in C_c^\infty$ with $\theta(0)=1$.
First try:

$$\|f^-\|_{L^2(P_{0,1})}^2 = \int_0^t\int_{\mathbb{R}^3}\int |1-\theta(t^{\frac12}\xi)|^2|\hat{f}(t,\xi)|^2 \,\mathrm{d}t\,\mathrm{d}\xi \lesssim \int_0^1\int_{\mathbb{R}^3} t|\xi|^2|\hat{f}(t,\xi)|^2\,\mathrm{d}t\,\mathrm{d}\xi?$$

$$|f^-(t,x)| \le \int_{\mathbb{R}^3} |1-\theta(t^{\frac12\xi)}|?$$

Note:

$$\|\mathcal{L}(\mathbb{1}_{B_2^C}f)\|_{L^\infty((0,1)\times B_1)} \lesssim \|f\|_Y$$

$$\|t^{\frac32}\theta(t^{\frac12}\xi)f\|_Y \lesssim \|f\|_Y$$

follows the same estimate for $\Gamma^{(1)} * f$ before. It remains to consider $\mathbb{1}_{B_2}f$, i.e. we can assume that $f(t,\bullet)$ is compactly supported in $y \in B_2$.

TODO?

$$|\mathcal{L}f(t,x)| = |\Gamma * f(t,x)| \lesssim \left|\int_0^t\int_{\mathbb{R}^3}\frac{1}{|t-s|^2+|x-y|^4}f(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|, \quad \operatorname{supp} f \subseteq B_2$$

$$\lesssim \int_0^t\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_2}(y)}{|t-s|^2+|y|^4}\underbrace{|f(s,y)|}_{\le\sqrt{|f(s,y)|}\|f\|_Y^{\frac12}}\,\mathrm{d}s\,\mathrm{d}y$$

$$\le \|f\|_Y^{\frac12}\left(\int_0^t\int_{\mathbb{R}^3}\frac{\mathbb{1}_{B_2}}{}\right)$$

□