LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis
Final exam

Family name: . . . . Matriculation no.: . . . .

First name: . . . . Semester: . . . .

Study course: . . . .

Signature: . . . .

You have **180 minutes** official working time and **one additional hour** to prepare and finalize your upload. Solutions must be uploaded until the deadline at 13:00 o'clock on 20.07.2021 via Uni2Work in PDF-format. Make sure to follow these rules:

- Solutions must be **handwritten** (pen on paper & scanned, or digital pen tablet). Do **not** use the colours **red** or **green**.
  If you do not use the official final exam preprint (this file), you must follow the official formatting instructions for "plain-paper submissions" given in uni2work.
- Solve each problem on the respective sheet. If you need more space, you can use the extra sheets at the end. In any case, extra sheets for technical reasons must state your name and the problem they refer to.
- All answers and solutions must provide sufficiently detailed arguments.
- Only one solution to each problem will be accepted. Please cross out everything which is not supposed to count.
- With your signature you agree to the rules of the exam.
  Specifically: You may use all materials and notes from the lecture, homeworks and tutorials, and any books (paper or e-book). Solutions must be prepared by yourself. You are not allowed to share information about any of the problems or their solutions of this exam with others before the hand-in deadline.

Before uploading please check whether your pdf-scan is readable and contains all your solutions (in total there are **four problems**). Do not forget to write your name on each sheet. Good luck!

Problem 1	Problem 2	Problem 3	Problem 4	Bonus	$\sum$	GRADE
(max 30)	(max 20)	(max 30)	(max 20)	(max 22)		

Problem Overview (you do not have to include this page in your submission).

Problem 1 (15+5+10 points). Define

$$X = \{f \in C^2(\mathbb{R}) : f(0) = 0, f'(0) = 1\}, \quad [f] = \{g \in X : f(n^{-1}) = g(n^{-1}), \forall n \in \mathbb{N}\}.$$

Consider the quotient space $Y = \{[f] : f \in X\}$ and the function $d : Y \times Y \to \mathbb{R}$:

$$d([f],[g]) = \sum_{n\in\mathbb{N}} |f(n^{-1}) - g(n^{-1})|, \quad \forall [f],[g] \in Y.$$

(a) Prove that d is well-defined and that (Y, d) is a metric space.
(Hint: If $f \in X$, $|f(x) - x| \leq C_f x^2$ for all $|x| \leq 1$ with a constant $C_f \in (0,\infty)$.)

(b) Given that for every $m \in \mathbb{N}$, there exists a function $f_m \in X$ such that

$$f_m(n^{-1}) = n^{-1} + n^{-3/2}\mathbb{1}_{\{n\leq m\}} = \begin{cases} n^{-1} + n^{-3/2} & \text{if } n \leq m, \\ n^{-1} & \text{if } n > m. \end{cases}$$

Prove that $\{[f_m]\}_{m=1}^{\infty}$ is a Cauchy sequence in (Y, d).
(You do not need to prove the existence of $\{f_m\}_{m=1}^{\infty}$.)

(c) Does the sequence $\{[f_m]\}_{m=1}^{\infty}$ converge in (Y, d)? Justify your claim.

Problem 2 (10+10 points). Let $d \geq 1$. Let Ω be a measurable, bounded subset of $\mathbb{R}^d$. Assume that $u_n \rightharpoonup 0$ weakly in $L^2(\Omega)$ when $n \to \infty$.

(a) Prove that $u_n \rightharpoonup 0$ weakly in $L^p(\Omega)$ for all $1 < p \leq 2$.

(b) For every $n \geq 1$, define $v_n : \mathbb{R}^d \to \mathbb{C}$ by

$$v_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Prove that $\widehat{v}_n \rightharpoonup 0$ weakly in $L^q(\mathbb{R}^d)$ for all $2 \leq q < \infty$.
(Hint: You can use the Hausdorff–Young inequality. Here $\widehat{v}_n$ is the Fourier transform.)

Problem 3 (5+5+10+10 points). Recall the Hilbert space

$$\ell^2(\mathbb{N}) = \Big\{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{C}, \sum_{n=1}^{\infty} |x_n|^2 < \infty\Big\}.$$

Let $\lambda \in (0,\infty)$ and define the operator T on $\ell^2(\mathbb{N})$ by

$$Tx = (\lambda x_n + x_{n+1})_{n=1}^{\infty}, \quad \forall x = (x_n)_{n=1}^{\infty} \in \ell^2(\mathbb{N}).$$

(a) Prove that T is a linear bounded operator.

(b) Is T a self-adjoint operator? Justify your claim.

(c) Prove that if $\lambda > 1$, then $0 \notin \sigma(T)$.

(d) Prove that if $\lambda = 1$, then $0 \in \sigma(T)$.

Problem 4 (10+10 points). Let $d \geq 1$. Let $a, g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let A and G be operators on $L^2(\mathbb{R}^d)$ defined by

$$(Af)(x) = a(x)f(x), \quad \widehat{Gf}(k) = \widehat{g}(k)\widehat{f}(k), \quad \forall f \in L^2(\mathbb{R}^d).$$

Consider the operator $K = A^*GA$ on $L^2(\mathbb{R}^d)$.

(a) Prove that K is a compact operator.
(Hint: You can prove that the kernel of K is $\overline{a(x)}g(x-y)a(y)$.)

(b) Prove that if $a(x) = g(x) = e^{-\pi|x|^2}$, then $K > 0$, namely $\langle u, Ku\rangle > 0$ for all $u \in L^2(\mathbb{R}^d)\setminus\{0\}$. Deduce that K has infinitely many strictly positive eigenvalues.