Chapter 4. Dual spaces and weak topologies

Recall: If X is a normed space, then

$$X^* = \mathcal{L}(X, \mathbb{C})$$

Heuristically, X^* is "nicer" than X.

① If X is a normed space, then X^* is always a Banach space.

② $$\|f\|_{X^*} = \sup_{\|x\| \leq 1} |f(x)| \quad , \forall f \in X^*$$

$$\|x\| = \sup_{\|f\|_{X^*} \leq 1} |f(x)| = \max_{\|f\|_{X^*} \leq 1} |f(x)| \quad \forall x \in X$$

(by Hahn-Banach theorem)

Q: Is there some maximizer for

$$\|f\|_{X^*} = \sup_{\|x\| \leq 1} |f(x)| \quad , \forall f \in X^* ?$$

In general, not! However, it is true if X is "reflexive" ($X = X^{**}$).

Given the Banach space X, we can define $\hat{X} \subset X^{**}$ as:

$$x \in X \to T_x \in X^{**} \quad \text{by}$$

$$T_x : X^* \to \mathbb{C}$$
$$f \to f(x)$$

Then: $X \simeq \hat{X} \subset X^{**}$

$\searrow$ isometry since

$$\|x\| = \sup_{\|f\|_{X^*} \leq 1} |f(x)| = \sup_{\|f\|_{X^*} \leq 1} |T_x(f)| = \|T_x\|_{X^{**}}$$

The fact that X is reflexive $\Leftrightarrow$ $\hat{X} = X^{**}$.

We will discuss the reflexive spaces later.

__Def__ (Weak convergence) Let X be a Banach space and $\{x_n\}_{n=1}^{\infty}$, x in X. Then we say

$$x_n \rightharpoonup x \qquad (x_n \text{ converges to } x \text{ weakly})$$

iff $f(x_n) \longrightarrow f(x)$, $\forall f \in X^*$.

<u>Remark:</u> If $x_n \to x$ in X, then:

$x_n \rightharpoonup x$ weakly.

However, the reverse is not correct.

<u>Example:</u> $X = \ell^p(\mathbb{N})$ where $1 < p < \infty$.

Denote $e_n = (0, 0, \dots, 1, 0, \dots)$
$\searrow$ n-th position

Then $\|e_n\|_{\ell^p} = 1 \Rightarrow e_n \not\to 0$ in norm.

However, we can prove that $e_n \rightharpoonup 0$ weakly.

Assume $\exists f \in X^*$ s.t. $f(e_n) \not\to 0$. Thus $\exists \varepsilon > 0$ and a subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ s.t.

$$|f(e_{n_k})| \geq \varepsilon, \ \forall k = 1, 2, \dots$$

Take $(a_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} a_n e_n \in \ell^p(\mathbb{N})$. Since $f \in X^*$, we know that $\exists C > 0$:

$$\underbrace{\left| f\left(\sum_n a_n e_n \right) \right|} \leq C \left\| \sum_n a_n e_n \right\|_{\ell^p}$$

$$\Rightarrow \quad \left| \sum_n a_n f(e_n) \right| \leq C \left(\sum_n |a_n|^p \right)^{1/p}$$

Since $|\rho(e_{n_k})| \geq \varepsilon$, choose a_{n_k} s.t.

$$a_{n_k} \rho(e_{n_k}) \geq \varepsilon |a_{n_k}|$$

Then:

$$\left| \rho\left(\sum_k a_{n_k} e_{n_k} \right) \right| \leq C \left(\sum |a_{n_k}|^p \right)^{1/p}$$

$$\begin{array}{c} \| \\ \left| \sum_k a_{n_k} \rho(e_{n_k}) \right| \\ \mathbin{\rotatebox[origin=c]{90}{$\leq$}} \\ \varepsilon \sum_k |a_{n_k}| \end{array}$$

$$\Rightarrow \quad \sum_k |a_{n_k}| \leq \frac{C}{\varepsilon} \left(\sum |a_{n_k}|^p \right)^{1/p}.$$

We get a contradiction by $a_{n_k} = \frac{1}{k} \longrightarrow$

LHS $= \infty$ while RHS $< \infty$ since $p > 1$.

<u>Remark:</u> If $\dim X < \infty$, then weak c.v. $\Leftrightarrow$ strong c.v.

When $\dim X = N$, we can find a basis $\{e_i\}_{i=1}^N$ s.t. $\forall x \in X$, we can write

$$x = \sum_{i=1}^{N} a_i(x)\, e_i$$

Thus:

$$\varphi : X \to \mathbb{C}^N$$

$$x \mapsto \left(a_i(x)\right)_{i=1}^{N} \quad \text{linear, bijective}$$

When $x_n \rightharpoonup x \Rightarrow f(x_n) \to f(x), \forall f \in X^*$

$$\Rightarrow a_i(x_n) \to a_i(x) \quad , \forall i = 1, \dots, N$$

$$\Rightarrow x_n = \sum_{i=1}^{N} a_i(x_n) e_i \to x = \sum_{i=1}^{N} a_i(x) e_i .$$

<u>Remark:</u> When $X = \ell^1(\mathbb{N})$, $\dim X = \infty$, but weak c.v. $\Leftrightarrow$ norm c.v. (come later)

<u>Remark:</u> Weak limit is unique! Namely if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$.

Indeed, $\forall f \in X^*$, then $f(x_n) \to f(x)$ and $f(x_n) \to f(y) \Rightarrow f(x) = f(y) \Rightarrow f(x-y) = 0$.

Then:
$$\|x-y\| = \sup_{\|f\| \le 1} |f(x-y)| = 0$$

$\Rightarrow x = y$.

Weak topology:

Recall: $(\underset{\swarrow \text{set}}{X}, \underset{\swarrow \text{collection of subsets of } X}{\mathcal{O}})$ is called a topological space if

- $\phi, X \in \mathcal{O}$
- $\bigcup_{i \in I \text{ arbitrary}} A_i \in \mathcal{O}$, $\forall A_i \in \mathcal{O}$
- $\bigcap_{i \in I \text{ finite}} A_i \in \mathcal{O}$, $\forall A_i \in \mathcal{O}$

Then $\mathcal{O}$ is called the collection of "open sets".

If $(X, \mathcal{O}_1)$ and $(X, \mathcal{O}_2)$ are two topological spaces, and $\mathcal{O}_1 \supset \mathcal{O}_2$, then we say that $\mathcal{O}_1$ is stronger (finer) than $\mathcal{O}_2$.

Remark: "Stronger" means $\exists$ more open sets

This implies that if $x_n \to x$ in $\mathcal{O}_1$, then $x_n \to x$ in $\mathcal{O}_2$.

i.e. $\exists$ less convergent sequences in $\mathcal{O}_1$ than $\mathcal{O}_2$

This also implies that if $f : (X, \mathcal{O}_2) \to \underset{\swarrow \text{top. space}}{Y}$

is continuous, then: $f : (X, \mathcal{O}_1) \to Y$ is continuous

[because f is continuous $\Leftrightarrow$ f^{-1}(open) is open]

Thus $\mathcal{O}_1$ is stronger than $\mathcal{O}_2$, then $\exists$ more continuous functions in $\mathcal{O}_1$ than $\mathcal{O}_2$.

Let X be a Banach space $\rightsquigarrow$ $(X, \mathcal{O}_{\|\cdot\|})$ is the strong topology (i.e. $\mathcal{O}_{\|\cdot\|} = \{$open sets in $\|\cdot\|\} = \{ \bigcup_{\text{arbitrary}}$ open balls $\}$).

The weak topology $\sigma(X, X^*)$ is the weakest topology in X s.t. $f \in X^*$ is continuous

$$(X, \sigma(X, X^*)) \longrightarrow \mathbb{C}.$$

<u>Construction of $\sigma(X, X^*)$:</u>

- $f^{-1}(A) \in \sigma(X, X^*)$, $\forall f \in X^*$, $\forall A$ open in $\mathbb{C}$
- $\sigma(X, X^*)$ is stable under $\bigcup_{\text{arbitrary}}$ & $\bigcap_{\text{finite}}$

<u>Def:</u>

$$\sigma(X, X^*) = \left\{ \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \{ f^{-1}(A) ; f \in X^*, A \overset{\text{open}}{\subset} \mathbb{C} \} \right\}$$

<u>Thm</u>: $\sigma(X, X^*)$ is a topology in X.

<u>Proof</u>: $\emptyset, X \in \sigma(X, X^*)$ ✓

$\sigma(X, X^*)$ is stable under $\bigcup_{\text{arbitrary}}$ ✓

<u>Why</u>: $\sigma(X, X^*)$ is stable under $\bigcap_{\text{finite}}$?

We prove that if $A, B \in \sigma(X, X^*)$, then $A \cap B \in \sigma(X, X^*)$. We can write:

$$A = \bigcup_{i \text{ arbitrary}} A_i, \quad B = \bigcup_{j \text{ arbitrary}} B_j,$$

A_i, B_j are $\bigcap_{\text{finite}} \{ f^{-1}(\omega) : f \in X^*, \omega \overset{\text{open}}{\subset} \mathbb{C} \}$

Then:

$$A \cap B = \Big(\bigcup_{i \text{ arbitrary}} A_i\Big) \cap \Big(\bigcup_{j \text{ arbitrary}} B_j\Big)$$
$$= \bigcup_{i,j \text{ arbitrary}} \underbrace{(A_i \cap B_j)}_{\bigcap_{\text{finite}}} \in \sigma(X, X^*)$$

□

<u>Remark</u>: $\bigcap_{\text{finite}} \bigcup_{\text{arbitrary}}$ does not work!

Theorem: Let X be a Banach space and $\{x_n\}_{n=1}^{\infty}$ and x in X. Then x_n converges to x in $\sigma(X, X^*)$ $\Leftrightarrow$ $x_n \to x$ weakly

(i.e. $f(x_n) \to f(x)$, $\forall f \in X^*$).

Proof: If $x_n \to x$ in $\sigma(X, X^*)$

$\Rightarrow$ since $f : (X, \sigma(X, X^*)) \to \mathbb{C}$ is continuous $\forall f \in X^*$

$\Rightarrow$ $f(x_n) \to f(x)$ in $\mathbb{C}$, $\forall f \in X^*$.

Reversely, assume $f(x_n) \to f(x)$ $\forall f \in X^*$. We prove $x_n \to x$ in $\sigma(X, X^*)$. Take an open set $U \in \sigma(X, X^*)$ and $x \in U$. We prove that $\exists N$: $x_n \in U$, $\forall n \geq N$.

Recall $U \in \sigma(X, X^*) \Rightarrow U = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} f^{-1}(\text{open})$

$\Rightarrow \exists V$ is $\bigcap_{\text{finite}} \{ f^{-1}(\omega) : f \in X^*, \omega \text{ open in } \mathbb{C} \}$

s.t. $x \in V$. We prove $\exists N$: $x_n \in V$, $\forall n \geq N$.

We write $V = \bigcap_{i,j \text{ finite}} f_i^{-1}(\omega_j) \ni x$

where $f_i \in X^*$, ω_j open in $\mathbb{C}$. Since

$\forall i,j$: $f_i(x_n) \xrightarrow{n\to\infty} f_i(x) \in \omega_j$ open $\mathbb{C}$

$\Rightarrow \exists N_{ij}$: $f_i(x_n) \in \omega_j$, $\forall n \geq N_{ij}$.

Define $N = \max_{ij} N_{ij}$, then:

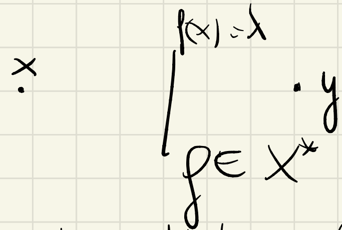
$$f_i(x_n) \in \omega_j, \forall n \geq N, \forall i,j$$

$\Rightarrow$ $x_n \in f_i^{-1}(\omega_j)$, $\forall n \geq N$, $\forall i,j$

$\Rightarrow$ $x_n \in \bigcap_{ij} f_i^{-1}(\omega_j) = V$ $\forall n \geq N$. $\square$

<u>Theorem:</u> Let X be a Banach space. Then $(X, \sigma(X, X^*))$ is a Hausdorff topological space, i.e. $\forall x \neq y \in X$, $\exists A_x, A_y$ open s.t. $x \in A_x$, $y \in A_y$, $A_x \cap A_y = \phi$.

Proof.



Since $x \neq y$, by Hahn-Banach theorem,

$\exists f \in X^*$ s.t. $f(x) < \lambda < f(y)$

Define

$$A_x = f^{-1}((-\infty, \lambda)) \ni x$$

$$A_y = f^{-1}((\lambda, \infty)) \ni y$$

$\Rightarrow A_x, A_y$ open since $f \in X^*$ continuous

$$A_x \cap A_y = f^{-1}((-\infty, \lambda)) \cap f^{-1}((\lambda, \infty))$$

$$= f^{-1}(\underbrace{(-\infty, \lambda) \cap (\lambda, \infty)}_{\phi}) = \phi.$$

Remark: In general, $(X, \sigma(X, X^*))$ is not metrizable, i.e. $\nexists$ metric $d : X \times X \to [0, \infty)$ s.t. $\sigma(X, X^*)$ is the same with the topology defined by the metric d.

(Homework)

Remark: Let X be a Banach space and $A \subset X$. If $[(\forall x_n \to x$ and $x_n \in A) \Rightarrow x \in A]$, then it is not enough to conclude that A is closed. (Example: exercise)

Theorem: Let X be a Banach space and let $\overline{B(0,1)}$ be the closed ball in $(X, \|\cdot\|)$. Then: $\overline{B(0,1)}$ is closed in $(X, \sigma(X, X^*))$.

In particular, if $x_n \to x$ weakly in X, then:

$$\|x\| \leq \liminf_{n \to \infty} \|x_n\|.$$

Proof: Let $x_n \to x$. Then: $\exists f_0 \in X^*, \|f_0\| \leq 1$

$$\|x\| = \sup_{\|f\|_{X^*} \leq 1} |f(x)| = |f_0(x)|$$

Then: $$|f_0(x)| = \liminf_{n \to \infty} \underbrace{|f_0(x_n)|}_{\leq \|f_0\| \|x_n\| \leq \|x_n\|}$$

$$\Rightarrow \|x\| \leq \liminf_{n \to \infty} \|x_n\|.$$

Consequence, if $x_n \in \overline{B(0,1)}$ and $x_n \to x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\| \leq 1 \Rightarrow x \in \overline{B(0,1)}$.

To prove that $\overline{B(0,1)}$ is closed in $\sigma(X, X^*)$, we write:

$$\overline{B(0,1)} = \{ x \in X : \|x\| \leq 1 \}$$

$$= \{ x \in X : \sup_{\|f\|_{X^*} \leq 1} |f(x)| \leq 1 \}$$

$$= \bigcap_{\|f\|_{X^*} \leq 1} \{ x \in X : |f(x)| \leq 1 \}$$

$$= \bigcap_{\|f\|_{X^*} \leq 1} \underbrace{f^{-1}\big(\overline{B_{\mathbb{C}}(0,1)}\big)}$$

closed in $\sigma(X, X^*)$ as $f : (X, \sigma(X, X^*)) \to \mathbb{C}$ is continuous $\square$

The $\bigcap_{\text{arbitrary}}$ (closed sets) is closed set in any topological space

$\Longleftrightarrow$ $\bigcup_{\text{arbitrary}}$ (open sets) is open set.

<u>Theorem</u>. Let X be a Banach space. Then the following statements are equivalent:

(a) X is inf-dim.

(b) $S(0,1) = \{ x \in X : \|x\| = 1 \}$ is <u>not</u> closed in $(X, \sigma(X,X^*))$. Indeed

$$\overline{S(0,1)}^{\sigma(X,X^*)} = \overline{B(0,1)}.$$

(c) $B(0,1)$ is <u>not</u> open in $(X, \sigma(X,X^*))$.

<u>Proof</u>: If $\dim X < \infty$, then $S(0,1)$ closed and $B(0,1)$ is open.

Assume that $\dim X = \infty$. We prove that

$$\overline{S(0,1)}^{\sigma(X,X^*)} = \overline{B(0,1)}.$$

Since $S(0,1) \subset \overline{B(0,1)}$ and $\overline{B(0,1)}$ is closed in $\sigma(X,X^*)$, then it suffices to show that

$$\overline{S(0,1)}^{\sigma(X,X^*)} \supset \overline{B(0,1)}.$$

Since $S(0,1) = \overline{B(0,1)} \setminus B(0,1)$, we only need

$$\overline{S(0,1)}^{\sigma(X,X^*)} \supset B(0,1).$$

Take $\|x_0\| < 1$ and prove $x_0 \in \overline{S(0,1)}^{\sigma(X,X^*)}$.

We prove that if $x_0 \in V^{\text{open}}$ in $\sigma(X,X^*)$, then $V \cap S(0,1) = \phi$.

As before, we can assume that

$$x_0 \in V = \bigcap_{\text{finite}} f_i^{-1}(\omega_j), \quad f_i \in X^*, \omega_j \overset{\text{open}}{\subset} \mathbb{C}$$

<u>Claim:</u> $\exists\, y_0 \neq 0$ s.t. $f_i(y_0) = 0, \forall i$ (why?)

Then: $g(t) = \|x_0 + t y_0\|$ continuous in $\|.\|$

and $g(0) = \|x_0\| < 1$, $g(t) \to \infty$ as $t \to \infty$

$\Rightarrow \exists\, t_0 : \|x_0 + t_0 y_0\| = 1$

but $f_i(x_0 + t_0 y_0) = f_i(x_0) \in \omega_j$

$\Rightarrow x_0 + t_0 y_0 \in V \cap S$

$\Rightarrow V \cap S(0,1) \neq \phi.$

Thus if $\dim X = \infty$, then

$$\overline{S(0,1)}^{\sigma(X,X^*)} = \overline{B(0,1)}$$

$\Rightarrow$ $S(0,1)$ is not closed in $\sigma(X,X^*)$.

Consequently,

$$B(0,1) = \underbrace{\overline{B(0,1)}}_{\text{closed in } \sigma(X,X^*)} \setminus \underbrace{S(0,1)}_{\text{not closed in } \sigma(X,X^*)}$$

$\rightsquigarrow$ $B(0,1)$ is not open in $\sigma(X,X^*)$ (why) $\square$

<u>Mazur lemma</u>: Let X be a Banach space. Let $A \subset X$ be a convex and closed subset of X. Then A is closed in $\sigma(X,X^*)$.

<u>Proof</u>: We prove $X \setminus A$ is open in $\sigma(X,X^*)$. Take $x_0 \in X \setminus A$. We find $x_0 \in U \subset X \setminus A$ s.t. U is open in $\sigma(X,X^*)$.

By Hahn-Banach theorem, $\exists f \in X^*$ s.t.

$$f(x_0) < \lambda < f(a), \ \forall a \in A.$$

Define $\quad U = f^{-1}(-\infty, \lambda) \to$ open in $\sigma(X, X^*)$.

and $\quad x_0 \in U \subset X \setminus A$. $\square$

<u>Corollary</u>: Let X be a Banach space. Assume $x_n \to x$ weakly in X. Then $\exists \{y_n\}$ in the convex combination of $\{x_n\}$ s.t. $y_n \to x$ strongly.

<u>Proof</u>:

<u>Recall</u>: A convex combination of $\{x_n\}$ is of the form

$$\sum_n \theta_n x_n \ , \quad 0 \leq \theta_n \leq 1, \ \sum_n \theta_n = 1$$

(finite sum)

Define $\quad A = \overline{\{\text{convex combinations of } \{x_n\}\}}^{\|\cdot\|}$

Then A is convex and closed in $\|\cdot\|$. Then by Mazur Lemma, A is closed in $\sigma(X, X^*)$.

On the other hand, $x_n \to x$ weakly in $\sigma(X, X^*)$ and $x_n \in A, \forall n \Rightarrow x \in A$.

Thus $\exists \{y_n\} \subset \{\text{convex combinations of } \{x_n\}\}$ s.t. $\quad y_n \to x$ in $\|\cdot\|$. $\square$

Thm (Eberlein – Smulien) Let X be a Banach space and $A \subset X$. Then TFAE:

(1) A is compact in $(X, \sigma(X, X^*))$, namely if $A \subset \bigcup_{i \in I} O_i$, with O_i open in $\sigma(X, X^*)$, then $A \subset \bigcup_{i \in I'} O_i$ with $I' \subset I$, $|I'| < \infty$.

(2) A is sequentially compact in $(X, \sigma(X, X^*))$, i.e. if $\{x_n\} \subset A$, then $\exists$ subsequence $\{x_{n_k}\}$ s.t. $x_{n_k} \to x$ in A.

(Homework / tutorial).

Weak-* topology:

Let X be a Banach space. Then we can define $X^* = \mathcal{L}(X, \mathbb{C})$ and $X^{**} = \mathcal{L}(X^*, \mathbb{C})$.

On X^*, we have 2 topologies:

- $(X^*, \|\cdot\|_*)$ strong topology

$$\|f\|_{X^*} = \sup_{\|x\|_X \leq 1} |f(x)|$$

- $(X^*, \sigma(X^*, X^{**}))$ weak topology

$f_n \to f$ weakly in $X^* \Leftrightarrow g(f_n) \to g(f)$
$\forall g \in X^{**}$.

Now we can define another topology, called weak-* topology $(X^*, \sigma(X^*, X))$.

Here

$$f_n \xrightarrow{*} f \text{ weakly-* in } X^*$$

$$\Leftrightarrow \quad f_n(x) \longrightarrow f(x), \quad \forall x \in X.$$

<u>Recall:</u> $X \hookrightarrow \hat{X} \subset X^{**}$ where

$$\hat{X} = \{ \varphi_x : x \in X \} \quad \text{and} \quad \varphi_x : X^* \to \mathbb{C}$$

$$\varphi_x(f) = f(x), \ \forall f \in X^*.$$

Def: The weak-* topology $\sigma(X^*, X)$ on X^* is the weakest topology, i.e. the topology with fewest open sets, such that all φ_x in $\hat{X}$ are continuous from $(X^*, \sigma(X^*, X)) \to \mathbb{C}$.

Recall: The weak topology $\sigma(X^*, X^{**})$ on X^* is the weakest topology s.t. all $g \in X^{**}$ are continuous from $(X, \sigma(X^*, X^{**})) \to \mathbb{C}$.

Thus: the weak-* topology is weaker than the weak topology, i.e.

$$\text{if } \ f_n \to f \ \text{ in } X^* \Rightarrow f_n \xrightarrow{*} f \ \text{ in } X.$$

$$(g(f_n) \to g(f), \forall g \in X^{**}) \qquad (f_n(x) \to f(x), \forall x \in X)$$

$$\varphi_x(f_n) \to \varphi_x(f), \forall x \in X$$

Moreover, if $X = X^{**}$ (more precisely $\hat{X} = X^{**}$) then weak & weak-* topology on X^* are the same. The property $X = X^{**}$ is called the reflexibility of X.

<u>Remark:</u> We can construct $\sigma(X^*, X)$ by

$$\sigma(X^*, X) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \{ \varphi_x^{-1}(A) : A \text{ open in } \mathbb{C},\ x \in X \}$$

<u>Example:</u> $X = c_0(\mathbb{N}) = \{ (x_n)_{n=1}^{\infty},\ x_n \to 0 \text{ as } n \to \infty \}$

Then $X^* \overset{(?)}{=} \ell^1(\mathbb{N}) = \{ (x_n)_{n=1}^{\infty} : \sum_n |x_n| < \infty \}$

$X^{**} \overset{(?)}{=} \ell^\infty(\mathbb{N}) = \{ (x_n)_{n=1}^{\infty} : \sup_n |x_n| < \infty \}$

We can see that the weak topology on $\ell^1(\mathbb{N})$ is really different from the weak-* topology.

Take $e_n = (0, 0, \dots, 1, 0, \dots) \in \ell^1(\mathbb{N})$

$1 \to$ n-th position

<u>Claim $e_n \xrightarrow{*} 0$:</u> Take $x \in X = c_0(\mathbb{N})$, we prove $e_n(x) \to 0$.

Here: $(\{f_n\})(\{x_n\}) = \sum_n f_n x_n$, $\forall \{f_n\} \in \ell^1$, $\{x_n\} \in c_0$

Then clearly $\forall x = (x_k)_{k=1}^{\infty} \in c_0(\mathbb{N})$

$$\underbrace{e_n}_{\in \ell^1(\mathbb{N})} (x) = x_n \to 0 \text{ as } n \to \infty .$$

Thus $e_n \xrightarrow{*} 0$ in $X^* = \ell^1(\mathbb{N})$.

<u>Claim:</u> $e_n \not\rightharpoonup 0$ in X^*.

Take $g = (1, -1, 1, -1, \dots) \in \ell^\infty(\mathbb{N}) = X^{**}$.

Then $g(e_n) = (-1)^n \not\to 0$ as $n \to \infty$

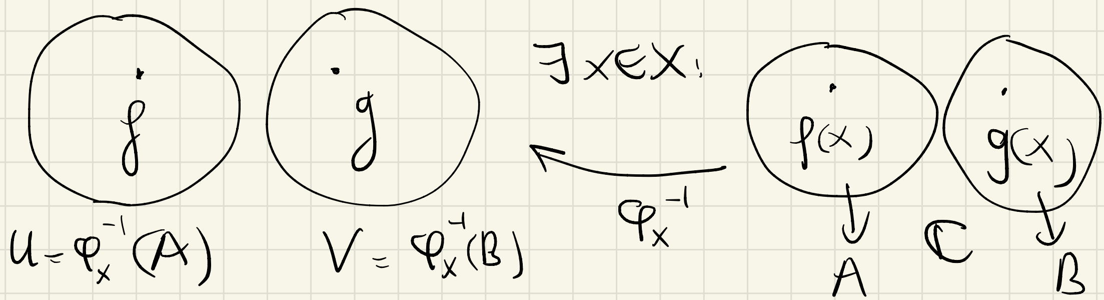
Thus $e_n \not\rightharpoonup 0$ in X^*.

<u>Theorem:</u> Let X be a Banach space. Then $(X^*, \sigma(X^*, X))$ is Hausdorff topological space. As a consequence, if

$$f_n \xrightarrow{*} f, \quad f_n \xrightarrow{*} g, \text{ then } f = g.$$

<u>Proof:</u> If $f_n \xrightarrow{*} f$ and $f_n \xrightarrow{*} g$, then:
$\forall x \in X$: $f_n(x) \to f(x)$ and $f_n(x) \to g(x)$ in $\mathbb{C}$ $\Rightarrow$ $f(x) = g(x)$ $\Rightarrow$ $f = g$ in X^*.

Assume $f \neq g$ in X^*. We find U, V open $\sigma(X^*, X)$ s.t. $f \in U$, $g \in V$, $U \cap V = \emptyset$.



Since $f \neq g$ in X^*, $\exists x \in X$: $f(x) \neq g(x)$ in $\mathbb{C}$.

Then $\exists A, B$ open in $\mathbb{C}$ s.t.

$$f(x) \in A, \quad g(x) \in B, \quad A \cap B = \emptyset$$

Define

$$U = \varphi_x^{-1}(A), \quad V = \varphi_x^{-1}(B), \quad \begin{pmatrix} \varphi_x(f) \\ = f(x) \end{pmatrix}$$

$\rightsquigarrow$ U & V open $\sigma(X^*, X)$ since

$$\varphi_x : \text{continuous } (X^*, \sigma(X^*, X)) \to \mathbb{C}$$

Moreover, $f \in U$, $g \in V$ and $U \cap V = \varphi_x^{-1}(A) \cap \varphi_x^{-1}(B)$

$$= \varphi_x^{-1}(\underbrace{A \cap B}_{\emptyset}) = \emptyset.$$

$\square$

Theorem: Let X be a Banach space. Assume

$$f_n \xrightarrow{*} f \quad \text{in } X^*. \quad \text{Then}$$

$\{f_n\}$ is bounded in $(X^*, \|\cdot\|_{X^*})$ and

$$\liminf_{n\to\infty} \|f_n\|_{X^*} \geqslant \|f\|_{X^*}.$$

Proof. Recall $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}(X, \mathbb{C}) = X^*$ and $\forall x \in X$, then:

$$f_n(x) \longrightarrow f(x) \quad \text{by weak * c.v.}$$

$$\Rightarrow \quad \sup_n |f_n(x)| < \infty, \ \forall x \in X$$

$$\Rightarrow \quad \sup_n \|f_n\|_{X^*} < \infty$$

by the Uniform boundedness Principle.

Then, take a subsequence of $\{f_n\}$ s.t.

$$\liminf_{n\to\infty} \|f_n\|_{X^*} = \lim_{k\to\infty} \|f_{n_k}\|_{X^*}.$$

We have:

$$\begin{cases} f_{n_k}(x) \longrightarrow f(x) \\ |f_{n_k}(x)| \leq \|f_{n_k}\|_{X^*} \|x\| \end{cases}$$

Thus $|f(x)| = \lim_{k\to\infty} \underbrace{|f_{n_k}(x)|}_{\leq \|f_{n_k}\|_{X^*} \|x\|}$

$$\leq \lim_{k\to\infty} \|f_{n_k}\|_{X^*} \|x\|$$

$$= \left(\liminf_{n\to\infty} \|f_n\|_{X^*}\right) \|x\|, \quad \forall x \in X$$

$$\Rightarrow \|f\|_{X^*} = \sup_{\|x\|\leq 1} |f(x)| \leq \liminf_{n\to\infty} \|f_n\|_{X^*}.$$

<u>Theorem:</u> (Banach - Alaoglu) let X be a Banach Space. Then $\overline{B(0,1)}^{\|\cdot\|_{X^*}} \subset X^*$ is a compact set in $(X^*, \sigma(X^*, X))$.

Moreover, if X is separable, then $\overline{B(0,1)}^{\|\cdot\|_{X^*}}$ is sequentially compact, i.e. $\forall \{f_n\}_{n=1}^{\infty}$ bounded in X^*, $\exists$ subsequence $\{f_{n_k}\}$ s.t.

$$f_{n_k} \xrightarrow{*} f \quad \text{in } X^*.$$

Remark: The result is in general not correct for the weak topology. Here the weak-* topology is weaker than the weak topology $\rightsquigarrow$ it has fewer open sets $\rightsquigarrow$ it has more compact sets.

Remark: In general, the compactness in a topological space is different from the sequential compactness. The important point here is that if X is separable, then

$$\left(\overline{B(0,1)}^{\|\cdot\|_{X^*}}, \ \sigma(X^*, X) \right) \quad \text{is}$$

metrizable. (Exercise)

Proof of the Banach-Alaoglu theorem when X is separable: Take $\{f_n\}$ bounded in X^*. We prove that $\exists$ subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \xrightarrow{*} f$

i.e. $$f_{n_k}(x) \longrightarrow f(x), \forall x \in X.$$

Take $\{x_n\}_{n=1}^{\infty}$ be dense in X. We find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ s.t. $\{f_{n_k}(x_\ell)\}_k$ converges, $\forall \ell$.

For $\ell = 1$: $\{f_n(x_1)\}_n$ is bounded in $\mathbb{C}$

$\Rightarrow$ $\exists$ a subsequence $\{f_{a_1(n)}\}_n$ of $\{f_n\}$ s.t.

$$f_{a_1(n)}(x_1) \longrightarrow \underbrace{f(x_1)}_{\text{limit}} \text{ as } n \to \infty.$$

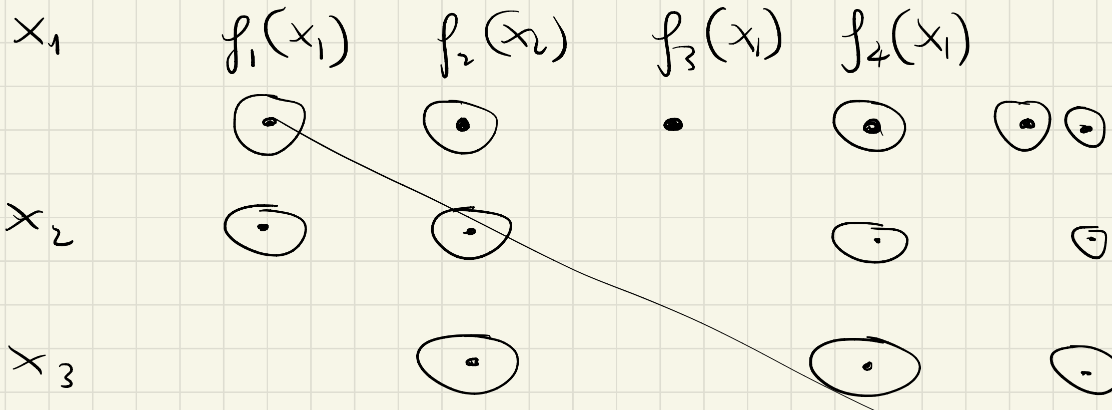
For $\ell = 2$: $\{f_{a_1(n)}(x_2)\}_n$ b.d. in $\mathbb{C}$

$\Rightarrow$ $\exists$ a subsequence $\{f_{a_2(n)}\}_n$ of $\{f_{a_1(n)}\}_n$ s.t.

$$f_{a_2(n)}(x_2) \longrightarrow f(x_2).$$

$\vdots$

By induction, $\forall k$, $\exists$ $\{f_{a_{k+1}(n)}\}_n$ subsequence of $\{f_{a_k(n)}\}_n$ s.t. $f_{a_{k+1}(n)}(x_{k+1}) \xrightarrow{n\to\infty} f(x_{k+1})$



By Cantor's diagonal argument

↓

Consider $\{f_{a_n(n)}\}_n$ a subsequence $\{f_n\}$

(a subsequence of $\{f_{a_k(n)}\}_k$)

Then
$$f_{a_n(n)}(x_i) \xrightarrow{n\to\infty} f(x_i), \quad \forall i = 1, 2, \dots$$

Since $\{x_n\}_{n=1}^{\infty}$ is dense in X, we can define $f : X \to \mathbb{C}$ from $\{f(x_n) : n = 1, 2 \dots\}$ by the continuity. More precisely, $\forall y \in X$, $\exists \{y_n\} \subset \{x_n\}_{n=1}^{\infty}$ s.t. $y_n \to y$ in X.

Then $\{f(y_n)\}$ is a Cauchy sequence in $\mathbb{C}$, indeed:

$$|f(y_i) - f(y_j)| \leq |f(y_i) - f_{a_n(n)}(y_i)|$$

$$+ |f_{a_n(n)}(y_i) - f_{a_n(n)}(y_j)| + |f_{a_n(n)}(y_j) - f(y_j)|$$

$$\leq |f(y_i) - f_{a_n}(y_i)| + \underbrace{\|f_{a_n(n)}\|_{X^*}}_{\leq C} \|y_i - y_j\|$$

$$+ |f_{a_n}(y_j) - f(y_j)|$$

$$\overset{n\to\infty}{\leq} C \|y_i - y_j\| \to 0 \quad \text{as } i,j \to \infty$$

since $\{y_i\}$ is Cauchy in X (as $y_j \to y$).

Thus $\{f(y_i)\}_{i=1}^{\infty}$ is Cauchy in $\mathbb{C}$ and $\mathbb{C}$ complete

$\Rightarrow \exists$ the limit $\lim_{j\to\infty} f(y_j) =: f(y)$.

We can see that

$$f_{a_n(n)}(y) \longrightarrow f(y), \quad \forall y \in X.$$

Indeed, take $\{y_n\} \subset \{x_n\}_{n=1}^{\infty}$ s.t. $y_n \to y$, then

by the triangle inequality

$$|f_{a_n(n)}(y) - f(y)| \leq |f_{a_n(n)}(y_j) - f_{a_m(m)}(y_j)|$$
$$+ |f_{a_m(m)}(y_j) - f_{a_m(m)}(y_i)| + |f_{a_m(m)}(y_i) - f(y_i)|$$
$$+ |f(y_i) - f(y)|$$
$$\leq |f_{a_n(n)}(y_j) - f_{a_m(m)}(y_j)| + \underbrace{\|f_{a_m(m)}\|_{X^*}}_{\leq C} \|y_j - y_i\|$$
$$+ |f_{a_m(m)}(y_i) - f(y_i)| + |f(y_i) - f(y)|$$
$$\overset{m \to \infty}{\leq} |f_{a_n(n)}(y_j) - f(y_j)| + C \|y_j - y_i\|$$
$$+ |f(y_i) - f(y)|$$
$$\overset{n \to \infty}{\leq} C \|y_j - y_i\| + \|f(y_i) - f(y)\|$$
$$\overset{i,j \to \infty}{\leq} 0$$
$$\Rightarrow \limsup_{n \to \infty} |f_{a_n(n)}(y) - f(y)| \leq 0$$
$$\Rightarrow f_{a_n(n)}(y) \longrightarrow f(y) \quad , \forall y \in X.$$

<u>Conclusion:</u> We proved $\exists$ a subsequence $\{f_{a_n(n)}\}$ of $\{f_n\}_{n=1}^{\infty}$ in X^* s.t.

$$f_{a_n(n)}(y) \longrightarrow f(y) \quad , \forall y \in X$$

Here $f : X \to \mathbb{C}$. Now we prove that $f \in X^*$.

First, $f_{a_n(n)}$ is linear $\forall n \Rightarrow f$ is linear.

Second,

$$|f(y)| = \lim_{n\to\infty} \underbrace{|f_{a_n(n)}(y)|}$$

$$\leq \|f_{a_n(n)}\|_{X^*} \|y\|$$

$$\leq C \|y\| \quad , \forall y \in X.$$

$\Rightarrow f$ is continuous.

Thus: $f_{a_n(n)} \xrightarrow{*} f$ in X^*.

Proof of the Banach - Alaoglu theorem (general case)

Theorem (Tychonoff): If $\{X_i\}_{i\in I}$ is a collection of topological spaces, and X_i is compact for any $i \in I$. Then the product space

$$Y = \prod_{i\in I} X_i = \{ (x_i)_{i\in I} : x_i \in X_i \}$$

with the product topology is compact.

(The proof requires Axiom of Choice / Zorn Lemma)

Now we define

$$\tilde{Y} = \mathbb{C}^X = \{ f : X \to \mathbb{C} \}$$
$$= \prod_{x\in X} \mathbb{C}$$

$$Y = \prod_{x\in X} D_x \quad , \quad D_x = \{ z \in \mathbb{C} : |z| \leq \|x\| \}$$

Thus $\forall x \in X$, D_x is compact $\Rightarrow$ Y is compact with the product topology by Tychonoff theorem.

Define $\Phi : X^* \to \tilde{Y}$ by trivial embedding

$$f \mapsto \Phi(f) = f : X \to \mathbb{C}.$$

Then $\Phi(X^*) \subset \tilde{Y}$ and Φ is bijective from $X^* \to \Phi(X^*)$. Moreover, Φ, Φ^{-1} are continuous:

$$\begin{cases} \text{in } \tilde{Y}, & f_n \to f \iff f_n(x) \to f(x), \forall x \in X. \\ \text{in } X^*, & f_n \xrightarrow{*} f \iff f_n(x) \to f(x), \forall x \in X. \end{cases}$$

<u>Claim</u>: $\Phi\left(\overline{B(0,1)}^{\|\cdot\|_{X^*}}\right)$ is a compact set in $\tilde{Y}$. This implies that $\overline{B(0,1)}^{\|\cdot\|_{X^*}}$ is compact in X^* with the weak-$*$ topology because Φ^{-1}(compact) is compact.

Write $\Phi\left(\overline{B(0,1)}^{\|\cdot\|_{X^*}}\right) = K_1 \cap K_2$

$$K_1 = \{ f : X \to \mathbb{C} : \ |f(x)| \le \|x\|, \forall x \in X \}$$

$$K_2 = \{ f : X \to \mathbb{C} : \ f(x+y) = f(x) + f(y) \text{ and } f(\lambda x) = \lambda f(x), \forall x, y \in X, \forall \lambda \in \mathbb{C} \}$$

Note that $K_1 = \{ f : X \to \mathbb{C} : |f(x)| \leq \|x\|, \forall x \in X \}$

$$= \prod_{x \in X} D_x \quad , \quad D_x = \{ z \in \mathbb{C} ; |z| \leq \|x\| \}$$

is compact in $\tilde{Y}$

We prove that K_2 is closed in $\tilde{Y}$.

$$K_2 = \{ f : X \to \mathbb{C} ; \quad f(x+y) = f(x) + f(y), \; f(\lambda x) = \lambda f(x), \forall x, y \in X, \forall \lambda \in \mathbb{C} \}$$

$$= \bigcap_{\substack{x, y \in X \\ \lambda \in \mathbb{C}}} \{ f : X \to \mathbb{C} : |f(x+y) - f(x) - f(y)| + |f(\lambda x) - \lambda f(x)| = 0 \}$$

We see that $\forall x, y \in X, \forall \lambda \in \mathbb{C}$, then

$$\{ f : X \to \mathbb{C} : |f(x+y) - f(x) - f(y)| + |f(\lambda x) - \lambda f(x)| = 0 \}$$

is closed in $\tilde{Y}$ because it is $g^{-1}(\{0\})$ where

$g : \tilde{Y} \to \mathbb{R}$ defined by

$$f \longmapsto g(f) = |f(x+y) - f(x) - f(y)| + |f(\lambda x) - \lambda f(x)|$$

and g is continuous. Thus $K_2 = \bigcap$ closed set

$\Rightarrow K_2$ is closed.

Thus $\Phi\left(\overline{B(0,1)}^{\|\cdot\|_{X^*}}\right) = \underbrace{K_1}_{\text{compact}} \cap \underbrace{K_2}_{\text{closed}}$

is compact $\Rightarrow$ $\overline{B(0,1)}^{\|\cdot\|_{X^*}} = \Phi^{-1}$ (a compact set)

is compact in X^* with the weak-$*$ topology. $\square$

<u>Recall:</u>

- On X with weak topology $\sigma(X, X^*)$, we do not have the compactness of $\overline{B(0,1)}^{\|\cdot\|_X}$.

  On X^* with weak $*$ topology $\sigma(X^*, X)$, we have the compactness of $\overline{B(0,1)}^{\|\cdot\|_{X^*}}$.
- On X, if $A \subset X$ is weakly compact

$\Leftrightarrow$ A is sequentially compact (Eberlein-Šmulian theorem, exercise / tutorial)

On X^*, if $A \subset X^*$ is weakly $*$ compact then $\not\Rightarrow$ A is sequentially compact.

(but if X is separable, then we have "$\Rightarrow$").

Reflexive spaces:

Recall if X is a Banach space, then

$$X \hookrightarrow \hat{X} = \{ T_x : X^* \to \mathbb{C} ; T_x(f) = f(x), \forall f \in X^* \}$$

and $\hat{X} \subset X^{**}$.

<u>Def:</u> X is reflexive if $X = X^{**}$ (or more precisely $\hat{X} = X^{**}$).

<u>Examples:</u>

- $\ell^p(\mathbb{N})$ or $L^p(\Omega)$ with $1 < p < \infty$ are reflexive.
- $\ell^p(\mathbb{N})$ or $L^p(\Omega)$ with $p = 1$ or $p = \infty$ are not reflexive.

$C(K)$ with K compact set is <u>not</u> reflexive.

<u>Motivation:</u>

- If X is reflexive, then on X^*, the weak topology $\sigma(X^*, X^{**})$ and the weak* topology $\sigma(X^*, X)$ are <u>the same</u>.
- In this case, we obtain the compactness of the weak topology and also the sequential

compactness.

The reverse is also correct.

<u>Theorem</u>: Let X be a Banach space. Then TFAE:

(i) X is reflexive.

(ii) $\overline{B(0,1)}^{\|\cdot\|_X}$ is weakly compact. (Kakutani)

(iii) $\overline{B(0,1)}^{\|\cdot\|_X}$ is weakly sequentially compact

i.e. if $\{x_n\}_{n=1}^{\infty} \subset \overline{B(0,1)}$, then $\exists$ a subsequence

$$x_{n_k} \to x \text{ weakly in } \overline{B(0,1)}.$$

(More generally, (iii) $\Leftrightarrow$ $\forall \{x_n\}_{n=1}^{\infty}$ bounded in X, $\exists$ a subsequence $x_{n_k} \to x$ weakly).

<u>Proof</u>:

(i) $\Rightarrow$ (ii) Assume that X is reflexive, i.e.

$$X \simeq \hat{X} = X^{**}$$

Thus $\overline{B(0,1)}$ in X is the same $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$ which is compact with the weak $*$ topology $\sigma(X^{**}, X^{*})$ by the Banach–Alaoglu theorem. But $X = X^{**}$, hence $\sigma(X^{**}, X^{*}) = \sigma(X, X^{*}) =$ the weak topology in X. Thus $\overline{B(0,1)}$ is compact in the weak topology $\sigma(X, X^*)$.

(i) $\Rightarrow$ (iii) First, assume that X is separable. Then X^* is also separable. By the Banach-Alaoglu theorem, $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$ is sequentially compact with the weak-* topology $\sigma(X^{**}, X^*)$. Since $X = X^{**} \Rightarrow \overline{B(0,1)}^{\|\cdot\|_X} = \overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$ is sequentially compact with the weak topology $\sigma(X, X^*) = \sigma(X^{**}, X^*)$.

In general, if X is not separable, then take $\{x_n\}_{n=1}^{\infty} \subset \overline{B(0,1)}^{\|\cdot\|_X}$. Define

$$\tilde{X} = \overline{\operatorname{Span}\{x_n\}_{n=1}^{\infty}}^{\|\cdot\|_X} \subset X$$

closed subspace

Then $\overline{B(0,1)}^{\|\cdot\|_{\tilde{X}}} \subset \overline{B(0,1)}^{\|\cdot\|_X}$

closed in strong topology

& convex $\Longrightarrow$ closed in weak topology

by Mazur lemma

By (i) $\Rightarrow$ (ii) i.e. $\overline{B(0,1)}^{\|\cdot\|_X}$ is weakly compact

$\Rightarrow$ $\overline{B(0,1)}^{\|\cdot\|_{\tilde{X}}}$ is weakly compact

$\Rightarrow$ $\overline{B(0,1)}^{\|\cdot\|_{\tilde{X}}}$ is weakly sequentially compact

($\tilde{X}$ is separable) $\Rightarrow$ $\exists$ a subsequence $x_{n_k} \to x$ weakly.

Note: By the same argument, we can prove (ii) $\Rightarrow$ (iii)

(ii) $\Rightarrow$ (i) We will need a key Lemma.

<u>Lemma (Goldstine)</u> Let X be a Banach space and let $T: X \to X^{**}$, $T_x(f) = f(x)$, $\forall f \in X^*$.

Then: $T(X)$ is dense in X^{**} with the weak-$*$ topology, i.e. $\phi \neq V \subset X^{**}$ open in weak-$*$ topo. then $T(X) \cap V \neq \phi$.

In particular, $T\left(\overline{B(0,1)}^{\|\cdot\|_X}\right)$ is dense $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$

<u>Proof:</u> Let $\phi \neq V \subset X^{**}$ be open in weak $*$ topology. Then we prove that $T(X) \cap V \neq \phi$.

By definition of the weak $*$ topology

$$V = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} \{ \xi \in X^{**} : \xi(\underset{f \in X^*}{f}) \in \underset{\text{open in } \mathbb{C}}{\omega} \}$$

For our purpose, we can assume that

$$V = \bigcap_{i=1}^{K} \{ \xi \in X^{**} : \xi(f_i) \in \omega_i \}$$

where $f_i \in X^*$ and ω_i open ball in $\mathbb{C}$.

Then $T(X) \cap V \neq \phi \iff \exists x \in X : T_x \in V$

$\iff \exists x \in X : \quad T_x(f_i) = f_i(x) \in \omega_i \;, \forall i = 1,..K$

Define $\varphi : X \to \mathbb{C}^K$ by

$$x \mapsto \varphi(x) = \left(f_i(x)\right)_{i=1}^{K}$$

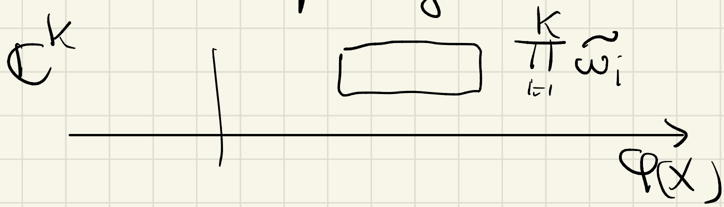
$\leadsto \varphi$ is linear and continuous. Then the fact

$\exists x \in X : f_i(x) \in \omega_i \;, \; \forall i = 1, \dots, K$

$\iff \exists x \in X : \varphi(x) \in \prod_{i=1}^{K} \omega_i$.

Assume by contradiction that

$$\underbrace{\varphi(X)}_{\text{closed subspace of } \mathbb{C}^k} \cap \underbrace{\prod_{i=1}^{K} \omega_i}_{\text{open, convex}} = \phi .$$



By the Hahn-Banach theorem, $\exists \beta \in \mathbb{C}^k$ s.t.

$$\underbrace{\beta \cdot \varphi(x)}_{\sum_{i=1}^{k} \beta_i f_i(x)} < \underbrace{\beta \cdot y}_{\sum_{i=1}^{k} \beta_i \cdot y_i},$$

$\forall x \in X$, $\forall y = (y_i) \in \prod_{i=1}^{k} \omega_i$

This implies that $\beta \cdot \varphi(x) = 0$, $\forall x \in X$

and $\beta \cdot y > 0$, $\forall y \in \prod_{i=1}^{k} \omega_i$.

From $\beta \cdot \varphi(x) = 0$, $\forall x \in X$

$$\Rightarrow \sum_{i=1}^{k} \beta_i f_i(x) = 0, \forall x \in X$$

$$\Rightarrow \underbrace{\left(\sum_{i=1}^{k} \beta_i f_i\right)}_{X^*}(x) = 0, \forall x \in X$$

$$\Rightarrow \sum_{i=1}^{k} \beta_i f_i = 0 \quad \text{in } X^*.$$

Because $V \neq \emptyset \Rightarrow \exists \xi \in V : \xi(f_i) \in \omega_i$, $\forall i$

$$\Rightarrow \sum_{i=1}^{k} \beta_i \cdot \xi(f_i) > 0$$

contradiction

$$\Rightarrow \xi\left(\sum_{i=1}^{k} \beta_i f_i\right) > 0 \rightsquigarrow \sum_{i=1}^{k} \beta_i f_i = 0.$$

The conclusion "$T\left(\overline{B(0,1)}^{\|\cdot\|_X}\right)$ is dense in $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$" is left as an exercise.

<u>Conclusion of (ii) $\Rightarrow$ (i)</u>

The mapping $T: X \to X^{**}$ is continuous from the weak topology in X to the weak* topology in X^{**}.

For example, if $x_n \rightharpoonup x$ in X

$$\Rightarrow f(x_n) \to f(x), \forall f \in X^*$$

$$\Rightarrow T_{x_n}(f) \to T_x(f), \forall f \in X^*$$

$$\Rightarrow T_{x_n} \xrightarrow{*} T_x \text{ in } X^{**}.$$

More generally, we prove that if V is open in X^{**} with the weak * topology, then $T^{-1}(V)$ is open in X with the weak topology.

In fact, by definition of the weak * topology,

$$V = \underbrace{\bigcup}_{\text{arbitrary}} \underbrace{\bigcap}_{\text{finite}} \{\xi \in X^{**} : \xi(\underbrace{f}_{f \in X^*}) \in \underbrace{\omega}_{\text{open in } \mathbb{C}}\}$$

Then

$$T^{-1}(V) = \bigcup_{\text{arbitrary}} \bigcap_{\text{finite}} T^{-1}\{\xi \in X^{**} : \xi(\overset{\substack{f \in X^* \\ \nearrow}}{f}) \in \overset{\substack{\omega \\ \text{open } \mathbb{C}}}{\omega}\}$$

By definition of the weak topology in X, we can conclude that $T^{-1}(V)$ is open if $\forall f \in X^*$ and $\forall \omega$ open in $\mathbb{C}$, then

$$T^{-1}\{\xi \in X^{**} : \xi(f) \in \omega\} \text{ is open}$$

$$= \{x \in X : T_x(f) \in \omega\}$$

$$= \{x \in X : f(x) \in \omega\}$$

$$= f^{-1}(\omega)$$ (this is open in X, in both strong & weak topologies)

Now (ii), we know that $\overline{B(0,1)}$ is compact in the weak topology. And since

$$\underbrace{T : (X, \sigma(X, X^*)) \longrightarrow (X^{**}, \sigma(X^{**}, X^*))}_{\text{continuous}}$$

$\Rightarrow T(\overline{B(0,1)})$ is compact in $(X^{**}, \sigma(X^{**}, X^*))$

$\Rightarrow T(\overline{B(0,1)})$ is closed in $(X^{**}, \sigma(X^{**}, X^*))$

On the other hand, by Goldstine lemma, $T(\overline{B(0,1)})$ is dense in $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$ with the weak * topology. Thus:

$$T(\overline{B(0,1)}) = \overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$$

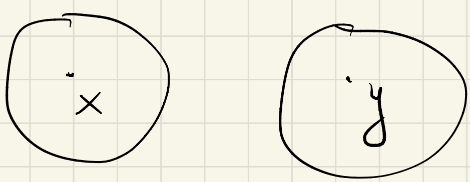
$$\Rightarrow \quad T(X) = X^{**}$$

$\Rightarrow$ X is reflexive.

The proof of (ii) $\Rightarrow$ (i).

The direction from (iii) $\Rightarrow$ (ii) or (i) is more difficult and left as an exercise. $\square$

<u>Remark:</u> In general, a compact set in a topological space is not necessarily closed. However, this is OK for Hausdorff spaces.



U_x $\qquad$ U_y

$\forall x \neq y$

$\exists\, U_x, U_y$ open

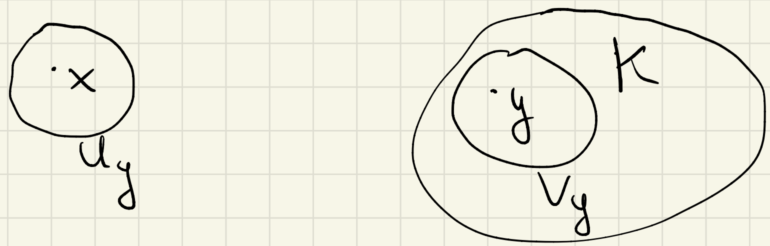
$x \in U_x$

$y \in U_y$

$U_x \cap U_y \neq \emptyset$

Lemma: Let X be a Hausdorff topological space. Then any compact set in X is closed.

Proof: Let $K \subset X$ be a compact set. We prove that $X \setminus K$ is open. Take $x \in X \setminus K$ and find an open set U: $x \in U \subset X \setminus K$.



$\forall y \in K$ then $x \neq y$. Since X is Hausdorff, $\exists$ U_y and V_y open sets such that

$$x \in U_y, \quad y \in V_y, \quad U_y \cap V_y = \emptyset.$$

Then:
$$K = \bigcup_{y \in K} \{y\} \subset \bigcup_{y \in K_y} V_y \quad \text{open}$$

Since K is compact, $\exists$ finite sets

$$K \subset \bigcup_{i=1}^{m} V_{y_i}$$

Define $U = \bigcap_{i=1}^{m} U_{y_i}$. Then U is open and $x \in U$, $U \cap K = \emptyset$ i.e. $U \subset X \setminus K$ $\square$

<u>Theorem:</u> Let X be a Banach space. Then

X is reflexive $\iff$ X^* is reflexive.

<u>Proof:</u> "$\Rightarrow$" Assume that X is reflexive. Then on X^*, the weak and weak$*$ topology are the same. But by the Banach Alaoglu theorem, $\overline{B(0,1)}^{\|.\|_{X^*}}$ is weakly $*$ compact. Thus $\overline{B(0,1)}^{\|.\|_{X^*}}$ is weakly compact. Then X^* is reflexive by Kakutani theorem.

<u>Remark:</u> There is a "fast proof"

$$X \text{ reflexive} \Rightarrow X = X^{**} \Rightarrow X^* = \underbrace{X^{***}}_{\nearrow (X^{**})^*} \underset{(?)}{=} (X^*)^{**}$$

$$\Rightarrow X^* \text{ is reflexive.}$$

"$\Leftarrow$" Assume X^* is reflexive. Then on X^{**}, the weak and and the weak$*$ topology are the same.

We know that $T: X \to X^{**}$, $T_x(f) = f(x)$, $\forall f \in X^*$ is a linear map and $T(X)$ is dense in X^{**}

with the weak* topology by Goldstine Lemma. Then $T(X)$ is dense in X^{**} with the weak topology. (why?)

However, $T(X)$ is convex, closed in the strong topology. Then by Mazur Lemma, $T(X)$ is closed in the weak topology. Thus $T(X) = X^{**}$, i.e. X is a reflexive space.

<u>Exercise:</u> Let X be a Banach space. Prove that $T(X)$ is closed in X^{**} with the strong topology.

<u>Remark:</u> If X is <u>not</u> reflexive, then:

$$X \simeq T(X) = \hat{X} \subsetneq \underset{\substack{\downarrow \\ \text{not reflexive}}}{X^{**}} \simeq \widehat{X^{**}} \subsetneq X^{****} \cdots$$

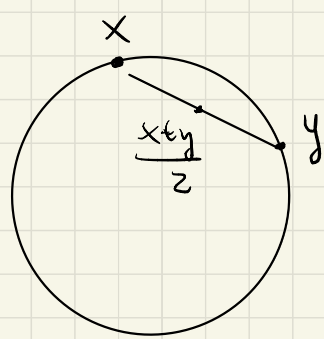
Exercise: Let X be a Banach space. Let M be a closed subspace of X. Prove that if X is reflexive, then M is reflexive.

Exercise: Let X be a Banach reflexive space. Let $A \subset X$ be a convex, closed set in the strong topology. Prove that A is compact in the weak topology.

Def (Uniformly convex spaces)

Let X be a Banach space. Then X is uniformly convex if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\sup_{\substack{\|x\| \leq 1, \|y\| \leq 1 \\ \|x-y\| \geq \varepsilon}} \left\| \frac{x+y}{2} \right\| \leq 1 - \delta$$



Note: By the triangle inequality, we only know that LHS ≤ 1.

Examples: • $\mathbb{R}^d$ with $|x| = \sqrt{\sum |x_i|^2}$, $x = (x_i)_{i=1}^d$ is uniformly convex for all $d \geq 2$.

• $\mathbb{R}^d$ with $|x|_1 = \sum_{i=1}^d |x_i|$, $x = (x_i)_{i=1}^d$

or $|x|_\infty = \max_{1 \leq i \leq d} |x_i|$

is not uniform convex.

• $\ell^p(\mathbb{N})$ is uniformly convex if $1 < p < \infty$

& not uniformly convex if $p = 1$ or $p = \infty$

Theorem (Milman - Pettis) Let X be a Banach space. If X is uniformly convex, then X is reflexive.

Remark: Note that the uniform convexity is a geometric property of the norm, i.e. it might happen that $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ are equivalent but $\|\cdot\|_1$ is uniform convex while $\|\cdot\|_2$ is not.

On the other hand, the reflexive property is a topological property, i.e if $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are equivalent, then $(X, \|\cdot\|_1)$ is reflexive $\iff$ $(X, \|\cdot\|_2)$ is reflexive.

<u>Remark</u>: There are examples of reflexive spaces s.t. $\nexists$ equivalent norm that is uniformly convex.

<u>Proof</u>: Recall: $T: X \to X^{**}$,

$$T_x(f) = f(x), \quad \forall x \in X, \ \forall f \in X^*.$$

We need to prove $T(X) = X^{**}$. Take $\xi \in X^{**}$ and we find $x \in X$ s.t. $\xi = T_x$. Assume $\|\xi\|_{X^{**}} = 1$

<u>Claim</u>: $\forall \varepsilon > 0$, $\exists x_\varepsilon \in X$ s.t. $\|x_\varepsilon\| \leq 1$ and

$$\|\xi - T_{x_\varepsilon}\|_{X^{**}} \leq \varepsilon.$$

Define: $\delta = \delta_\varepsilon = \inf\limits_{\substack{\|x\|, \|y\| \leq 1 \\ \|x - y\| \geq \varepsilon}} \left(1 - \left\|\frac{x+y}{2}\right\|_X\right) > 0.$

By def, $1 = \|\xi\|_{X^{**}} = \sup\limits_{\|f\|_{X^*} \leq 1} |\xi(f)| = \sup\limits_{\|f\|_{X^*} = 1} |\xi(f)|$

$\Rightarrow \quad \exists f \in X^*, \ \|f\|_{X^*} = 1, \ |\xi(f)| \geq 1 - \frac{\delta}{2}.$

Define:

$$\xi \in V = \{ \eta \in X^{**} : | \eta(f) - \xi(f) | < \delta/2 \}$$

$$= T_f^{-1}\left(B_{\mathbb{C}}\left(\xi(f), \frac{\delta}{2}\right)\right), \quad T_f : X^{*} \to (X^{*})^{**}$$

$$T_f(\eta) = \eta(f)$$

Since T_f is continuous from X^{**} with weak $*$ topology to $\mathbb{C}$ $\Rightarrow$ V is open in X^{**} with weak $*$ topology. Moreover, we know that $T(X)$ is dense in X^{**} with weak-$*$ topology

$$\rightsquigarrow \quad V \cap T(X) \neq \emptyset$$

$$\rightsquigarrow \quad \exists x \in X : \quad | T_x(f) - \xi(f) | < \delta/2$$

$$\Longleftrightarrow \quad | f(x) - \xi(f) | < \delta/2.$$

Actually, we can take $\|x\| \leq 1$ since $T\left(\overline{B(0,1)}^{\|\cdot\|_X}\right)$ is dense in $\overline{B(0,1)}^{\|\cdot\|_{X^{**}}}$.

We claim that $\| T_x - \xi \|_{X^{**}} \leq \varepsilon$. Assume that $\| T_x - \xi \|_{X^{**}} > \varepsilon$. Thus: $\xi \in \overline{B_{X^{**}}(T_x, \varepsilon)}^{c}$.

Exercise: $\overline{B(T_x, \varepsilon)}^{\|\cdot\|_{X^{**}}}$ is closed in X^{**} with the weak * topology.

Thus: $\xi \in \underbrace{V}_{\text{open}} \cap \underbrace{\left(\overline{B(T_x, \varepsilon)}^{\|\cdot\|_{X^{**}}}\right)^c}_{\text{open}}$ open in X^{**} with the weak* topology

$$\Rightarrow \quad V \cap \left(\overline{B(T_x, \varepsilon)}^{\|\cdot\|_{X^{**}}}\right)^c \cap \underbrace{T(X)}_{\text{dense in } X^{**}} \neq \emptyset$$

$$\Rightarrow \exists y : \quad T_y \in V \cap \left(\overline{B(T_x, \varepsilon)}^{\|\cdot\|_{X^{**}}}\right)^c$$

i.e. $\quad |f(y) - \xi(f)| < \delta/2$ and

$$\|T_y - T_x\|_{X^{**}} > \varepsilon \quad (\Leftrightarrow \|x-y\|_X > \varepsilon)$$

And we can also take $\|y\|_X \leq 1$. In summary,

$$\exists x, y \in X : \quad \|x\|, \|y\| \leq 1, \quad \exists f \in X^*, \|f\| = 1$$

$$|f(x) - \xi(f)| < \frac{\delta}{2}, \quad |f(y) - \xi(f)| < \frac{\delta}{2}$$

and $\quad \|x-y\|_X > \varepsilon, \quad |\xi(f)| > 1 - \dfrac{\delta}{2}$

$$|f(x) - \xi(f)| < \frac{\delta}{2}, \quad |f(y) - \xi(f)| < \frac{\delta}{2}$$

and $\|x-y\|_X > \varepsilon$, $\quad |\xi(f)| > 1 - \frac{\delta}{2}$

Consequently:

$$2\left(1-\frac{\delta}{2}\right) < 2|\xi(f)| \leq |f(x) - \xi(f)| + |f(y) - \xi(f)| + |f(x) + f(y)|$$

$$< \delta + |f(x+y)| \leq \delta + \underbrace{\|f\|}_{\leq 1} \|x+y\|$$

$$\Rightarrow \|x+y\| > 2 - 2\delta$$

$$\Rightarrow \left\|\frac{x+y}{2}\right\| > 1 - \delta \Leftrightarrow \delta > 1 - \left\|\frac{x+y}{2}\right\|$$

However, this contradicts the fact that

$$\delta = \inf_{\substack{\|x'\|, \|y'\| \leq 1 \\ \|x'-y'\| \geq \varepsilon}} \left\{ 1 - \left\|\frac{x'+y'}{2}\right\| \right\}$$

Thus $\forall \varepsilon > 0$, $\exists x \in X$, $\|x\| \leq 1$, $\|T_x - \xi\|_{X^{**}} \leq \varepsilon$.

Since $T(\overline{B(0,1)})$ is closed in $\overline{B(0,1)}^{\|.\|_{X^{**}}}$ with the strong topology and we have proved that $T(\overline{B(0,1)})$ is dense in $\overline{B(0,1)}^{\|.\|_{X^{**}}}$

$\rightsquigarrow$ $T(\overline{B(0,1)}) = \overline{B(0,1)}^{\|.\|_{X^{**}}}$

$\rightsquigarrow$ $\exists\, x_\xi \in \overline{B(0,1)}$ s.t. $Tx_\xi = \xi$.

__Exercise:__ Prove that $T\left(\overline{B(0,1)}^{\|.\|_X}\right)$ is closed in X^{**} with the strong topology. (Hint: $\|T_x\|_{X^{**}} = \|x\|_X$ & X is Banach space.)

__Exercise.__ Let X be a uniformly convex Banach space. Then TFAE:

(i) $x_n \to x$ strongly in X as $n \to \infty$

(ii) $x_n \rightharpoonup x$ weakly in X and $\|x_n\| \to \|x\|$.

__Exercise:__ Let X be a Banach space. Prove that TFAE: (i) X is uniformly convex.

(ii) $\forall$ sequences $\{x_n\}, \{y_n\}$, if $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\left\|\frac{x_n+y_n}{2}\right\| \to 1$, then $\|x_n - y_n\| \to 0$.