LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Winter Semester 2023-2024

# Fourier Analysis and Nonlinear PDE

**Final Exam**

Family name: . . . . . . . . . . . . . . . . . . . First name: . . . . . . . . . . . . . . . . . . . .

You have **180 minutes** to solve **5 problems**. Electronic devices are not allowed.

**Problem 1.** (15 points) Define the function $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=e^x$ if $x<0$, $f(x)=1$ if $0\le x\le 1$, and $f(x)=x^2$ if $x\ge 1$. Prove that $f\in\mathcal{D}'(\mathbb{R})$ and compute the distributional derivative of $f$. Is $f$ a tempered distribution?

**Problem 2.** (5+5+10 points) Define $f(x)=x^{-1}|\ln|x||^{-\beta}$ for $x\in\mathbb{R}\backslash\{0\}$.

(a) Show that there exists $\beta>0$ such that $f\notin L^1_{\mathrm{loc}}(\mathbb{R})$.

(b) Show that for every $\varphi\in\mathcal{S}(\mathbb{R})$ we have

$$|\varphi(x)-\varphi(-x)|\le\frac{2|x|}{1+|x|^2}C_\varphi,\quad\forall x\in\mathbb{R},\quad C_\varphi=\sup_{y\in\mathbb{R}}(1+|y|^2)(|\varphi(y)|+|\varphi'(y)|).$$

(c) Let $\beta>0$ as in (a). Show that the mapping $T:\mathcal{S}(\mathbb{R})\to\mathbb{C}$ defined by

$$T(\varphi)=\lim_{\varepsilon\to0^+}\int_{|x|\ge\varepsilon}f(x)\varphi(x)\mathrm{d}x,\quad\forall\varphi\in\mathcal{S}(\mathbb{R})$$

is a tempered distribution. (Hint: $f$ is an odd function and (b) is helpful.)

**Problem 3.** (15 points) Let $d\ge1$. Is there a function $u_0\in L^2(\mathbb{R}^d)\backslash\{0\}$ such that $e^{t\Delta}u_0\to u_0$ in $L^2(\mathbb{R}^d)$ faster than $O(\sqrt{t})_{t\to0}$, namely

$$\lim_{t\to0^+}t^{-1/2}\|e^{t\Delta}u_0-u_0\|_{L^2(\mathbb{R}^d)}=0?$$

You may either show an example or prove the nonexistence.

**Problem 4.** (10+15 points) Consider the nonlinear equation for $u(t,x)$, $t\in\mathbb{R}$, $x\in\mathbb{R}^d$

$$\mathbf{i}\partial_t u=-\Delta u+(1-e^{-|u|})u,\quad u(0,x)=u_0(x).$$

(a) Use Duhamel formula and a fixed point argument to show that for every initial data $u_0\in L^2(\mathbb{R}^d)$, there exists a local solution $u\in L^\infty_t L^2_x$.

(b) Write the conservation law of mass (you do not need to prove it). Use this to prove the existence of a global solution in $L^\infty_t L^2_x$ for every $u_0\in L^2(\mathbb{R}^d)$. Is this solution unique?

**Problem 5.** (10+15 points) Consider the nonlinear equation for $u(t,x)$, $t\in\mathbb{R}_+$, $x\in\mathbb{R}^3$

$$\partial_t u-\Delta u+(1-e^{-|u|})(\sqrt{-\Delta}u)=0,\quad u(0,x)=u_0(x).$$

(a) Show that $\|(1-e^{-|u|})(\sqrt{-\Delta}v)\|_{\dot H^{-1/2}}\lesssim\|u\|_{\dot H^1}\|v\|_{\dot H^1}$.

(b) Show that for every $u_0\in\dot H^{1/2}(\mathbb{R}^3)$, there exists a unique local solution $u\in L^4_t\dot H^1_x$. (Hint: Problem E9.2 is helpful for a fixed point argument.)