LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. Dr. Phan Thành Nam
Dr. Maximilian Duell

Summer Term 2021
04.06.2021

Functional Analysis

Midterm

Family name: .... *Matriculation no.:* ....

First name: .... *Semester:* ....

Study course: ....

Signature: ....

You have **120 minutes** official working time and one **additional** hour to prepare and finalize your upload. Solutions must be uploaded until the **deadline** at 13:00 o'clock on 04.06.2021 via Uni2Work in PDF-format. Make sure to to follow these rules:

- Solutions must be **handwritten**. (pen on paper & scanned, or digital pen tablet)

  Do **not** use the colours **red** or **green**.

  If you do not use the official midterm preprint (this file), you must follow the official formatting instructions for "plain-paper submissions" given in uni2work.

- Solve each problem on the respective sheet. If you need more space, you can use the extra sheets at the end. In any case, extra sheets for technical reasons must state your name and the problem they refer to.

- All answers and solutions must provide sufficiently detailed arguments.

- Only one solution to each problem will be accepted. Please cross out everything which is not supposed to count.

- With your signature you agree to the rules of the exam.

  Specifically: You may use all materials and notes from the lecture, homeworks and tutorials, and any books (paper or e-book). Solutions must be prepared by yourself. You are not allowed to share information about any of the problems or their solutions of this midterm with others before the hand-in deadline on 04.06.2021, 13:00 CEST.

Before uploading check whether your pdf-scan is **readable** and contains all your solutions (in total there are **four problems**). Do not forget to write your name on each sheet.

Good luck!

Problem	1	2	3	4	$\sum$
Points	/2	/2	/2	/4	/10

Problem 1. [1+1 Points]

Let (X, d) be a metric space. Let A and B be non-empty and closed subsets of X such that $A \cap B = \emptyset$.

(a) Consider the function $f : X \to \mathbb{R}$ given by

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)},$$

where $d(x, A) := \inf_{a \in A} d(x, a)$.

Prove that f is well-defined and continuous, with $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

(b) Prove that there exist open subsets $U, V \subseteq X$ such that

$$A \subseteq U, \quad B \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

Problem 2. [1+1 Points]

Let X be a Banach space. Let A, B be two non-empty compact subsets of X.

(a) Prove that $A - B := \{a - b : a \in A,\ b \in B\}$ is a compact set.

(b) Prove that there exist $a_0 \in A$ and $b_0 \in B$, such that $\|a_0 - b_0\| = \inf_{\substack{a \in A \\ b \in B}} \|a - b\|$.

Problem 3. [1+1 Points]

Let X, Y be Banach spaces and $f : X \to Y$ a linear mapping. Prove that the following statements are equivalent:

(i) f is continuous.

(ii) For all sequences $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, we have

$$\sum_{n=1}^{\infty} \|f(x_n)\|_Y < \infty.$$

Problem 4. [1+1+2 Points]

Let X be a Banach space.

(a) Prove that if $f \in X^*$, then $\operatorname{codim}(\ker f) \leq 1$.

(b) Prove that if $\dim X = \infty$ and $\{f_1, \ldots, f_M\} \subseteq X^*$ $(M \in \mathbb{N})$ is a finite set of functionals, then $\dim \left(\bigcap_{i=1}^{M} \ker f_i \right) = \infty$.

Hint: You can use induction and $\ker(f_1) \cap \ker(f_2) = \ker(f_2\big|_{\ker f_1})$.

(c) Assume that $\dim X = \infty$. Let A be a non-empty subset of X. Prove that if A is open in the weak topology, then A is unbounded. (That is, $\sup_{x \in A} \|x\| = \infty$.)

Problem 1. [1+1 Points]

Let (X, d) be a metric space. Let A and B be non-empty and closed subsets of X such that $A \cap B = \emptyset$.

(a) Consider the function $f : X \to \mathbb{R}$ given by

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)},$$

where $d(x, A) := \inf_{a \in A} d(x, a)$.

Prove that f is well-defined and continuous, with $f(x) = 0$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$.

(b) Prove that there exist open subsets $U, V \subseteq X$ such that

$$A \subseteq U, \quad B \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$

Solution:

Continuation of the solution for Problem 1.

Problem 2. [1+1 Points]

Let X be a Banach space. Let A, B be two non-empty compact subsets of X.

(a) Prove that $A - B := \{a - b : a \in A,\ b \in B\}$ is a compact set.

(b) Prove that there exist $a_0 \in A$ and $b_0 \in B$, such that $\|a_0 - b_0\| = \inf\limits_{\substack{a \in A \\ b \in B}} \|a - b\|$.

Solution:

Continuation of the solution for Problem 2.

Problem 3. [1+1 Points]

Let X, Y be Banach spaces and $f : X \to Y$ a linear mapping. Prove that the following statements are equivalent:

(i) f is continuous.

(ii) For all sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, we have

$$\sum_{n=1}^{\infty} \|f(x_n)\|_Y < \infty.$$

Solution:

Continuation of the solution for Problem 3.

Problem 4. [1+1+2 Points]

Let X be a Banach space.

(a) Prove that if $f \in X^*$, then $\operatorname{codim}(\ker f) \leq 1$.

(b) Prove that if $\dim X = \infty$ and $\{f_1, \ldots, f_M\} \subseteq X^*$ ($M \in \mathbb{N}$) is a finite set of functionals, then $\dim \left(\bigcap_{i=1}^{M} \ker f_i \right) = \infty$.

Hint: You can use induction and $\ker(f_1) \cap \ker(f_2) = \ker(f_2\big|_{\ker f_1})$.

(c) Assume that $\dim X = \infty$. Let A be a non-empty subset of X. Prove that if A is open in the weak topology, then A is unbounded. (That is, $\sup_{x \in A} \|x\| = \infty$.)

Solution:

Continuation of the solution for Problem 4.

Name: .... *Page:* ....../......

Extra sheet. *Continuation of the solution to Problem* ____ :