Chapter 6: Hilbert spaces

<u>Def:</u> Let X be a Banach space. Then X is called a (complex) Hilbert space if the norm $\|\cdot\|_X$ is induced by an inner product $\langle \cdot, \cdot \rangle$, namely

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in X.$$

Here $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{C}$ is an inner prod

if $(x, y) \to \langle x, y\rangle$ is linear in y
& anti-linear in x

i.e.
$$\begin{cases} \langle x, \lambda_1 y_1 + \lambda_2 y_2\rangle = \lambda_1 \langle x, y_1\rangle + \lambda_2 \langle x, y_2\rangle \\ \langle \lambda_1 x_1 + \lambda_2 x_2, y\rangle = \overline{\lambda_1}\langle x_1, y\rangle + \overline{\lambda_2}\langle x_2, y\rangle \end{cases}$$

<u>Remark:</u> We implicitly require

$$\begin{cases} \langle x, x\rangle \geq 0, \quad \forall x \in X \\ \langle x, x\rangle = 0 \iff x = 0 \end{cases}$$

such that $\sqrt{\langle x, x\rangle}$ is a norm.

<u>Remark:</u> From $\langle x, x\rangle \in \mathbb{R}, \forall x$ we get

$$\overline{\langle x, y\rangle} = \langle y, x\rangle, \quad \forall x, y \quad \text{(Exercise)}$$

Remark: From $\langle x,x\rangle \geq 0$

$\Rightarrow \langle x+y, x+y\rangle \geq 0$

$\Rightarrow \|x\|^2 + \|y\|^2 + \underbrace{\langle x,y\rangle + \langle y,x\rangle}_{2\,\mathrm{Re}\langle x,y\rangle} \geq 0$

$\Rightarrow |2\,\mathrm{Re}\langle x,y\rangle| \leq \|x\|^2 + \|y\|^2, \ \forall x,y$

Replacing y by λy for $\lambda > 0$

$\Rightarrow \quad 2\lambda\, |\mathrm{Re}\langle x,y\rangle| \leq \|x\|^2 + \lambda^2 \|y\|^2$

Optimizing over $\lambda > 0$

$\Rightarrow \quad |\mathrm{Re}\langle x,y\rangle| \leq \|x\|.\|y\|.$

Replacing y by zy with $z \in \mathbb{C}$, $|z| = 1$,

$$z \underbrace{\langle x,y\rangle}_{\in \mathbb{C}} = |\langle x,y\rangle|$$

we obtain:

Lemma: (Cauchy-Schwarz inequality) X Hilbert space. Then:

$$|\langle x,y\rangle| \leq \|x\|.\|y\|, \ \forall x,y \in X$$

Example: Let (Ω, μ) be a measure space. Then $L^2(\Omega)$ is a Hilbert space with

$$\|f\|^2_{L^2(\Omega)} = \int_\Omega |f(x)|^2 \, d\mu(x)$$

$$= \int_\Omega \overline{f(x)}\, f(x) \, d\mu(x)$$

$$= \langle f, f \rangle_{L^2(\Omega)}$$

where

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega \overline{f(x)}\, g(x)\, d\mu(x).$$

Remark: Any separable Hilbert space is isometry to $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^d)$ or $L^2(\Omega)$ with Ω open in $\mathbb{R}^d$).

Theorem (Parallelogram identity) If X is a Hilbert space, then:

$$(*) \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \ \forall x,y.$$

Reversely, if X is a Banach space and the norm satisfies the above identity, then X is a Hilbert space.

Proof: If X is a Hilbert space, then:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x,y\rangle$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\mathrm{Re}\langle x,y\rangle$$

$\to$ identity.

Reversely, if X is a Banach space & $(*)$ holds, then we can define $\langle , \rangle$ as follows:

$$2\mathrm{Re}\,\langle x,y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2$$

$$\underbrace{2\mathrm{Re}\,\langle x,iy\rangle}_{-2\mathrm{Im}\langle x,y\rangle} = \|x+iy\|^2 - \|x\|^2 - \|y\|^2$$

$$\Rightarrow 2\langle x,y\rangle = \left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) - i\left(\|x+iy\|^2 - \|x\|^2 - \|y\|^2\right)$$

$\leadsto$ we can check that $\langle , \rangle$ is an inner product.

Theorem: Let X be a Hilbert space. Then X is uniformly convex. In particular, X is reflexive.

Proof: Let $\{x_n\}$, $\{y_n\}$ be two sequences s.t.

$$\|x_n\| \le 1, \quad \|y_n\| \le 1, \quad \|x_n + y_n\| \to 2.$$

We need to prove that $(x_n - y_n) \to 0$. Using

$$\underbrace{\|x_n + y_n\|^2}_{\to\, 2^2 = 4} + \|x_n - y_n\|^2 = 2\|x_n\|^2 + 2\|y_n\|^2$$

$$\le 2 + 2 = 4$$

$$\Rightarrow \limsup_{n\to\infty} \|x_n - y_n\|^2 \le 0$$

$$\Rightarrow x_n - y_n \to 0.$$

Thus X is uniformly convex. Hence, X is reflexive.

Remark: X is reflexive $\Leftrightarrow$ $X \simeq X^{**}$

For the Hilbert spaces, we actually have

$$X \simeq X^{*}.$$

Theorem (Projection onto closed convex sets)

Let X be a Hilbert space. Let M be a closed convex subset of X. Then $\exists!$ mapping

$$P_M : X \to M$$

(*) s.t. $\|x - P_M x\| = \inf_{y \in M} \|x - y\| = \mathrm{dist}(x, M)$

$\forall x \in X$.

Moreover, the vector $u = P_M x \in M$ is characterized uniquely by the inequality

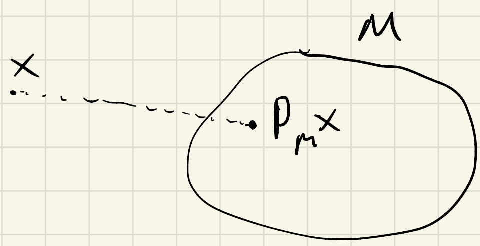
(**) $\mathrm{Re}\langle x - P_M x, P_M x - y \rangle \geq 0, \; \forall y \in M$

which is equivalent:

$$\|P_M x - x\|^2 + \|P_M x - y\|^2 \leq \|y - x\|^2, \; \forall y \in M.$$

Remark: In general, P_M may be not linear.

But if M is a subspace of X, then P_M is linear.



Remark: In general, if M is closed convex subset of a Banach space X, then:

- If X is reflexive, $\forall x \in X$, $\exists u \in M$ s.t.
$$\|x-u\| = \inf_{y\in M} \|x-y\| = \operatorname{dist}(x, M).$$

- If X is uniformly convex, the vector $u = u_x$ defined as above is unique. (exercise)

Proof of the theorem:

The existence & uniqueness of P_M are left as an exercise.

Proof of $(*) \Leftrightarrow (**)$

Assume $(*)$ holds. Then:

$$\|x - P_M x\| \leq \|x-y\|, \ \forall y \in M$$

$$\Leftrightarrow \quad \|x - P_M x\| \leq \|x - (1-t) P_M x - t y\|, \ \forall y \in M, \ \forall t \in [0,1]$$

$$\Leftrightarrow \ 0 \leq \|x - P_M x + t(P_M x - y)\|^2 - \|x - P_M x\|^2$$

$$= t^2 \|P_M x - y\|^2 + 2t \operatorname{Re} \langle x - P_M x, P_M x - y \rangle$$

$\forall y \in M$, $\forall t \in [0,1]$.

Thus $\quad \operatorname{Re}\langle x - P_M x, P_M x - y\rangle \geq 0.$

Reversely, assume $(*)$ holds. Then from the above computation:

$$\|x - P_M x + t(P_M x - y)\|^2 - \|x - P_M x\|^2$$
$$= t^2 \|P_M x - y\|^2 + 2t \underbrace{\operatorname{Re}\langle x - P_M x, P_M x - y\rangle}_{\geq 0}$$

$$\Rightarrow \|x - P_M x + t(P_M x - y)\|^2 \geq \|x - P_M x\|^2 + t^2 \|P_M x - y\|^2$$
$$\forall y \in M, \forall t \in [0,1].$$

Taking $t = 1$, then:

$$\|x - y\|^2 \geq \|x - P_M x\|^2 + \|P_M x - y\|^2, \forall y \in M$$

The latter implies $(*)$

$$\|x - y\|^2 \geq \|x - P_M x\|^2. \qquad \square$$

<u>Remark</u>: P_M is a projection, i.e. $P_M^2 = P_M$ since $P_M x = x$ if $x \in M$.

Theorem: Let X be a Hilbert space. Let M be a closed subspace of X. Then the projection

$$P_M : X \to M$$

is a linear mapping with $\|P_M\|_{\mathcal{L}(X,M)} = 1$.
Moreover, $\forall x \in X$, the vector $P_M x \in M$ is characterized uniquely by the identity

$$\langle x - P_M x , y \rangle = 0, \ \forall y \in M.$$

Put differently, we have the decomposition

$$x = P_M x + P_{M^\perp} x \quad , \forall x \in X$$

where $P_M x \in M$, $P_{M^\perp} x \in M^\perp$ where

$$M^\perp = \{ y \in X : y \perp M \}$$
$$= \{ y \in X : \langle y, m \rangle = 0, \forall m \in M \}.$$

(Here $M^\perp$ is also a closed subspace of X.)

Proof: We know that P_M is well-defined and $\forall x \in X$, then $P_M x \in M$ is characterized uniquely by the inequality

$$\langle x - P_M x, \; P_M x - y \rangle \geq 0, \; \forall y \in M$$

This holds when M is closed & convex. Now we know that M is a subspace of X. Then $P_M x - M = M = -M$, and hence

$$\begin{cases} \langle x - P_M x, \; y \rangle \geq 0, \; \forall y \in M \\ \langle x - P_M x, \; -y \rangle \geq 0 \end{cases}$$

$$\Rightarrow \quad \langle x - P_M x, \; y \rangle = 0, \; \forall y \in M.$$

Thus $\quad x - P_M x \in M^\perp = \{ z \in X : \langle z, y \rangle = 0, \forall y \in M \}$

<u>Exercise:</u> $M^\perp$ is a closed subspace of X.

Hence, we can conclude that $\forall x \in X$,

$$x = \underbrace{P_M x}_{\in M} + \underbrace{(x - P_M x)}_{\in M^\perp} = P_M x + P_{M^\perp} x.$$

Here we used the fact that $P_{M^\perp} P_M = 0$.

Actually, $\quad P_{M^\perp} = 1 - P_M$ and P_M is a projection.

$\square$

Theorem (Riesz representation theorem).

Let X be a Hilbert space. Then $X^* \simeq X$ in the sense that $\forall f \in X^*$, $\exists!\, u = u_f \in X$ s.t.

$$f(\varphi) = \langle u, \varphi \rangle \quad , \forall \varphi \in X.$$

Moreover,

$$\|f\|_{X^*} = \|u\|_X .$$

Proof: Abstract proof: Define $T : X \to X^*$

$X \ni x \mapsto T_x \in X^*$ s.t. $T_x(\varphi) = \langle x, \varphi \rangle, \forall y \in X$.

Then $\|T_x\|_{X^*} = \|x\|_X$.

Thus $T : X \to T(X) \subset X^*$ is an isometry.

It remains to prove that $T(X) = X^*$.

Assume by contradiction that $T(X) \neq X^*$.

Then $\exists f_0 \in X^* \setminus T(X)$. Since $T(X)$ is closed subspace and $f_0 \notin T(X)$, by the Hahn-Banach theorem, $\exists h \in X^{**}$ s.t.

$$h(f_0) \neq 0 = h(T(X)).$$

Since X is reflexive $\to X^{**} = X \to h$ can be identified with $x_0 \in X$ in the sense that

$$h(g) = g(x_0), \quad \forall g \in X^*.$$

Consequently

$$0 \neq h(f_0) = f_0(x_0)$$

and $0 = h(T_x) = \langle x_0, x\rangle, \forall x \in X$

$\Rightarrow x_0 = 0 \Rightarrow$ a contradiction.

<u>A more direct proof:</u>

Let $f \in X^*$. If $f = 0$, then there is nothing to prove ($f \in X^* \leftrightarrow 0 \in X$). Assume that $f \neq 0$. Then $M = \operatorname{Ker} f = \{x \in X : f(x) = 0\}$.

We know that $M \subsetneq X$ since $f \neq 0$.

Take $x_0 \in X \setminus M$. Define

$$x_1 = x - P_M x \in M^\perp$$

$$\Rightarrow f(x_1) = f(x) - \underbrace{f(P_M x)}_{=0 \text{ since } P_M x \in \operatorname{Ker} f} \neq 0$$

Now $\forall y \in X$, then:

$$y - \frac{f(y)}{f(x_1)} x_1 \in M = \ker f$$

since

$$f\left(y - \frac{f(y)}{f(x_1)} x_1\right) = f(y) - \frac{f(y)}{f(x_1)} f(x_1) = 0$$

Combining with $x_1 \in M^\perp$, then:

$$0 = \left\langle x_1, y - \frac{f(y)}{f(x_1)} x_1 \right\rangle$$

$$= \langle x_1, y \rangle - \frac{f(y)}{f(x_1)} \|x_1\|^2$$

$$\Leftrightarrow \quad \langle x_1, y \rangle = \frac{\|x_1\|^2}{f(x_1)} f(y)$$

$$\Leftrightarrow \quad f(y) = \Big\langle \underbrace{\frac{f(x_1)}{\|x_1\|^2} x_1}_{u_f}, y \Big\rangle, \quad \forall y \in X.$$

Thus $\forall f \in X^*$, if $f \neq 0$, then $\exists u_f \in X$ s.t.

$$f(y) = \langle u_f, y \rangle, \quad \forall y \in X.$$

This ends the proof. $\square$

Two remarks on $X \simeq X^*$.

<u>Remark 1:</u> Remember that $X \ni u_f \longleftrightarrow f \in X^*$:

$$f(\varphi) = \langle u_f, \varphi \rangle, \ \forall \varphi \in X.$$

Thus $f \mapsto u_f$ is anti-linear, i.e.

$$u_{\lambda f} = \bar{\lambda} u_f, \ \forall \lambda \in \mathbb{C}.$$

Hence, the identification $X \simeq X^*$ should be taken with a "complex conjugation", i.e.

$$f, g \in X^* \longleftrightarrow u_f, u_g \in X$$

then

$$\langle f, g \rangle_{X^*} = \langle u_g, u_f \rangle_X = \overline{\langle u_f, u_g \rangle_X}$$

<u>Remark 2:</u> If X is a Hilbert space and M is a closed subspace of X and

$$\|x\|_M \geq \|x\|_X.$$

Then $M \subset X$ continuous embedding

$$\Rightarrow \quad X^* \subset M^*$$

In this case, it will be confusing if we

want to identify both $X \simeq X^*$ & $M \simeq M^*$.
The normal way is
$$M \subset X \simeq X^* \subset M^*$$
<u>Example:</u> Consider
$$X = \ell^2(\mathbb{N}) = \{ (x_n)_{n=1}^\infty : x_n \in \mathbb{C}, \ \sum |x_n|^2 < \infty \}$$
$$M = \{ (x_n)_{n=1}^\infty ; x_n \in \mathbb{C}, \ \sum n^2 |x_n|^2 < \infty \}$$
Then $X = X^*$ but
$$M^* = \{ (x_n)_{n=1}^\infty : x_n \in \mathbb{C} : \sum \frac{1}{n^2} |x_n|^2 < \infty \}.$$
<u>Remark:</u> let X be a Hilbert space. Then,
$$x_n \to x \quad \text{weakly in } X$$
$$\Leftrightarrow \langle x_n, \varphi \rangle \to \langle x, \varphi \rangle, \ \forall \varphi \in X.$$

Theorem (Lax-Milgram). Let X be a Hilbert space, let $a: X \times X \to \mathbb{C}$ satisfy

- $a(x,y)$ is linear in y and anti-linear in x
- $|a(x,y)| \leq C \|x\| \|y\|$, $\forall x, y$ (a is bounded)
- $a(x,x) \geq c_0 \|x\|^2$, $\forall x$, with a constant $c_0 > 0$ (a is coercive)

Then: $\forall f \in X$, $\exists!\, u = u_f$, s.t.

$$a(u, \varphi) = \langle f, \varphi \rangle_X, \quad \forall \varphi \in X.$$

Proof: $(X, \|.\|_a)$ is a Hilbert space where

$$\|x\|_a := \sqrt{a(x,x)}$$

Moreover, $C\|x\|^2 \geq \|x\|_a^2 \geq c_0 \|x\|^2$, $\forall x \in X$.

For any given $f \in X$, then

$$X \ni \varphi \mapsto \langle f, \varphi \rangle \in \mathbb{C}$$

this map belongs to $(X, \|.\|)^* = (X, \|.\|_a)^*$.

Applying Riesz theorem for the Hilbert space $(X, \|.\|_a)$ we find that $\exists!\, u = u_f$ s.t.

$$a(u, \varphi) = \langle f, \varphi \rangle, \quad \forall \varphi \in X. \qquad \square$$

Remark: The vector $u = u_f$ is characterized by

$$a(u,u) - 2\mathrm{Re}\langle f, u\rangle = \min_{v \in X}\left(a(v,v) - 2\mathrm{Re}\langle f, v\rangle\right)$$

(exercise).

Direct sums & orthogonal bases:

Def: Let X be a Hilbert space. Let $(E_n)_n$ be a family of closed subspace of X s.t.

(1) $E_n \perp E_m$, $\forall n \neq m$

i.e. $\langle x, y\rangle = 0$, $\forall x \in E_n$, $\forall y \in E_m$

(2) $\overline{\mathrm{Span}\left(\bigcup_n E_n\right)} = X$.

Then we write $X = \bigoplus_n E_n$ as a direct sum of orthogonal spaces.

Theorem (Bessel-Parseval identity)

Assume $X = \bigoplus_{n=1}^{\infty} E_n$ as a direct sum of orthogonal spaces. For $x \in X$, define

$$x_n := P_{E_n} x \quad \text{and} \quad S_n = \sum_{i=1}^{n} x_i .$$

Then $S_n \to x$ strongly, i.e. $x = \sum_{n=1}^{\infty} x_n$ and

$$\|x\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 .$$

Remark: If we only know that $(E_n)_n$ are orthogonal (but we don't assume that $\mathrm{Span}(\bigcup_n E_n)$ is dense) , then

$$\|x\|^2 \geqslant \sum_{n=1}^{\infty} \|x_n\|^2 \quad , \quad x_n = P_{E_n} x .$$

(Bessel inequality)

Proof: $\quad S_n = \sum_{i=1}^{n} x_n = \sum_{i=1}^{n} P_{E_i} x$

$$\Rightarrow \langle x, S_n\rangle = \sum_{i=1}^{n} \langle x, P_{E_i} x\rangle$$

$$= \sum_{i=1}^{n} \underbrace{\langle P_{E_i} x, P_{E_i} x\rangle}_{\|P_{E_i} x\|^2} = \sum_{i=1}^{n} \|x_i\|^2$$

Moreover, $\quad \|S_n\|^2 = \langle \sum_{i=1}^{n} P_{E_i} x , \sum_{j=1}^{n} P_{E_j} x\rangle$

$$= \sum_{i,j=1}^{n} \underbrace{\langle P_{E_i} x , P_{E_j} x\rangle}_{=0 \text{ if } i\neq j} = \sum_{i=1}^{n} \|P_{E_i} x\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Thus:

$$\sum_{i=1}^{n} \|x_i\|^2 = \|S_n\|^2 = \langle x, S_n\rangle \overset{\text{by Cauchy-Schwarz}}{\leq} \|x\| \, \|S_n\|$$

Thus we obtain Bessel inequality

$$\sum_{i=1}^{n} \|x_i\|^2 = \|S_n\|^2 \leq \|x\|^2.$$

Consequently,

$$\sum_{i=1}^{\infty} \|x_i\|^2 \leq \|x\|^2.$$

Consider $m > n$:

$$\|S_m - S_n\|^2 = \Big\|\sum_{i=n+1}^{m} \overset{P_{E_i}x}{x_i}\Big\|^2 = \sum_{i=n+1}^{m} \|x_i\|^2 \to 0$$

when $m, n \to \infty$. Thus $(S_n)_{n\geq 1}$ is a Cauchy sequence in X $\Rightarrow$ S_n has a strong limit y when $n \to \infty$. We prove that $y = x$. In fact,

$$P_{E_i} S_n = P_{E_i}\Big(\sum_{j=1}^{n} P_{E_j} x\Big) = P_{E_i} x = x_i \quad \forall n \geq i$$

$$\overset{n\to\infty}{\Rightarrow} \quad P_{E_i} y = x_i = P_{E_i} x$$

Thus: $\quad P_{E_i}(x-y) = 0, \ \forall i.$

$$\Rightarrow \quad x - y \perp E_i, \forall i$$

i.e. $\langle x-y, \varphi\rangle = 0, \ \forall \varphi \in E_i, \ \forall i$

$$\Rightarrow \quad \langle x-y, \varphi\rangle = 0, \ \forall \varphi \in \overline{\operatorname{Span}\big(\bigcup_i E_i\big)} = X$$

Thus $x = y$, i.e. $S_n \to x$ strongly in X.

Consequently,

$$\|x\|^2 = \lim_{n\to\infty} \|S_n\|^2 = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)$$

$$= \sum_{i=1}^{\infty} \|x_i\|^2 . \qquad \square$$

<u>Def:</u> Let X be a Hilbert space. A sequence $(e_n)_{n\geq 1}$ is called an orthonormal basis (ONB) if $\langle e_n, e_m \rangle = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$

and $\overline{\operatorname{Span}\{e_n, n\geq 1\}} = X$.

<u>Remark:</u> $(e_n)_{n\geq 1}$ is an ONB

$\iff X = \bigoplus_n E_n$ where $E_n = \operatorname{Span}(e_n)$.

Consequence: if $(e_n)_{n\geq 1}$ is an ONB, then $\forall x \in X$, we have:

$$\begin{cases} x = \sum_{n\geq 1} \langle e_n, x \rangle e_n & \text{strongly in } X \\ \|x\|^2 = \sum_{n\geq 1} |\langle e_n, x \rangle|^2 & (\text{Parseval identity}) \end{cases}$$

<u>Example</u> If $X = L^2(0,1)$, then we can take

$$e_n(x) = e^{2\pi i n x}, \quad n = 0,1,2,\dots$$

or $$e_n(x) = \sqrt{2}\,\sin(\pi n x), \quad n = 1,2,3,\dots$$

or $$e_n(x) = \sqrt{2}\,\cos(\pi n x), \quad n = 0,1,2,\dots$$

In this case, we know that $\forall u \in L^2(0,1)$, then

$$u = \underbrace{\lim_{n\to\infty} S_n}_{= \sum_{j=1}^{\infty} \langle e_n, u\rangle e_n}, \qquad S_n = \sum_{j=1}^{n} \langle e_n, u\rangle e_n$$

(Fourier series)

<u>Remark:</u> From the fact that $S_n \to u$ strongly in $L^2(0,1)$, by Dominated C.V., $\exists$ subsequence S_{n_k} s.t. $S_{n_k} \to u$ a.e.

<u>Theorem (Carleson)</u> $\forall u \in L^2(0,1)$, then

$$S_n \to u \quad \text{a.e.}$$

(This is a deep result.)

<u>Theorem:</u> Let X be a Hilbert space. Then X has an orthonormal basis $(e_n)_{n\geq 1}$ if and only if X is separable.

<u>Proof:</u> • Assume that X has an orthonormal basis $(e_n)_{n\geq 1}$. Then $\forall x \in X$, we have:

$$x = \sum_{n\geq 1} \langle e_n, x \rangle e_n$$

Then the (countable) set of finite sums $\sum_{n\geq 1} r_n e_n$ where r_n rational and $r_n = 0$ except finite n.

Thus X is separable.

• Assume that X is separable, i.e. $\exists$ sequence $\{v_n\}_{n=1}^{\infty}$ which is dense in X. We can construct the ONB $(e_n)_{n=1}^{\infty}$ by induction.

1. $e_1 = \frac{v_1}{\|v_1\|}$ if $v_1 \neq 0$.

2. if $(e_n)_{n=1}^{N}$ is ONB for $\mathrm{Span}(v_1, \dots, v_m)$.

If $v_{m+1} \notin \mathrm{Span}(v_1 \dots v_m)$, then take e_{N+1} s.t. $(e_n)_{n=1}^{N+1}$ is ONB for $\mathrm{Span}(v_1 \dots v_{m+1})$

Remark: If X is an inf-dim separable Hilbert space, then $X \simeq \ell^2(\mathbb{N})$ with an isometry. More precisely, X has an ONB $(e_n)_{n=1}^{\infty}$ and $\ell^2(\mathbb{N})$ also has an ONB $(\tilde{e}_n)_{n=1}^{\infty}$ where $\tilde{e}_n = (0, \dots, \underset{n\text{-th}}{1}, 0, \dots)$.

$$T: X \to \ell^2(\mathbb{N})$$
$$e_n \mapsto Te_n = \tilde{e}_n \qquad \forall n$$

$$x = \sum_{n=1}^{\infty} \langle e_n, x \rangle e_n \mapsto Tx = \sum_{n=1}^{\infty} \langle e_n, x \rangle \tilde{e}_n.$$

Thus: $\|x\|^2 = \sum_{n=1}^{\infty} |\langle e_n, x \rangle|^2 = \|Tx\|^2.$

Consequently, we also have

$$X \simeq L^2(0,1)$$

or $X \simeq L^2(\Omega)$, Ω open set in $\mathbb{R}^d$ with Lebesgue measure.

Operators in Hilbert spaces: X a Hilbert space

Def. $\mathcal{L}(X) = \{ A : X \to X \text{ linear \& bounded} \}$

$$\|A\| = \sup_{\|x\|=1} \|A(x)\| = \sup_{\|x\|\leq 1} \|A(x)\|$$

$$= \sup_{x \neq 0} \frac{\|A(x)\|}{\|x\|} < \infty .$$

Def. (Spectrum) Let $A \in \mathcal{L}(X)$, X Hilbert space.

We define the resolvent set

$$\rho(A) = \{ \lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ exists in } \mathcal{L}(X) \}$$

and the spectrum

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

$$= \{ \lambda \in \mathbb{C} : (\lambda - A)^{-1} \notin \mathcal{L}(X) \}$$

Remark: Here we say that $B = A^{-1}$ if

$$BA = AB = 1 \quad (\text{identity})$$

Remark: If $\exists\, x_0 \neq 0$ s.t. $(\lambda - A) x_0 = 0$, then clearly $\lambda \in \sigma(A)$. In this case, λ is an eigenvalue and x_0 is an eigenvector.

In general, the spectrum could be much bigger than the set of e.v.

<u>Example</u>: $X = L^2(0,1)$

$$A : X \to X$$

$$(Af)(x) = x f(x), \quad \forall x \in (0,1)$$
$$\forall f \in L^2(0,1).$$

Then $\sigma(A) = [0,1]$ but A has <u>no</u> eigenvalue.

In fact, if $\exists \lambda \in \mathbb{C}$, $\exists f \in L^2(0,1)$, $f \neq 0$ s.t.

$$(Af)(x) = \lambda f(x), \quad \text{a.e. } x \in (0,1)$$

$$\Rightarrow \quad x f(x) = \lambda f(x), \quad \text{a.e. } x$$

$$\Rightarrow \quad (\lambda - x) f(x) = 0 \quad \text{a.e. } x$$

$$\Rightarrow \quad f(x) = 0 \quad \text{a.e. } x \rightarrow \text{contradiction.}$$

<u>Theorem</u>: (Multiplication operators) Let (Ω, μ) be a measure space. Let $a \in L^\infty(\Omega)$. Define $A : X = L^2(\Omega) \to L^2(\Omega)$ by

$$(Af)(x) = a(x) f(x) \quad \text{a.e. } x \in \Omega.$$

Then $A \in \mathcal{L}(X)$ and $\|A\| = \|a\|_{L^\infty}$ and

$$\sigma(A) = \text{ess-rang}(a)$$
$$= \{ \lambda \in \mathbb{C} : \mu\left(a^{-1}\left(B_{\mathbb{C}}(\lambda, \varepsilon)\right)\right) > 0 \text{ for all } \varepsilon > 0 \}$$

In particular, if $\Omega \subset \mathbb{R}^d$ and a is continuous, then $\sigma(A) = \overline{\text{rang}(a)} = \overline{a(\Omega)}$.

(Exercise).

<u>Theorem.</u> Let X be a Hilbert space and $A \in \mathcal{L}(X)$. Then $\sigma(A)$ is closed subset of $\overline{B_{\mathbb{C}}(0, \|A\|)}$.

<u>Proof:</u> 1) Take $\lambda \in \mathbb{C}$ and $|\lambda| > \|A\|$. We prove $\lambda \notin \sigma(A) \Leftrightarrow \lambda \in \rho(A) \Leftrightarrow (\lambda - A)^{-1} \in \mathcal{L}(X)$.

We have:

$$\lambda - A = \lambda(1 - \lambda^{-1}A) \quad \text{where } \|\lambda^{-1}A\| < 1$$

$$\Rightarrow (\lambda - A)^{-1} = \lambda^{-1}(1 - \lambda^{-1}A)^{-1}$$
$$= \lambda^{-1}\left[1 + (\lambda^{-1}A) + (\lambda^{-1}A)^2 + (\lambda^{-1}A)^3 + \dots\right]$$

The right side is well-defined as an operator in $\mathcal{L}(X) \leadsto (\lambda - A)^{-1} \in \mathcal{L}(X)$.

<u>Exercise</u>: Let X be a Hilbert space. Let $B \in \mathcal{L}(X)$ and $\|B\| < 1$. Define

$$S_n = 1 + B + B^2 + \dots + B^n.$$

(a) Prove that S_n is a Cauchy sequence in $\mathcal{L}(X)$ and hence $S_n \to S_\infty = \sum_{n=0}^{\infty} B^n$.

(b) Prove that $S_n(1-B) = (1-B)S_n \to 0$ strongly in $\mathcal{L}(X)$. Deduce that $S_\infty = (1-B)^{-1}$.

2) We prove that $\sigma(A)$ is closed $\Leftrightarrow$ $\rho(A)$ is open. Take $\lambda_0 \in \rho(A)$. We prove that if $\lambda \neq \lambda_0$ but $|\lambda - \lambda_0|$ is small, then $\lambda \in \rho(A)$.
We have:

$$\lambda - A = \lambda - \lambda_0 + \lambda_0 - A$$
$$= (\lambda_0 - A)\left((\lambda - \lambda_0)(\lambda_0 - A)^{-1} + 1 \right)$$

Since $(\lambda_0 - A)^{-1} \in \mathcal{L}(X)$, if $|\lambda - \lambda_0|$ is small, then: $B = -(\lambda - \lambda_0)(\lambda_0 - A)^{-1}$ satisfies $\|B\| < 1$.
Thus $(1-B)^{-1} \in \mathcal{L}(X)$.

Thus: $$\lambda - A = (\lambda_0 - A)(1 - B)$$
$$\Rightarrow \quad (\lambda - A)^{-1} = (1-B)^{-1}(\lambda_0 - A)^{-1}. \qquad \square$$

<u>Exercise</u>: Let X be a Hilbert space. Let $A, B \in \mathcal{L}(X)$ s.t. $A^{-1}, B^{-1} \in \mathcal{L}(X)$. Then
$$(AB)^{-1} = B^{-1}A^{-1} \text{ in } \mathcal{L}(X).$$

<u>Self-adjoint operators</u>

<u>Def</u> (Adjoint operator) Let X be a Hilbert space. Let $A \in \mathcal{L}(X)$. Define $A^* \in \mathcal{L}(X)$ by
$$\langle x, Ay\rangle = \langle A^*x, y\rangle, \ \forall x, y \in X.$$

<u>Remark</u>: The existence of A^* follows from Riesz representation theorem. In fact, we know that $\forall x \in X$,
$$y \longmapsto f(y) = \langle x, Ay\rangle, \quad f \in X^*$$
$$|f(y)| \leq \|x\| \, \|Ay\| \leq \underbrace{\|x\| \, \|A\|}_{<\infty} \|y\|$$

By Riesz theorem, $\exists z \in X$ s.t.

$$\underset{\underset{\langle x, Ay\rangle}{\|}}{f(y)} = \langle z, y\rangle, \quad \forall y \in X$$

Thus $\forall x \in X$, $\exists z \in X$ s.t.

$$\langle x, Ay\rangle = \langle z, y\rangle, \quad \forall y \in X$$

$$A^*x := z$$

Thus $\exists A^*: X \to X$ s.t.

$$\langle x, Ay\rangle = \langle A^*x, y\rangle, \quad \forall x, y \in X.$$

·) A^* is linear i.e.

$$\langle x_1 + x_2, Ay\rangle = \langle x_1, Ay\rangle + \langle x_2, Ay\rangle$$

$$\langle x_1 + x_2, Ay\rangle = \langle A^*(x_1+x_2), y\rangle$$

$$\langle x_1, Ay\rangle + \langle x_2, Ay\rangle = \langle A^*x_1, y\rangle + \langle A^*x_2, y\rangle = \langle A^*x_1 + A^*x_2, y\rangle$$

$$\langle A^*(x_1+x_2), y\rangle \longleftrightarrow \langle A^*x_1 + A^*x_2, y\rangle \quad \forall x_1, x_2, y \in X.$$

Thus $\quad A^*(x_1+x_2) = A^*x_1 + A^*x_2.$

Moreover: $\langle \lambda x, Ay\rangle = \bar{\lambda}\, \langle x, Ay\rangle = \bar{\lambda}\, \langle A^*x, y\rangle$

$$\langle \lambda x, Ay\rangle = \langle A^*(\lambda x), y\rangle \qquad \bar{\lambda}\,\langle A^*x, y\rangle = \langle \lambda A^*x, y\rangle$$

$$\Rightarrow \quad A^*(\lambda x) = \lambda A^* x, \quad \forall x \in X.$$

Therefore A^* is linear.

<u>Remark:</u> Here we used the fact that if $a, b \in X$ s.t. $\langle a, y\rangle = \langle b, y\rangle$, $\forall y \in X$

$$\Rightarrow \quad a = b.$$

This is because $\langle a-b, y\rangle = 0$, $\forall y \in X$

$\Rightarrow a - b = 0$ (e.g. we can take $y = a-b$)

<u>Exercise:</u> Let X be a Hilbert space, let M be a subspace of X. Then TFAE:

1) M is dense in X

2) $M^\perp = \{0\}$, i.e. if $x \perp M$ ($\langle x, y\rangle = 0, \forall y \in M$) then $x = 0$.

·) A^* is bounded and $\|A^*\| = \|A\|$

From $\langle A^* x, y\rangle = \langle x, Ay\rangle$, $\forall x, y \in X$

$$\Rightarrow \|A^*\| = \sup_{\|x\| \leq 1} \|A^* x\| = \sup_{\|x\|, \|y\| \leq 1} |\langle A^* x, y\rangle|$$

$$= \sup_{\|x\|, \|y\| \leq 1} |\langle x, Ay\rangle| = \sup_{\|y\| \leq 1} \|Ay\| = \|A\|.$$

Def: let X be a Hilbert space, let $A \in \mathcal{L}(X)$. Then A is called a self-adjoint operator if $A = A^*$.

Theorem: let X be a (complex) Hilbert space, let $A \in \mathcal{L}(X)$. Then TFAE:

1) A is self-adjoint, i.e. $A = A^*$.
2) $\langle x, Ay \rangle = \langle Ax, y \rangle, \ \forall x, y \in X$.
3) $\langle x, Ax \rangle \in \mathbb{R}, \ \forall x \in X$.

Proof: $\boxed{1) \Rightarrow (2)}$ obvious

$$\langle x, Ay \rangle \underset{\text{def of } A^*}{=} \langle A^* x, y \rangle \underset{A^* = A}{=} \langle Ax, y \rangle$$

$\boxed{(2) \Rightarrow (3)}$ $\langle x, Ax \rangle = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle}$

$\Rightarrow \langle x, Ax \rangle \in \mathbb{R}, \ \forall x \in X$.

$\boxed{(3) \Rightarrow (2)}$ From $\langle x, Ax \rangle \in \mathbb{R}, \forall x \in X$

$$\underbrace{\langle x + y, A(x+y) \rangle}_{\in \mathbb{R}} = \overbrace{\langle x, Ax \rangle}^{\mathbb{R}} + \overbrace{\langle y, Ay \rangle}^{\mathbb{R}}$$
$$+ \langle x, Ay \rangle + \langle y, Ax \rangle$$

$$\Rightarrow \quad \langle x, Ay\rangle + \langle y, Ax\rangle \in \mathbb{R}$$

$$\Rightarrow \quad \langle x, Ay\rangle + \overline{\langle Ax, y\rangle} \in \mathbb{R}, \quad \forall x, y \in X$$

$$\Rightarrow \quad \operatorname{Im}\left(\langle x, Ay\rangle + \overline{\langle Ax, y\rangle} \right) = 0$$

$$\Rightarrow \quad \operatorname{Im}\langle x, Ay\rangle - \operatorname{Im}\langle Ax, y\rangle = 0.$$

Replacing $y \mapsto iy$

$$\langle x, A(iy)\rangle + \overline{\langle Ax, iy\rangle} \in \mathbb{R}$$

$$\Rightarrow \quad i\left(\langle x, Ay\rangle - \overline{\langle Ax, y\rangle} \right) \in \mathbb{R}$$

$$\Rightarrow \quad \operatorname{Re}\left(\langle x, Ay\rangle - \overline{\langle Ax, y\rangle} \right) = 0$$

$$\Rightarrow \quad \operatorname{Re}\langle x, Ay\rangle - \operatorname{Re}\langle Ax, y\rangle = 0$$

Thus, $\langle x, Ay\rangle = \langle Ax, y\rangle, \quad \forall x, y \in X.$

$\boxed{(2) \Rightarrow (1)}$

$$\underset{\langle A^* x, y\rangle}{\underset{\parallel}{\langle x, Ay\rangle}} \underset{(2)}{=} \langle Ax, y\rangle$$

$$\Rightarrow \quad \langle A^* x, y\rangle = \langle Ax, y\rangle, \quad \forall x, y \in X$$

$$\Rightarrow \quad A^* x = Ax, \ \forall x \in X \Rightarrow A^* = A.$$

Theorem: Let X be a Hilbert space. Let $A \in \mathcal{L}(X)$ be a self-adjoint operator. Then:

1) $\sigma(A) \subset \mathbb{R}$.

2) If $A \geq 0$, i.e. $\langle x, Ax \rangle \geq 0 \quad \forall x \in \mathbb{R}$

then: $\sigma(A) \subset [0, \infty)$.

Remark: Actually it holds that

$$\sigma(A) \subset \left[\inf_{\|x\|=1} \langle x, Ax \rangle , \sup_{\|x\|=1} \langle x, Ax \rangle \right]$$

Proof. 1) Why $\sigma(A) \subset \mathbb{R}$? We have to prove that if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\lambda \in \rho(A)$, i.e. $(\lambda - A)^{-1} \in \mathcal{L}(X)$. Assume

$$\lambda = a + ib, \quad a, b \in \mathbb{R}, \quad b \neq 0$$

Then
$$\lambda - A = (a - A) + ib$$
$$= b \Big[\underbrace{\frac{a - A}{b}}_{\text{self-adjoint}} + i \Big]$$

Therefore, it suffices to consider the case when $a = 0$, $b = 1$, i.e. we prove that if $A = A^*$, then

$(A+i)^{-1} \in \mathcal{L}(X)$.

Key observation:

$$\|(A+i)x\|^2 = \langle (A+i)x, (A+i)x\rangle$$
$$= \langle Ax + ix, Ax + ix\rangle$$
$$= \|Ax\|^2 + \|x\|^2 + 2\underbrace{\mathrm{Re}\langle x, Aix\rangle}_{\underbrace{\mathrm{Re}(i \underbrace{\langle x, Ax\rangle}_{\in\mathbb{R}})}_{=0}}$$
$$= \|Ax\|^2 + \|x\|^2$$

Similarly,

$$\|(A-i)x\|^2 = \|Ax\|^2 + \|x\|^2 \quad , \forall x \in X.$$

Why $(A+i)^{-1} \in \mathcal{L}(X)$?

We prove that $(A+i)^{-1}$ is well-defined

$\Leftrightarrow$ $(A+i)$ is surjective, i.e. $(A+i)X = X$.

•) $(A+i)X$ is closed in X.

Take $(A+i)x_n \to y$, then $(A+i)x_n$ is a Cauchy sequence $\Rightarrow$ $(A+i)(x_n - x_m) \to 0$ as $m, n \to \infty$. Since $\|(A+i)x\| \geq \|x\| \Rightarrow x_n - x_m \to 0$ as $m, n \to \infty$ $\Rightarrow$ x_n is a Cauchy sequence

$\Rightarrow x_n \to x \Rightarrow (A+i)x_n \to (A+i)x = y$

•) $(A+i)X$ is dense in X. Take $x \in X$,

s.t. $x \perp (A+i)X$, prove that $x = 0$.

In fact, $0 = \langle x, (A+i)y \rangle$, $\forall y \in X$

$$\| $$

$$\langle (A-i)x, y \rangle$$

Thus $(A-i)x = 0$. But $\|(A-i)x\| \geq \|x\|$, hence

$x = 0$.

<u>Conclusion:</u> $(A+i)X$ is closed & dense in X

$\Rightarrow$ $(A+i)X = X$ $\Rightarrow$ $A+i$ is a surjective.

Thus $(A+i)^{-1}$ is well-defined.

Moreover,

$$\|\underbrace{(A+i)x}_{y}\| \geq \|x\|, \quad \forall x \in X$$

$$\Rightarrow \quad \|y\| \geq \|(A+i)^{-1}y\|, \quad \forall y \in X$$

$\Rightarrow$ $(A+i)^{-1}$ is bounded.

Thus $-i \in \rho(A)$. This completes the proof

of (1).

(2) Assume further that $A \geq 0$, i.e.

$$\langle x, Ax \rangle \geq 0, \quad \forall x \in X.$$

We prove that $\sigma(A) \subset [0, \infty)$. Take $\lambda \in \sigma(A)$. From (1), we know that $\lambda \in \mathbb{R}$. We need to prove that $\lambda \geq 0$.

By the contradiction, we prove that if $\lambda < 0$ then $\lambda \in \rho(A)$. By the Cauchy-Schwarz

$$\|(A-\lambda)x\| \geq \langle x, (A-\lambda)x \rangle \quad , \forall x \in X, \ \|x\| = 1$$

$$= \underbrace{\langle x, Ax \rangle}_{\geq 0} - \lambda \|x\|^2$$

$$\geq |\lambda| \|x\| \quad , \forall \|x\| = 1$$

Thus: $\left\| \left(\frac{A}{\lambda} - 1\right) x \right\| \geq \|x\|, \ \forall x \in X.$

Exercise. Let X be a Hilbert space. Let $A \in \mathcal{L}(X)$ s.t. $A = A^*$ and

$$\|Ax\| \geq \|x\|, \ \forall x \in X.$$

Then: $\quad A^{-1} \in \mathcal{L}(X)$.

(Hint: You can prove AX is closed and dense in X)

Thus we proved that if $\lambda < 0$, then $\lambda \in \rho(A)$.
This means that $\sigma(A) \subset [0, \infty)$. $\square$

<u>Compact operators:</u>

<u>Def.</u> Let X be a Hilbert space. Let $A \in \mathcal{L}(X)$.
Then A is a compact operator if
the set $\overline{A(B(0,1))}$ is compact in X.

<u>Remark:</u> By the linearity, A is compact
$\Longleftrightarrow$ $\overline{A(\omega)}$ is compact, $\forall \omega$ bounded in X

<u>Exercise:</u> A is compact
$\Longleftrightarrow$ $\Big((x_n \rightharpoonup x \text{ in } X) \Rightarrow (Ax_n \to Ax \text{ in } X) \Big)$

Theorem (Spectral theorem for compact & self-adjoint operators) Let X be a separable Hilbert space. Let $A \in \mathcal{L}(X)$ be a compact & self-adjoint operator. Then: $\exists$ an ONB $\{x_n\}$ of X and real numbers $\{\lambda_n\}_{n\geq 1}$ s.t. $\lambda_n \to 0$ and

$$A x_n = \lambda_n x_n, \quad \forall n \geq 1.$$

Proof: We use the variational technique.

Step 1: Consider the problem

$$m_1 := \sup_{\|x\| \leq 1} |\langle x, Ax \rangle|.$$

Since A is bounded, $m_1 < \infty$. We prove that $\exists\, x_1$ s.t. $\|x_1\| \leq 1$ & $|\langle x_1, Ax_1 \rangle| = m_1$.

Take a minimizing sequence $(y_n)_{n=1}^{\infty}$ s.t. $\|y_n\| \leq 1$ and $|\langle y_n, Ay_n \rangle| \to m_1$.

Since y_n is bounded, by the Banach-Alaoglu theorem, $\exists$ a subsequence $y_{n_k} \to x_1$ weakly in X.

Since A is a compact operator, from $y_{n_k} \rightharpoonup x$, we have $Ay_{n_k} \to Ax_1$ strongly. Thus:

$$\langle y_{n_k}, Ay_{n_k}\rangle \longrightarrow \langle x_1, Ax_1\rangle$$

$$\Rightarrow \quad m_1 = \lim_{k\to\infty} |\langle y_{n_k}, Ay_{n_k}\rangle| = |\langle x_1, Ax_1\rangle|.$$

Moreover, $y_{n_k} \rightharpoonup x_1$ and $\|y_{n_k}\| \le 1$, we know that $\|x_1\| \le 1$.

Therefore, x_1 is an optimizer for

$$m_1 = |\langle x_1, Ax_1\rangle| = \sup_{\|y\|\le 1} |\langle y, Ay\rangle|.$$

<u>Exercise:</u> Let X be a Hilbert space. Assume $a_n \rightharpoonup a$ weakly and $b_n \to b$ strongly. Then:

$$\langle a_n, b_n\rangle \longrightarrow \langle a, b\rangle \quad \text{as } n\to\infty.$$

<u>Step 2:</u> We prove that $\exists \lambda_1 \in \mathbb{R}$ s.t.

$$Ax_1 = \lambda_1 x_1.$$

If $x_1 = 0$, then we can take $\lambda_1 = 0$.

Now consider the case when $x_1 \neq 0$.

Recall that

$$|\langle x_1, Ax_1\rangle| = \sup_{\|y\|\leq 1} |\langle y, Ay\rangle|$$

• <u>Case 1:</u> $\langle x_1, Ax_1\rangle \geq 0$. Then:

$$\langle x_1, Ax_1\rangle = \sup_{\|y\|\leq 1} \langle y, Ay\rangle$$

Take $\varphi \perp x_1$, $\|\varphi\| = 1$ and consider

$$y_\varepsilon = \frac{x_1 + \varepsilon\varphi}{\sqrt{1+\varepsilon^2}}, \quad \|y_\varepsilon\|^2 = \frac{\|x_1\|^2 + \varepsilon^2}{1+\varepsilon^2} \leq 1$$

Then when $\varepsilon \in \mathbb{R}$

$$f(\varepsilon) = \langle y_\varepsilon, Ay_\varepsilon\rangle = \frac{\langle x_1+\varepsilon\varphi, A(x_1+\varepsilon\varphi)\rangle}{1+\varepsilon^2}$$

attains the maximum when $\varepsilon = 0$. This implies that

$$0 = \frac{d}{d\varepsilon} f(\varepsilon)\Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon}\left(\frac{1}{1+\varepsilon^2}\left(\langle x_1, Ax_1\rangle + \varepsilon^2\langle\varphi, A\varphi\rangle + 2\varepsilon \operatorname{Re}\langle x_1, A\varphi\rangle\right)\right)\Big|_{\varepsilon=0}$$

$$= \left[\frac{-2\varepsilon}{(1+\varepsilon^2)^2}(\dots) + \frac{1}{1+\varepsilon^2}\left(2\varepsilon\langle\varphi, A\varphi\rangle + 2\operatorname{Re}\langle x_1, A\varphi\rangle\right)\right]\Big|_{\varepsilon=0}$$

$= \ 2\,\mathrm{Re}\,\langle x_1, A\varphi\rangle$

Thus

$$\mathrm{Re}\,\langle x_1, A\varphi\rangle = 0, \quad \forall \varphi \perp x_1, \|\varphi\| = 1$$

$$\Rightarrow \quad \mathrm{Re}\,\langle x_1, A\varphi\rangle = 0, \quad \forall \varphi \perp x_1$$

Replacing φ by $i\varphi$

$$\Rightarrow \quad \mathrm{Re}\,\langle x_1, A(i\varphi)\rangle = 0, \quad \forall i\varphi \perp x_1$$

i.e. $\quad \mathrm{Im}\,\langle x_1, A\varphi\rangle = 0, \quad \forall \varphi \perp x_1.$

Conclusion: $\quad \langle x_1, A\varphi\rangle = 0, \quad \forall \varphi \perp x_1.$

Since A is self-adjoint:

$$\langle Ax_1, \varphi\rangle = \langle x_1, A\varphi\rangle = 0, \quad \forall \varphi \perp x_1$$

$$\Rightarrow \quad Ax_1 \perp \varphi, \quad \forall \varphi \perp x_1.$$

Thus: $\quad Ax_1 \in \left(\mathrm{Span}(x_1)^{\perp}\right)^{\perp} = \underset{\mathbb{C}x_1}{\underset{\shortparallel}{\mathrm{Span}(x_1)}}$

(You can also write

$$Ax_1 = \alpha_1 x_1 + x_1^{\perp} \leadsto x_1^{\perp} = 0$$

)

In summary, $\exists \lambda_1 \in \mathbb{C}$ s.t. $Ax_1 = \lambda_1 x_1.$

Why $\lambda_1 \in \mathbb{R}$?

From $\quad Ax_1 = \lambda_1 x_1 \Rightarrow \underbrace{\langle x_1, Ax_1\rangle}_{\in \mathbb{R}} = \lambda_1 \underbrace{\|x_1\|^2}_{\neq 0}$

$$\Rightarrow \lambda_1 \in \mathbb{R}.$$

·) Case 2: $\langle x_1, Ax_1\rangle \leq 0$. Then:

$$\langle x_1, Ax_1\rangle = \inf_{\|y\| \leq 1} \langle y, Ay\rangle$$

Then $\forall \varphi \perp x_1$:

$$f(\varepsilon) = \langle y_\varepsilon, Ay_\varepsilon\rangle, \qquad y_\varepsilon = \frac{x_1 + \varepsilon\varphi}{\sqrt{1+\varepsilon^2}}$$

attains the minimum at $\varepsilon = 0$

$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\varepsilon) = 0 \Rightarrow$ the same conclusion!

Step 3:

Lemma: If A is a self-adjoint operator, and $\langle x, Ax\rangle = 0$, $\forall x$, then $Ax = 0$, $\forall x$.

Proof of the lemma. We have:

$$0 = \langle x+y, A(x+y)\rangle = \underbrace{\langle x, Ax\rangle}_{=0} + \underbrace{\langle y, Ay\rangle}_{=0} + 2\mathrm{Re}\langle x, Ay\rangle$$

$$\Rightarrow \quad \mathrm{Re}\langle x, Ay\rangle = 0 \quad , \forall x, y$$

Replacing y by iy

$$\mathrm{Im}\langle x, Ay\rangle = 0, \quad \forall x, y$$

Thus $\langle x, Ay\rangle = 0, \ \forall x, y \Rightarrow Ay = 0, \ \forall y. \ \square$

Step 4: Consider the case when $\mathrm{Ker}\, A = \{0\}$. By

Step 1: $|\langle x_1, Ax_1\rangle| = \sup\limits_{x \in X, \|x\| \leq 1} |\langle x, Ax\rangle|$

From the above lemma, either $A = 0$ or

$$\sup_{\|x\| \leq 1} |\langle x, Ax\rangle| > 0.$$

$\Rightarrow x_1 \neq 0 \Rightarrow \|x_1\| = 1$ and

$Ax_1 = \lambda_1 x_1$ for some $\lambda_1 \in \mathbb{R}, \lambda_1 \neq 0.$

Observation: $A : \mathrm{Span}(x_1) \to \mathrm{Span}(x_1)$

and $A : \mathrm{Span}(x_1)^{\perp} \longrightarrow \mathrm{Span}(x_1)^{\perp}$

Define $A_1 : X_1 \to X_1, \quad X_1 = \mathrm{Span}(x_1)^{\perp}$

where $A_1 = A|_{X_1} = P_{X_1} A P_{X_1}$.

Thus A_1 is compact & self-adjoint operator on X_1

Applying Step 1 to $A_1 \Rightarrow \exists\, x_2 \in X_1 : \|x_2\| \leq 1$

and
$$|\langle x_2, A_1 x_2\rangle| = \sup_{\|x\|\leq 1,\ x\in X_1} |\langle x, A_1 x\rangle|$$

$$\Leftrightarrow \quad |\langle x_2, A x_2\rangle| = \sup_{\|x\|\leq 1,\ x\perp x_1} |\langle x, Ax\rangle| .$$

Since $\operatorname{Ker} A = \{0\} \Rightarrow \operatorname{Ker} A_1 = \{0\} \Rightarrow$

$x_2 \neq 0 \Rightarrow \|x_2\| = 1$, and

$$A x_2 = \lambda_2 x_2 \quad \text{for } \lambda_2 \in \mathbb{R},\ \lambda_2 \neq 0$$

Define $\quad X_2 = \operatorname{Span}(x_1, x_2)^{\perp}$ and

$$A_2 = A|_{X_2} = P_{X_2} A P_{X_2}$$

By Step 1, $\exists\, x_3 \in X_2$ s.t.
$$|\langle x_3, A x_3\rangle| = \sup_{\|x\|\leq 1,\ x\perp x_1, x_2} |\langle x, A x\rangle|.$$

Since $\operatorname{Ker} A = \{0\} \Rightarrow \operatorname{Ker} A_2 = \{0\} \Rightarrow$
$x_3 \neq 0 \Rightarrow \|x_3\| = 1$, and

$$Ax_3 = \lambda_3 x_3 \quad , \text{ for } \lambda_3 \in \mathbb{R},\ \lambda_3 \neq 0.$$

By induction, $\exists \{x_n\}$ ON family & $\{\lambda_n\} \subset \mathbb{R}$
s.t. $\quad A x_n = \lambda_n x_n, \ \forall n$

$$|\lambda_n| = |\langle x_n, Ax_n \rangle| = \sup_{\substack{\|x\| \leq 1 \\ x \perp x_1, \dots, x_{n-1}}} |\langle x, Ax \rangle|$$

<u>Why $\lambda_n \to 0$?</u>

From $\quad Ax_n = \lambda_n x_n$ and A is a compact operator, we get $\lambda_n \to 0$. More precisely, since $\{x_n\}$ ON family $\Rightarrow$ $x_n \to 0$ weakly (exercise). Since A is a compact operator, $Ax_n \to 0$ strongly. Thus

$$\underset{\downarrow 0}{Ax_n} = \lambda_n x_n \Rightarrow |\lambda_n| = \|\lambda_n x_n\| = \|Ax_n\| \to 0.$$

Why $\{x_n\}$ is an ONB:

Assume that $\{x_n\}$ is <u>not</u> an ONB, i.e.

$$\overline{\mathrm{Span}(x_n : n \in \mathbb{N})} \subsetneq X.$$

Thus

$$Y := \overline{\mathrm{Span}(x_n : n \in \mathbb{N})}^{\perp} \neq \{0\}.$$

But $\forall \varphi \in Y$, $\|\varphi\| \leq 1$, we have

$$|\lambda_n| = |\langle x_n, Ax_n \rangle| = \sup_{\substack{\|x\| \leq 1, \\ x \perp x_1, \dots, x_{n-1}}} |\langle x, Ax \rangle| \geq |\langle \varphi, A\varphi \rangle|$$

$$\downarrow$$

$$0$$

$\Rightarrow \langle \varphi, A\varphi \rangle = 0, \quad \forall \varphi \in Y.$

By the lemma in Step 3 $\Rightarrow$ $A = 0$ on Y

But it contradicts to the fact that $\mathrm{Ker} A = \{0\}$.

Thus $\{x_n\}$ is an ON Basis for X.

Step 5: In general, define

$$\tilde{A} = A\big|_{(\operatorname{Ker} A)^\perp} : (\operatorname{Ker} A)^\perp \to (\operatorname{Ker} A)^\perp$$

Then $\operatorname{Ker}(\tilde{A}) = \{0\}$ and $\tilde{A}$ is compact and self-adjoint. Thus $\exists \{y_n\}_{n \geq 1}$ ONB for $(\operatorname{Ker} A)^\perp$ s.t.

$$A y_n = \tilde{A} y_n = \lambda_n y_n \quad, \forall n$$

Take $\{z_n\}$ ONB for $\operatorname{Ker} A$:

$$A z_n = 0 = 0 \cdot z_n \quad, \forall n$$

Take $\{x_n\} = \{y_n\} \cup \{z_n\}$

$$\Rightarrow \quad A x_n = \lambda_n x_n , \forall n.$$

$$(\text{with } \lambda_n = 0 \text{ if } x_n \in \operatorname{Ker} A) \qquad \square$$

Remark: There is also "Spectral theorem for self-adjoint operators".

Thm: Let X be a separable Hilbert space. Let A be a bounded, self-adjoint operator on X.

Then $\exists$ measure space (Ω, μ) & a unitary transformation $U : X \to L^2(\Omega)$ s.t.

$$U A U^{-1} = \text{multiplication operator on } L^2(\Omega)$$

$$= M_a \text{ where}$$

$$(M_a f)(x) = a(x) f(x), \; \forall f \in L^2(\Omega)$$

and $a \in L^\infty(\Omega)$, a is real-valued.

The proof of the spectral theorem is more complicated.

<u>Remark:</u> A short form of the spectral theorem for self-adjoint compact operator is:

$$A = \sum_{n \geq 1} t_n |u_n\rangle\langle u_n|$$

where $|u_n\rangle\langle u_n|$ is the projection on u_n, i.e.

$$\left(|u_n\rangle\langle u_n| f \right) = \langle u_n, f \rangle u_n.$$

Here we used the "bra-ket" notation:

$$\begin{cases} |u\rangle \in X & \text{with } u \in X \\ \langle u| \in X^* & \text{with } u \in X \end{cases}$$

In general, $|u\rangle\langle v|$ is an operator on X:

$$\begin{aligned} (|u\rangle\langle v|) f &= |u\rangle\langle v| \cdot |f\rangle \\ &= |u\rangle \langle v, f\rangle \\ &= \langle v, f\rangle u . \end{aligned}$$

<u>Remark:</u> There is also a result for compact operators.

<u>Theorem:</u> Let X be a separable Hilbert space. Let A be a bounded, compact operator on X. Then: $\exists$ ONB $\{u_n\}$ and ONB $\{v_n\}$ of X, $\{\lambda_n\} \subset \mathbb{R}$, s.t. $\lambda_n \to 0$ and

$$A = \sum_{n \geq 1} \lambda_n |u_n\rangle\langle v_n| .$$

(If $A = A^*$, then $u_n = v_n$ & $A = \sum \lambda_n |u_n\rangle\langle u_n|$.)

Outline of the proof:

Heuristically, if $A = \sum_{n\geq 1} \lambda_n |u_n\rangle\langle v_n|$

$$\Rightarrow A^* = \sum_{n\geq 1} \lambda_n |v_n\rangle\langle u_n|$$

$$\Rightarrow A^*A = \Big(\sum_{n\geq 1} \lambda_n |v_n\rangle\langle u_n|\Big)\Big(\sum_{m\geq 1} \lambda_m |u_m\rangle\langle v_m|\Big)$$

$$= \sum_{m,n} \lambda_n \lambda_m |v_n\rangle \underbrace{\langle u_n, u_m\rangle}_{=\begin{cases}1 & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}} \langle v_m|$$

$$= \sum_n \lambda_n^2 |v_n\rangle\langle v_n|$$

Since A is compact $\Rightarrow$ A^*A is compact and $\forall x$: $\langle x, A^*Ax\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$

$\Rightarrow$ $A^*A \geq 0$ & A^*A is self-adjoint.

By the spectral theorem for compact & self-adjoint operators, we can write

$$A^*A = \sum_n \lambda_n^2 |v_n\rangle\langle v_n|$$

$\{v_n\}$ ONB, $\lambda_n^2 \geq 0$, $\lambda_n \to 0$.

Then we can find u_n by

$$A = \Sigma \lambda_n |u_n\rangle\langle v_n|$$

$$\Rightarrow A v_n = \lambda_n u_n$$

$$\Rightarrow \boxed{u_n = \frac{A v_n}{\lambda_n} \quad \text{if } \lambda_n \neq 0.}$$

This leads to a proof of the theorem.

Remark: A trivial example of compact operators is the finite-rank operators.

Thm: Let X be a Hilbert space. Let $A \in \mathcal{L}(X)$ s.t. A is finite-rank i.e.

$$\dim(AX) < \infty.$$

Then A is a compact operator.

Proof: We need to prove that

$$\overline{A(B(0,1))} \text{ is compact.}$$

It is obvious in our case, $\overline{A(B(0,1))}$ is a closed bounded set of AX and $\dim(AX) < \infty$.

<u>Theorem:</u> let X be a Hilbert space. Assume A_n is a compact operator $\forall n \geq 1$ and $A_n \to A$ in $\mathcal{L}(X)$, i.e. $\|A_n - A\|_{\mathcal{L}(X)} \to 0$. Then A is a compact operator.

Consequently: If A_n is finite-rank & $A_n \to A$ in $\mathcal{L}(X)$ then A is compact.

<u>Another example on Compact operators:</u>

<u>Def:</u> (Kernels of operators). Let (Ω, μ) be a measure space. Let $K(x,y): \Omega \times \Omega \to \mathbb{C}$.

Define operator K on $L^2(\Omega)$ by:

$$(Kf)(x) = \int_\Omega K(x,y)\, f(y)\, dy.$$

<u>Theorem.</u> Assume (Ω, μ) is sigma-finite (s.t. $L^2(\Omega)$ is separable). If $K(x,y) \in L^2(\Omega \times \Omega)$. Then the operator K on $L^2(\Omega)$ defined by

$$(Kf)(x) = \int_\Omega K(x,y)\, f(y)\, dy$$

is a compact operator on $L^2(\Omega)$.

(In this case, K is called a Hilbert-Schmidt operator)

<u>Proof:</u> Let $\{u_n\}$ be an ONB for $L^2(\Omega)$. Then $\{u_n(x)\, u_m(y)\}_{m,n\geq 1}$ is an ONB for $L^2(\Omega\times\Omega)$.

This allows us to write by Parseval identity

$$K(x,y) = \sum_{m,n\geq 1} K_{mn}\, u_n(x)\, u_m(y)$$

where $K_{m,n} = \langle u_m(x)\, u_n(y), K(x,y)\rangle \in \mathbb{C}$

and $$\int |K(x,y)|^2 dx dy = \sum_{m,n} |K_{mn}|^2.$$

Then: $\forall f \in L^2(\Omega)$

$$(Kf)(x) = \int \sum_{mn} K_{mn}\, u_n(x)\, u_m(y) f(y)\, dy$$

$$= \sum_{mn} K_{mn}\, u_n(x) \langle \bar{u}_m, f\rangle$$

$$\Rightarrow \quad K = \sum_{mn} K_{mn} |u_n\rangle\langle \bar{u}_m|$$

Define:

$$K_N := \sum_{n \le N} \sum_{m=1}^{\infty} K_{mn} \, |u_n\rangle\langle \bar{u}_m|$$

Note that

$$K_N L^2(\Omega) \subset \mathrm{Span}\{u_n : n \le N\}$$

$\Rightarrow$ K_N is finite-rank $\rightarrow$ compact.

Moreover,

$$\|(K-K_N)f\|^2 = \left\| \sum_{n>N} \sum_{m=1}^{\infty} K_{mn} \, u_n \langle \bar{u}_m, f\rangle \right\|^2$$

$$= \left\| \sum_{n>N} u_n \underbrace{\left(\sum_{m=1}^{\infty} K_{mn} \langle \bar{u}_m, f\rangle \right)}_{\in \mathbb{C}} \right\|^2$$

$$= \sum_{n>N} \left| \underbrace{\sum_{m=1}^{\infty} K_{mn} \langle \bar{u}_m, f\rangle}_{} \right|^2$$

$$\le \sum_{n>N} \left(\sum_{m=1}^{\infty} |K_{mn}|^2 \right) \underbrace{\left(\sum_{m=1}^{\infty} |\langle \bar{u}_m, f\rangle|^2 \right)}_{\|f\|^2}$$

$$\le \left(\sum_{n>N} \sum_{m=1}^{\infty} |K_{mn}|^2 \right) \|f\|^2$$

Thus $\|K - K_N\|_{\mathcal{L}(X)} \leq \sum_{n>N} \sum_{m=1}^{\infty} |K_{mn}|^2 \xrightarrow[N\to\infty]{} 0$

since $\sum_{n} \sum_{m} |K_{mn}|^2 < \infty$.

Therefore, K is a compact operator.