Chapter 3. Banach Spaces

<u>Def.</u> Let X be a normed vector space. Then X is a Banach space if X is complete (i.e. any Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X has a limit in X).

Examples

·) $\mathbb{R}$, $|x|$ = absolute value $\longrightarrow$ Banach space

$\mathbb{C} \simeq \mathbb{R}^2$ $\longrightarrow$ Banach space

$\mathbb{R}^d$, $|x|$ Eucl. $\longrightarrow$ Banach space

$(X, \|.\|)$ finite dimensional normed space $\longrightarrow$ Banach space (why?)

·) Let K be a compact metric space and consider $X = C(K, \mathbb{C}) = \{f : K \to \mathbb{C}, \text{ continuous}\}$ with the norm

$$\|f\|_\infty := \sup_{x \in K} |f(x)|.$$

Then: $C(K, \mathbb{C})$ with $\|f\|_\infty$ is a Banach space.

Proof. First, $X = C(K, \mathbb{C})$ is a vector space.

Moreover, $\|.\|_\infty$ is a norm in X;

$$\|f\|_\infty = \sup_{x \in K} |f(x)| \geq 0$$

$$\|f\|_\infty = 0 \iff f(x) = 0, \forall x \in K \iff f = 0$$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \quad (\text{why?})$$

$$\iff |f(x) + g(x)| \leq \|f\|_\infty + \|g\|_\infty, \forall x \in K.$$

This follows from the triangle inequality

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$
$$\leq \|f\|_\infty + \|g\|_\infty$$

Next, let us prove that $(X, \|.\|_\infty)$ is complete.

Take a Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $(X, \|.\|_\infty)$

We need to find a limit f s.t.

$$f_n \to f \quad \text{in} \quad (X, \|.\|_\infty).$$

By assumption, $\{f_n\}$ is a Cauchy sequence i.e.

$$\|f_n - f_m\|_\infty \to 0 \text{ as } m,n \to \infty$$

Consequently, $\forall x \in K$, then:

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \to 0 \text{ as } m,n\to\infty$$

$\Rightarrow \{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$

$\Rightarrow \exists \lim_{n\to\infty} f_n(x) =: f(x)$ (def of f)

<u>Why $f \in X = C(K,\mathbb{C})$</u>; i.e. why is f cont?

Take $x_n \to x$ in K. We prove that

$$f(x_n) \to f(x) \text{ in } \mathbb{C}.$$

We have: $\forall k, m$

$$|f(x_n) - f(x)| \leq |f(x_n) - f_k(x_n)| + \overbrace{|f_k(x_n) - f_m(x_n)|}^{\leq \|f_k - f_m\|_\infty}$$

$$+ |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|$$

$$|f(x_n) - f(x)| \leq |f(x_n) - f_k(x_n)| + \|f_k - f_m\|_\infty$$
$$+ |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|$$

Take $k \to \infty$

$$|f(x_n) - f(x)| \leq \limsup_{k \to \infty} \|f_k - f_m\|_\infty$$
$$+ |f_m(x_n) - f_m(x)| + |f_m(x) - f(x)|$$

Take $n \to \infty$

$$\limsup_{n \to \infty} |f(x_n) - f(x)| \leq \limsup_{k \to \infty} \|f_k - f_m\|_\infty$$
$$+ |f_m(x) - f(x)|$$

Take $m \to \infty$

$$\limsup_{n \to \infty} |f(x_n) - f(x)| \leq \underbrace{\limsup_{m \to \infty} \limsup_{k \to \infty} \|f_k - f_m\|_\infty}_{= 0}$$

because $\{f_m\}$ is a Cauchy sequence

Thus we conclude that

$f(x_n) \to f(x)$ as $n \to \infty \Rightarrow f$ is cont.

Finally, we prove that $f_n \to f$ in $\|\cdot\|_\infty$, namely

$$\sup_x |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

We have:

$$|f_n(x) - f(x)| \le |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$
$$\le \|f_n - f_m\|_\infty + |f_m(x) - f(x)|$$

Take $m \to \infty$

$$|f_n(x) - f(x)| \le \limsup_{m\to\infty} \|f_n - f_m\|_\infty$$

$$\Rightarrow \quad \underbrace{\sup_x |f_n(x) - f(x)|}_{= \|f_n - f\|_\infty} \le \limsup_{m\to\infty} \|f_n - f_m\|_\infty$$

Take $n \to \infty$

$$\limsup_{n\to\infty} \|f_n - f\|_\infty \le \underbrace{\limsup_{n\to\infty} \limsup_{m\to\infty} \|f_n - f_m\|_\infty}_{= 0}$$

because $\{f_n\}$ is Cauchy sequence.

Thus $f_n \to f$.

Thus we have $\| f_n - f \|_\infty \to 0$ as $n \to \infty$.

We conclude that $X = C(K, \mathbb{C})$ with $\|.\|_\infty$ is a Banach space.

<u>Remark</u>: We use "K is compact" implicitly from the fact that we can define

$$\|f\|_\infty \text{ for } f \in C(K, \mathbb{C}).$$

Recall that if $f : K \to \mathbb{C}$ is continuous & K is compact $\Rightarrow$ $f(K)$ is compact $\subset \mathbb{C}$.

Example: $X = C_b(\mathbb{R}^d, \mathbb{C})$

$$= \left\{ f : \mathbb{R}^d \to \mathbb{C} \text{ continuous and } \|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)| < \infty \right\}$$

Then $(X, \|\cdot\|_\infty)$ is a Banach space.

Example. $X = C_c(\mathbb{R}^d, \mathbb{C})$

$$= \{ f : \mathbb{R}^d \to \mathbb{C} \text{ continuous and compactly supported} \}$$

Then $(X, \|\cdot\|_\infty)$ is a normed space but it is <u>not</u> complete.

Def. f is compactly support $\Leftrightarrow$ $\exists$ a compact set $K \subset \mathbb{R}^d$ s.t. $f(x) = 0$, $\forall x \in \mathbb{R}^d \setminus K$.

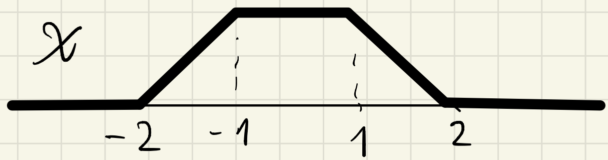
Proof. Let us find an example for a Cauchy sequence in $X = C_c(\mathbb{R}^d, \mathbb{C})$ with $\|\cdot\|_\infty$ which does not converge to any limit in X.

Take $f(x) = e^{-|x|^2} \in C_b(\mathbb{R}^d, \mathbb{C})$

$\Rightarrow f \notin C_c(\mathbb{R}^d, \mathbb{C})$

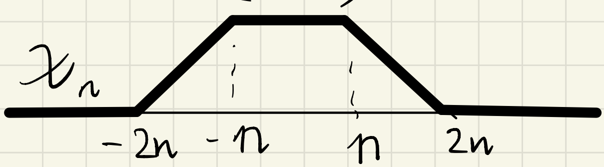
Take a function $\chi : \mathbb{R}^d \to [0,1]$ which is continuous and:

$$\begin{cases} \chi(x) = 1 & \text{if } |x| \leq 1 \\ \chi(x) = 0 & \text{if } |x| \geq 2 \end{cases}$$



Define

$$\chi_n(x) = \chi(x/n), \quad \forall n = 1, 2, 3, \dots$$



Define

$$f_n(x) = \chi_n(x) f(x) = \chi_n(x) e^{-|x|^2} \in C_c(\mathbb{R}^d)$$

We see that $\{f_n\}$ is a Cauchy sequence since $|f_n(x) - f_m(x)| = |\chi_n(x) - \chi_m(x)| |f(x)|$

$$\leq \begin{cases} |f(x)|, & \text{if } |x| \geq \min(n,m) \\ 0 & \text{if } |x| \leq \min(n,m) \end{cases}$$

$$\leq e^{-\min(m^2, n^2)}$$

Thus $\quad \|f_n - f_m\|_\infty \leq e^{-\min(m^2,n^2)} \to 0$

as $n, m \to \infty$

However, we can see that if $f_n \to g$ then $g = f$ but $f \notin C_c(\mathbb{R}^d, \mathbb{C}) \Rightarrow$ $\{f_n\}$ does not have a limit in $X = C_c(\mathbb{R}^d, \mathbb{C})$

Conclusion: $X = C_c(\mathbb{R}^d, \mathbb{C})$ is not a Banach space.

Exercise. $C_0(\mathbb{R}^d, \mathbb{C}) = \{ f : \mathbb{R}^d \to \mathbb{C}$ continuous and $\lim_{|x| \to \infty} |f(x)| = 0 \}$

Prove that $C_0(\mathbb{R}^d, \mathbb{C})$ is a Banach space with

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)| .$$ Moreover,

$$\overline{C_c(\mathbb{R}^d, \mathbb{C})}^{\|\cdot\|_\infty} = C_0(\mathbb{R}^d, \mathbb{C}).$$

<u>Remark:</u>

$$C_c(\mathbb{R}^d, \mathbb{C}) \subsetneq C_0(\mathbb{R}^d, \mathbb{C}) \subsetneq C_b(\mathbb{R}^d, \mathbb{C}) \subsetneq C(\mathbb{R}^d, \mathbb{C})$$

$C_c(\mathbb{R}^d, \mathbb{C})$: not complete with $\|\cdot\|_\infty$.

$C_0(\mathbb{R}^d, \mathbb{C})$: complete

$C_b(\mathbb{R}^d, \mathbb{C})$: complete

$C(\mathbb{R}^d, \mathbb{C})$: we cannot define $\|\cdot\|_\infty$ on $C(\mathbb{R}^d)$

<u>Exercise:</u> Let $\ell^p(\mathbb{N}) = \{ x = (x_1, x_2, \dots),\ x_i \in \mathbb{C}$ and $\|x\|_{\ell^p} < \infty \}$ $(1 \leq p \leq \infty)$

where
$$\|x\|_{\ell^p} := \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} & \text{if } p < \infty \\ \sup_n |x_n| & \text{if } p = \infty \end{cases}.$$

Prove that $\ell^p(\mathbb{N})$ is a Banach space.

<u>Exercise:</u> Let X be a Banach space and let Y be a subspace of X. Then Y is a Banach space $\iff$ Y is closed in X.

Thm (Banach) Let X be a normed space. Then X is locally compact (i.e. $\overline{B(0,1)}$ is a compact set) if and only if $\dim X < \infty$.

Proof. If $\dim X < \infty \Rightarrow \overline{B(0,1)}$ is compact ($\Leftarrow$ $\mathbb{R}^d$ is locally compact)

Assume that X is locally compact. Then we prove that $\dim X < \infty$.

Lemma. Let X be a normed space. Let Y be a subspace of X, Y is closed, $Y \neq X$. Then: $\forall \varepsilon \in (0,1)$, $\exists\, x_0 \in X \setminus Y$ s.t. $\|x_0\| = 1$ and

$$\operatorname{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\| \geq 1 - \varepsilon.$$

Proof. Because $Y \neq X \Rightarrow \exists\, x' \in X \setminus Y$.

Because Y is closed and $x' \notin Y$, then

$$d := d(x', Y) > 0.$$

We can find $y' \in Y$ s.t. $\|x' - y'\| \leq \frac{d}{1-\varepsilon}$.

Define $x_0 := \dfrac{x'-y'}{\|x'-y'\|}$. Then $\forall y \in Y$,

$$\|x_0 - y\| = \left\| \frac{x'-y'}{\|x'-y'\|} - y \right\|$$

$$= \frac{1}{\|x'-y'\|} \cdot \| x' - \underbrace{(y' + y\,\|x'-y'\|)}_{\in Y} \|$$

$$\geq \frac{1}{\|x'-y'\|} \quad d(x', Y)$$

$$\geq \frac{1}{\left(\frac{d}{1-\varepsilon}\right)} \cdot d = 1-\varepsilon. \qquad \square$$

We prove that if $\dim X = \infty$, then X is not locally compact. More precisely, we prove that if $\dim X = \infty$, then $\exists \{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| = 1$ but $\|x_n - x_m\| \geq 1-\varepsilon$ $\forall n \neq m$ (for any given $\varepsilon \in (0,1)$).

We choose $\{x_n\}$ by induction.

- $x_1 \in X$, $\|x_1\| = 1$.
- $\text{Span}(x_1) = Y_1 \subsetneq X$

  Y_1 is closed since it is finite-dim (why?)

By the Lemma, $\exists\, x_2 \in X \setminus Y_1$, $\|x_2\| = 1$

and $\text{dist}(x_2, Y_1) \geq 1-\varepsilon \Rightarrow \|x_1 - x_2\| \geq 1-\varepsilon$.

- $Y_2 = \text{Span}(x_1, x_2) \subsetneq X$

  Y_2 is closed

By the Lemma, $\exists\, x_3 \in X \setminus Y_2$ s.t.

$\|x_3\| = 1$, $\text{dist}(x_3, Y_2) \geq 1-\varepsilon$

$\Rightarrow \|x_3 - x_1\|$, $\|x_3 - x_2\| \geq 1-\varepsilon$.

$\vdots$

This gives the desired sequence $\{x_n\}$.
This sequence has <u>no</u> subsequence which is convergent $\rightsquigarrow$ X is not locally compact!

Exercise: Let X be a normed space and $\dim X = \infty$. Prove that $\exists \{x_n\}_{n=1}^{\infty} \subset X$ s.t. $\|x_n\| = 1 \ \forall n$ and $\|x_n - x_m\| \geq 1$, $\forall n \neq m$.

Exercise: Let X be a normed space and let $Y \subset X$ be a subspace with $\dim Y < \infty$. Prove that $\exists x_0 \in X \setminus Y$: $\|x_0\| = 1$ and

$$\operatorname{dist}(x_0, Y) \geq 1.$$

(Hint: you can proved that

$$\operatorname{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\|$$

is attained for some $y \in Y$, i.e. $\exists y_0 \in Y$:

$$\|x_0 - y_0\| = \inf_{y \in Y} \|x_0 - y\|$$)

Q: Let X be a Banach space and Y be a closed subspace of X. Let $x_0 \in X \setminus Y$. Can we expect $\exists$ minimiser for

$$\operatorname{dist}(x_0, Y) = \inf_{y \in Y} \|x_0 - y\| \quad ?$$

In general, it is not true! However, it is true if X is "reflexive" ($X^{**} = X$, come later)

<u>Def.</u> (Separability) Let X be a normed space. Then X is separable if $\exists A \subset X$, A is countable and A is dense in X.

<u>Thm.</u> Let X be a normed space. Then X is separable $\iff \exists \{x_n\}_{n=1}^{\infty} \subset X$ s.t.

$$X = \overline{\mathrm{Span}\{x_n : n \geq 1\}}.$$

<u>Proof</u>: "$\Rightarrow$" If X is separable, $\exists A$ countable s.t. $X = \overline{A}$. Then $A = \{x_n\}_{n=1}^{\infty}$ and

$$A \subset \mathrm{Span}\{x_n : n \geq 1\}$$

$$\Rightarrow X = \overline{A} \subset \overline{\mathrm{Span}\{x_n : n \geq 1\}}.$$

"$\Leftarrow$" If $X = \overline{\mathrm{Span}\{x_n : n \geq 1\}}$

$$\Rightarrow X = \overline{A} \text{ where}$$

$$A = \Big\{ \sum_{n \geq 1} \theta_n x_n, \ \theta_n \in \mathbb{Q} + i\mathbb{Q}, \ \{\theta_n\} \text{ has finite elements} \neq 0 \Big\}$$

$\leadsto A$ is countable $\Rightarrow X$ is separable.

Examples.

- $X = C([0,1], \mathbb{C}) = \{ f : [0,1] \to \mathbb{C} \text{ continuous} \}$

$$\|f\| = \sup_{x \in [0,1]} |f(x)| .$$

This Banach space is separable due to Weierstrass Theorem:

$$X = \overline{A} \quad \text{where} \quad A = \{ \text{pol. with rational coefficients} \}$$

- $X = C_0(\mathbb{R}^d, \mathbb{C})$ is separable. (why?)

- $X = \ell^p(\mathbb{N})$ is separable for $1 \leq p < \infty$.

Indeed $X = \overline{\operatorname{Span}\{ x_n : n \geq 1 \}}$ where

$$x_n = (0, 0, \ldots, \underbrace{1}_{n\text{-th position}}, 0, 0, \ldots)$$

- $X = \ell^\infty(\mathbb{N})$ is <u>not</u> separable. Indeed, define $x_B = (x_B^{(n)})_{n \geq 1}$, $x_B^{(n)} = \begin{cases} 1 & \text{if } n \in B \\ 0 & \text{otherwise} \end{cases}$ where $B \subset \mathbb{N}$.

Then: $\|x_B - x_{B'}\| = 1$ if $B \neq B'$.

Thus $\{x_B\}_{B \subset \mathbb{N}}$ is uncountable $(|2|^{\mathbb{N}} \sim |\mathbb{R}|)$

and this implies that $X = \ell^\infty$ is non-separable (why?)

Operators on Banach spaces:

Def. Let X, Y be two normed spaces. Then

$$\mathcal{L}(X,Y) = \{ f : X \to Y \text{ linear \& continuous} \}$$

with the norm:

$$\|f\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|f(x)\|_Y = \sup_{\|x\|_X = 1} \|f(x)\|_Y$$

In particular, $\mathcal{L}(X) = \mathcal{L}(X,X)$ and $X^* = \mathcal{L}(X,\mathbb{C})$.

Remark: .) $\|f(x)\|_Y \leq \|f\|_{\mathcal{L}(X,Y)} \|x\|_X$, $\forall x \in X$

.) If $f : X \to Y$ is linear, then f is continuous $\iff \sup_{\|x\| \leq 1} \|f(x)\|_Y < \infty$.

Theorem: Let X be a normed space and Y be a Banach space. Then $\mathcal{L}(X,Y)$ is a Banach space.

Proof. First $\mathcal{L}(X,Y)$ is a normed space ✓

Second, $\mathcal{L}(X,Y)$ is complete. Take $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$, i.e.

$$\|f_n - f_m\|_{\mathcal{L}(X,Y)} \to 0 \quad \text{as } m,n \to \infty.$$

For every $x \in X$, then:

$$\|f_n(x) - f_m(x)\|_Y = \|(f_n - f_m)(x)\|_Y$$

$$\leq \|f_n - f_m\|_{\mathcal{L}(X,Y)} \|x\| \to 0$$

as $m,n \to \infty$

Thus $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y.

Because Y is complete $\Rightarrow \exists\, f(x) = \lim_{n\to\infty} f_n(x)$.

We prove that $f \in \mathcal{L}(X,Y)$. Clearly f is linear. Moreover, we have if $\|x\|_X \leq 1$

$$\|f(x)\|_Y \leq \|f(x) - f_n(x)\|_Y + \|f_n(x)\|_Y$$

$$\leq \|f(x) - f_n(x)\|_Y + \|f_n\|_{\mathcal{L}(X,Y)}$$

Take $n \to \infty$

$$\|f(x)\|_Y \le \limsup_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)}$$

$$\Rightarrow \quad \sup_{\|x\|_X \le 1} \|f(x)\|_Y \le \limsup_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)} < \infty$$

Here we used the fact that $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}(X,Y) \Rightarrow \{f_n\}$ is bounded in $\mathcal{L}(X,Y)$. Thus $f \in \mathcal{L}(X,Y)$ and

$$\|f\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X \le 1} \|f(x)\|_Y$$

$$\le \limsup_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)}.$$

Actually, we can prove that (why?)

$$\|f\|_{\mathcal{L}(X,Y)} \le \liminf_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)}$$

Next, we prove that $f_n \to f$ in $\mathcal{L}(X,Y)$.

We have $\forall \|x\|_X \leq 1$

$$\|f_n(x) - f(x)\|_Y \leq \|f_n(x) - f_m(x)\|_Y + \|f_m(x) - f(x)\|_Y$$

$$\leq \|f_n - f_m\|_{\mathcal{L}(X,Y)} + \|f_m(x) - f(x)\|_Y$$

Take $m \to \infty$

$$\|f_n(x) - f(x)\|_Y \leq \limsup_{m\to\infty} \|f_n - f_m\|_{\mathcal{L}(X,Y)}$$

$$\Rightarrow \sup_{\|x\|_X \leq 1} \|f_n(x) - f(x)\|_Y \leq \dots$$

$$\Rightarrow \|f_n - f\|_{\mathcal{L}(X,Y)} \leq \limsup_{m\to\infty} \|f_n - f_m\|_{\mathcal{L}(X,Y)}$$

Take $n \to \infty$

$$\limsup_{n\to\infty} \|f_n - f\|_{\mathcal{L}(X,Y)} \leq \limsup_{n\to\infty} \limsup_{m\to\infty} \|f_n - f_m\|_{\mathcal{L}(X,Y)} = 0$$

because $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}(X,Y)$.

Thus $f_n \to f$ in $\mathcal{L}(X,Y)$.

Three fundamental Theorems:

Theorem (Uniform boundedness principle, Banach - Steinhaus theorem)

Let X, Y be two Banach spaces. Let $\{f_i\}_{i\in I} \subset \mathcal{L}(X,Y)$. If $\forall x \in X$,

$$\sup_{i\in I} \|f_i(x)\|_Y < \infty,$$

then

$$\sup_{i\in I} \sup_{\|x\|\leq 1} \|f_i(x)\|_Y < \infty$$

i.e.

$$\sup_{i\in I} \|f_i\|_{\mathcal{L}(X,Y)} < \infty.$$

Remark: The set I can be anything, might be uncountable.

Recall the Baire Category theorem:

Let X be a complete metric space. If $\{X_n\}_{n=1}^{\infty}$ be a sequence of closed set $X_n \subset X$ and $\bigcup_{n=1}^{\infty} X_n = X$. Then $\exists n: X_n \supset B(x,r)$.

Proof of the uniform boundedness principle:

$\forall n \geq 1$, define $X_n = \{x \in X : \sup_{i \in I} \|f_i(x)\|_Y \leq n\}$

Then X_n is a closed subset of X. In fact, if $a_k \to a$ in X and $\{a_k\} \subset X_n$, then $a \in X_n$ because:

$$\|f_i(a)\|_Y \leq \|f_i(a) - f_i(a_k)\|_Y + \|f_i(a_k)\|_Y$$
$$\leq \|f_i(a) - f_i(a_k)\|_Y + n$$

Take $k \to \infty$

$$\|f_i(a)\|_Y \leq n \quad , \quad \forall i \in I$$

$\Rightarrow a \in X_n \Rightarrow X_n$ is closed.

Moreover:

$$\bigcup_{n=1}^{\infty} X_n = X$$

since

$$\bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} \{x \in X : \sup_{i \in I} \|f_i(x)\|_Y \leq n\}$$
$$= \{x \in X : \sup_{i \in I} \|f_i(x)\|_Y < \infty\}$$
$$= X \quad \text{by assumption on } f_i$$

By Baire Category theorem, $\exists\, n \in \mathbb{N}$ and $B(x_0, r_0) \subset X$ s.t.

$$B(x_0, r_0) \subset X_n = \{ x \in X ; \sup_{i \in I} \|f_i(x)\|_Y \leq n \}$$

$$\Rightarrow \quad \|f_i(y)\|_Y \leq n, \ \forall y \in B(x_0, r_0), \ \forall i \in I$$

$$\Rightarrow \quad \|\underbrace{f_i(x_0 + r_0 x)}_{f_i(x_0) + r_0 f_i(x)}\|_Y \leq n, \ \forall \|x\|_X \leq 1, \ \forall i \in I$$

$$\Rightarrow \quad \|f_i(x)\|_Y \leq \frac{n + \|f_i(x_0)\|_Y}{r_0}, \ \forall \|x\|_X \leq 1, \ \forall i \in I$$

$$\Rightarrow \quad \|f_i\|_{\mathcal{L}(X,Y)} \leq \frac{n + \|f_i(x_0)\|_Y}{r_0}, \ \forall i \in I$$

$$\Rightarrow \quad \sup_{i \in I} \|f_i\|_{\mathcal{L}(X,Y)} \leq \frac{n + \sup_{i \in I} \|f_i(x_0)\|_Y}{r_0} < \infty \qquad \square$$

<u>Corollary</u>: let X, Y be Banach spaces. let $\{f_n\} \subset \mathcal{L}(X,Y)$ s.t. $f_n(x) \to f(x)$, $\forall x \in X$.

Then: $f \in \mathcal{L}(X, Y)$ and

$$\|f\|_{\mathcal{L}(X,Y)} \leq \liminf_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)}$$

<u>Proof.</u> Because $f_n(x) \to f(x)$ $\forall x \in X$ and f_n is linear $\forall n$ $\Rightarrow$ f is linear.

For every $x \in X$, $\{f_n(x)\}_{n=1}^{\infty}$ is bounded i.e.

$$\sup_{n\in\mathbb{N}} \|f_n(x)\|_Y < \infty$$

By the uniform boundedness principle

$$\sup_{n\in\mathbb{N}} \underbrace{\sup_{\|x\|\leq 1} \|f_n(x)\|_Y}_{\|f_n\|_{\mathcal{L}(X,Y)}} < \infty$$

$$\Leftrightarrow \quad \sup_{\|x\|\leq 1} \sup_{n\in\mathbb{N}} \|f_n(x)\|_Y < \infty$$

$$\Rightarrow \quad \sup_{\|x\|\leq 1} \|f(x)\|_Y < \infty$$

$\Rightarrow$ f is continuous.

Moreover,

$$\|f\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|\leq 1} \|f(x)\|_Y = \sup_{\|x\|\leq 1} \liminf_{n\to\infty} \underbrace{\|f_n(x)\|_Y}$$

$$\leq \|f_n\|_{\mathcal{L}(X,Y)} \|x\|_X$$

$$\leq \liminf_{n\to\infty} \|f_n\|_{\mathcal{L}(X,Y)} \quad \square$$

<u>Corrolary</u>: Let X be a Banach space. Let B be a subset of X. Then

B is bounded (i.e $\sup_{x\in B} \|x\| < \infty$)

$\Longleftrightarrow$ B is weakly-bounded (i.e $\sup_{x\in B} |f(B)| < \infty$ $\forall f \in X^* = \mathcal{L}(X,\mathbb{C})$)

<u>Proof</u>. "$\Rightarrow$" Obvious! $\left(|f(x)| \leq \|f\| \, \|x\|_X \right)$

"$\Leftarrow$" Non-trivial. We need

<u>Lemma</u>. (A consequence of Hahn-Banach theorem) Let X be a normed space. Then $\forall x \in X$,

$$\|x\|_X = \sup_{\substack{f \in X^* \\ \|f\| \leq 1}} |f(x)| = \max_{\substack{f \in X^* \\ \|f\| = 1}} |f(x)|$$

<u>Proof</u>. First, we have:

$$\sup_{\substack{f \in X^* \\ \|f\| \leq 1}} |f(x)| \leq \sup_{\substack{f \in X^* \\ \|f\| \leq 1}} \left(\|f\| \cdot \|x\|_X \right) = \|x\|_X$$

Second, we prove that $\exists f_0 \in X^*$ s.t. $\|f_0\| = 1$, and $f_0(x) = \|x\|_X$. We define

$$f_0 : \operatorname{Span}(x) = \mathbb{C}x \longrightarrow \mathbb{C} \quad \text{by}$$

$$f_0(zx) := z\|x\|_X$$

$\Rightarrow$ $f_0 \in \mathcal{L}(\operatorname{Span}(x), \mathbb{C})$ and $\|f_0\|_{\mathcal{L}(\operatorname{Span}(x))} = 1$.

By the Hahn-Banach theorem, we can extend the functional $f_0 : X \longrightarrow \mathbb{C}$ s.t.

$f_0 \in \mathcal{L}(X, \mathbb{C}) = X^*$, $f_0(zx) = z\|x\|_X, \forall z \in \mathbb{C}$

and
$$\|f_0\|_{\mathcal{L}(X,\mathbb{C})} = \|f_0\|_{\mathcal{L}(\mathrm{Span}(x),\mathbb{C})} = 1.$$

Then $f_0(x) = \|x\|_X$. $\square$

Conclusion of the Corrolary: "$\Leftarrow$"

Assume that $B \subset X$ is weakly bounded.

Then
$$\sup_{b \in B} |f(b)| < \infty, \quad \forall f \in X^*.$$

Define $Y = X^*$ (a Banach) and
$$\{T_b\}_{b \in B} \subset Y^* \quad (= X^{**})$$
$$T_b(f) := f(b), \quad \forall f \in Y = X^*, \forall b \in B$$

Then: $\forall f \in Y = X^*$, then by the weak b.d.
$$\sup_{b \in B} |T_b(f)| = \sup_{b \in B} |f(b)| < \infty$$

By the uniform boundedness principle,
$$\sup_{b \in B} \sup_{\|f\| \leq 1} |f(b)| < \infty$$

$$\sup_{b\in B} \underbrace{\sup_{\|f\|\le 1} |f(b)|}_{\|b\| \text{ by the lemma}} < \infty$$

$$\Rightarrow \quad \sup_{b\in B} \|b\| < \infty \quad \text{, i.e. } B \text{ is bounded} \ \square.$$

<u>Exercise</u> (Another proof of the uniform boundedness principle, due to Hahn 1922)

Let X, Y be Banach spaces. Let a family $\{f_i\} \subset \mathcal{L}(X,Y)$ s.t. $\sup_{i\in I} \|f_i\|_{\mathcal{L}(X,Y)} = \infty$.

Then we prove that $\exists x \in X$ s.t.

$$\sup_{i\in I} \|f_i(x)\| = \infty.$$

(a) Prove that $\exists$ a sequence $\{f_n\}_{n=1}^{\infty} \subset \{f_i\}_{i\in I}$ and a sequence $\{x_n\}_{n\ge 1}^{\infty} \subset X$ s.t.

$$\|x_n\| \le 2^{-n} \min\{\|T_j\|^{-1} : j<n\}$$

$$(\text{and } 2^{-1} \text{ if } n = 1)$$

and $\|T_n x_n\| \geq n + \sum_{j=1}^{n-1} \|T_n x_j\|$

$(\text{or } \geq 1 \text{ if } n=1)$.

(b) Prove that $\sum_{n=1}^{\infty} x_n = x$ in X.

(c) $\sum_{j=n+1}^{\infty} \|T_n x_j\| \leq 1$, $\forall n$

(d) $\|T_n x\| \geq n-1$, $\forall n \rightsquigarrow$ conclusion.

<u>Theorem</u> (Open mapping theorem, Banach-Schauder theorem)

Let X, Y be Banach spaces. Let $f \in \mathcal{L}(X,Y)$.
If f is surjective, i.e. $f(X) = Y$, then:

$f(A)$ is open in Y, $\forall A$ open in X.

<u>Remark</u>: If $f: X \to Y$ is continuous, then $f^{-1}(B)$ is open in X, $\forall B$ open in Y.

<u>Corrolary</u>: let X, Y be Banach spaces and $f \in \mathcal{L}(X,Y)$. If f is bijective, then $f^{-1} \in \mathcal{L}(Y,X)$.

(For that reason, the "open mapping theorem is also called the "inverse mapping theorem")

<u>Corrollary</u>: Let X be a normed space with $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces and

$$\|x\|_1 \geq \|x\|_2 \, , \quad \forall x \in X.$$

Then $\exists \, C > 0$ s.t.

$$\|x\|_1 \leq C \|x\|_2 \, , \quad \forall x \in X.$$

<u>Proof</u>: Define $f : (X, \|\cdot\|_1) \longrightarrow (X, \|\cdot\|_2)$ with $f(x) = x$. Then f is bijective and $\|f(x)\|_2 \leq \|x\|_1$, i.e. f is continuous. By the Open mapping theorem (by the above corrolary), f^{-1} is also continuous.

Thus: $\|f^{-1}(x)\|_1 \leq C\|x\|_2, \forall x \in X$

$$\|f^{-1}(x)\|_1 = \|x\|_1$$

□

Proof of the Open mapping theorem:

Step 1. We claim that it suffices to show that

(*) $$f(B_X(0,1)) \supset B_Y(0,r_0)$$

for some $r_0 > 0$.

Indeed, let U be an open set in X. Then we prove that $f(U)$ is open in Y. Take $y \in f(U)$ and we prove that $\exists r_y > 0$ s.t.

$$f(U) \supset B_Y(y, r_y).$$

Since $y \in f(U) \Rightarrow y = f(x)$ for $x \in U$.

And U is open $\Rightarrow \exists r_x > 0$ s.t. $B_X(x, r_x) \subset U$

Thus:

$$f(U) \supset f(B_X(x,r_x)) = f(x + B_X(0,r_x))$$

$$= f(x) + r_x f(B_X(0,1))$$

$$\supset y + r_x B_Y(0,r_0) = B(y, r_x r_0).$$

Step 2. We prove a weaker property:

$$\overline{f(B(0,1))} \supset B(0, 2r_0).$$

Define $\forall n \in \mathbb{N}$:

$$Y_n = n\,\overline{f(B(0,1))}$$
$$= \{ny : y \in \overline{f(B(0,1))}\}.$$

Then Y_n is closed $\forall n$ and

$$\bigcup_{n=1}^{\infty} Y_n = \bigcup_{n=1}^{\infty} n\,\overline{f(B(0,1))}$$

$$\supset \bigcup_{n=1}^{\infty} n\, f(B(0,1))$$

$$\supset f\left(\bigcup_{n=1}^{\infty} n\, B(0,1)\right)$$

$$= f\left(\bigcup_{n=1}^{\infty} B(0,n)\right) = f(X) = Y$$

By the Baire Category theorem, $\exists n \in \mathbb{N}$ s.t.

$$\underset{\displaystyle n\,\overline{f(B(0,1))}}{\overset{\displaystyle Y_n}{\|}} \supset B(y', r')$$

$$\Rightarrow \quad \overline{f(B(0,1))} \supset B(y_0, r_1)$$

where $y_0 = \frac{y'}{n}$, $r_1 = \frac{r'}{n}$.

Since $B(0,1)$ is convex

$\Rightarrow$ $f(B(0,1))$ is convex

$\Rightarrow$ $\overline{f(B(0,1))}$ is convex (why ?)

<u>Exercise</u>: If A is convex, then $\overline{A}$ is convex.

Moreover, $B(0,1)$ is even, namely

$$-B(0,1) = \{-x : x \in B(0,1)\} = B(0,1)$$

$\Rightarrow$ $\overline{f(B(0,1))}$ is also even.

Thus from

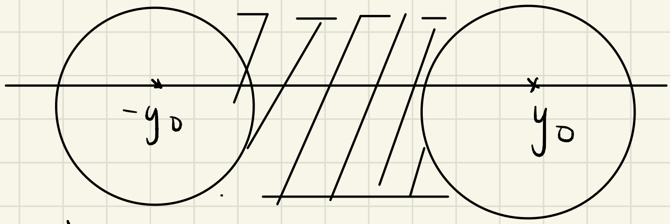
$$\overline{f(B(0,1))} \supset B(y_0, r_1)$$

$$\Rightarrow \quad \overline{f(B(0,1))} \supset -B(y_0, r_1) = B(-y_0, r_1)$$

By the convexity

$$\overline{f(B(0,1))} \supset \frac{1}{2} B(y_0, r) + \frac{1}{2} B(-y_0, r_1)$$

$$\supset B\left(0, \frac{r_1}{2}\right).$$



$\frac{1}{2}B(-y_0, r_0)$ $\qquad\qquad$ $\frac{1}{2}B(y_0, r_0)$

Thus we have the conclusion with $r_0 = \frac{r_1}{4}$.

<u>Step 3.</u> We prove that

$$f(B(0,1)) \supset B(0, r_0).$$

Recall from Step 2:

$$\overline{f(B(0,1))} \supset B(0, 2r_0).$$

Take $y_0 \in B(0, r_0)$. Then we need to find $x \in B(0,1)$ s.t. $f(x) = y_0$.

Because $2y_0 \in B(0, 2r_0) \subset \overline{f(B(0,1))}$, then $\exists\, x_0 \in B(0,1)$ s.t.

$$\| 2y_0 - f(x_0) \| < r_0$$

$$\Longleftrightarrow \quad \| y_0 - \tfrac{1}{2} f(x_0) \| < \frac{r_0}{2}$$

Then: $4\left(y_0 - \frac{1}{2}f(x_0)\right) \in B(0, 2r_0)$

$$\subset \overline{f(B(0,1))}$$

$\Rightarrow \exists\, x_1 \in B(0,1)$:

$$\left\| 4\left(y_0 - \frac{1}{2}f(x_0)\right) - f(x_1) \right\| < r_0$$

$$\Rightarrow \quad \left\| y_0 - \frac{1}{2}f(x_0) - \frac{1}{4}f(x_1) \right\| < \frac{r_0}{4}$$

Then $8\left(y_0 - \frac{1}{2}f(x_0) - \frac{1}{4}f(x_1)\right) \in B(0, 2r_0)$

$$\subset \overline{f(B(0,1))}$$

$\Rightarrow \exists\, x_2 \in B(0,1)$ s.t.

$$\left\| 8\left(y_0 - \frac{1}{2}f(x_0) - \frac{1}{4}f(x_1)\right) - f(x_2) \right\| < r_0$$

$$\Rightarrow \left\| y_0 - \frac{1}{2}f(x_0) - \frac{1}{4}f(x_1) - \frac{1}{8}f(x_2) \right\| < \frac{r_0}{8}$$

By induction, we can find a sequence $\{x_n\}_{n=1}^{\infty} \subset B(0,1)$ and

$$\left\| y_0 - \sum_{j=0}^{n} \frac{1}{2^{j+1}} f(x_j) \right\| \leq \frac{r_0}{2^{n+1}}, \forall n \geq 1$$

$$\Leftrightarrow \left\| y_0 - f\left(\sum_{j=0}^{n} \frac{1}{2^{j+1}} x_j \right) \right\| < \frac{r_0}{2^{n+1}}$$

Because $\{x_n\}_{n=0}^{\infty} \subset B(0,1)$ and X is a Banach space, we have the convergence

$$\lim_{n\to\infty} \underbrace{\sum_{j=0}^{n} \frac{1}{2^{j+1}} x_j}_{(z_n \text{ is a Cauchy sequence})} = x \quad \text{in } X$$

Thus: $\| y_0 - f(x) \| = 0 \Leftrightarrow f(x) = y_0$

and $\|x\| < \sum_{j=0}^{n} \frac{1}{2^{j+1}} \|x_j\| \leq 1$

Conclusion: $\exists x \in B(0,1)$ s.t. $f(x) = y_0$.

<u>Remark:</u> If $f \in \mathcal{L}(X,Y)$ and $f(X) = Y$, then in general, it does <u>not</u> hold that $f(A)$ is closed if A is closed.

Example: Take $f : \underset{=X}{\mathbb{R}^2} \to \underset{=Y}{\mathbb{R}}$

$$f(x,y) = y$$

$\Rightarrow f(\mathbb{R}^2) = \mathbb{R}$, $\quad f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$.

However, $A = \{(x,y) \in \mathbb{R}^2 : y \geq e^x\}$

Then A is closed in $\mathbb{R}^2$ but

$$f(A) = (0, \infty) \text{ is } \underline{\text{not}} \text{ closed in } \mathbb{R}.$$

Nevertheless: $\quad f\left(\overline{B(0,1)}\right) = [-1,1]$ closed.

<u>Q</u>: Can we say that $f\left(\overline{B(0,1)}\right)$ is closed?

<u>Exercise</u>: Let X, Y be Banach spaces. Let $f \in \mathcal{L}(X,Y)$ be a bijective. Then

(a) $f(A)$ is closed in Y, $\forall A$ closed in X

(b) $f\left(\overline{B(0,1)}\right) = \overline{f(B(0,1))}$.

<u>Remark</u>: The assumption "f is bijective" can be replaced by "f is surjective" if X is reflexive ($X = X^{**}$, come later).

Exercise. Let X, Y be Banach spaces. Let $f \in \mathcal{L}(X, Y)$. Assume $\overline{f(B(0,1))}$ contains a ball in Y. Then $f(B(0,1))$ contains a ball, and hence f is surjective.
(Hint: You can mimic the proof of the Open mapping Theorem).

Theorem (Closed Graph theorem) Let X, Y be Banach spaces. Let $f: X \to Y$ be a linear function. Then:

f is continuous $\iff$ the graph $\mathcal{G} = \{(x, f(x)) : x \in X\}$ is a closed set in $X \times Y$.

Remark: If X & Y are normed spaces, then $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is also a normed space where

$$\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y.$$

Actually, we can also use another norm, e.g.

$$\|x\|_X + \|y\|_Y \rightsquigarrow \max(\|x\|_X, \|y\|)$$

$$\text{or} \quad \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p}, \ 1 \le p < \infty$$

All of these norms are equivalent.

<u>Exercise</u> Let X, Y be Banach spaces. Prove $X \times Y$ is also a Banach space.

<u>Proof of the Closed Graph Theorem:</u>

"$\Rightarrow$" Assume that f is continuous. We prove that $\mathcal{G} = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

Take a sequence $(x_n, f(x_n))_{n=1}^{\infty}$ in $\mathcal{G}$ and assume $(x_n, f(x_n)) \to (x, y)$ in $X \times Y$.

Then we need to prove $(x, y) \in \mathcal{G}$.

Namely, if $x_n \to x$ and $f(x_n) \to y$, then $y = f(x)$

This is obvious since f is continuous,

$$x_n \to x \Rightarrow \left.\begin{array}{l} f(x_n) \to f(x) \\ \& \ f(x_n) \to y \end{array}\right\} \Rightarrow y = f(x).$$

"$\Leftarrow$" Assume that $\mathcal{G} = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$, i.e., if $x_n \to x$ and $f(x_n) \to y$, then $y = f(x)$. Now we prove that f is cont.

Consider the Banach space X with 2 norms

$$\|x\|_X \quad , \quad \|x\|_1 := \|x\|_X + \|f(x)\|_Y .$$

<u>Claim:</u> $(X, \|.\|_1)$ is a Banach space.

From the claim and $\|x\|_1 \geq \|x\|_X$, by the Open mapping theorem, $\exists$ a constant $C>0$:

$$\|x\|_1 \leq C \|x\|_X$$

$$\Rightarrow \quad \|x\|_X + \|f(x)\|_Y \leq C \|x\|_X \,, \forall x \in X$$

$$\Rightarrow \quad \|f(x)\|_Y \leq C \|x\|_X \,, \quad \forall x \in X$$

$\Rightarrow$ f is continuous.

<u>Proof of the claim:</u> We need to prove that $(X, \|.\|_1)$ is complete. Take a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in $(X, \|.\|_1)$, we need to prove that $x_n \to x$ in $(X, \|.\|_1)$.

We know that:

$$\|x_n - x_m\|_1 = \|x_n - x_m\|_X + \|f(x_n) - f(x_m)\|_Y$$

$$\to 0 \quad \text{as} \quad m, n \to \infty$$

$\Rightarrow$ $\{x_n\}$ is Cauchy in X and $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy in Y

$\Rightarrow$ Since X & Y are complete, $x_n \to x$ in X and $f(x_n) \to y$ in Y

$\Rightarrow$ $(x_n, f(x_n)) \to (x, y)$ in $X \times Y$

$\Rightarrow$ Since $\mathcal{G}$ is closed, we conclude that $y = f(x)$

Thus: $x_n \to x$ in X and $f(x_n) \to f(x)$ in Y

$$\Rightarrow \|x_n - x\|_1 = \|x_n - x\|_X + \|f(x_n) - f(x)\|_Y$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

Thus $(X, \|.\|_1)$ is complete. $\square$

<u>Direct sum:</u> If Y and Z are two subspaces of a vector space X and $Y \cap Z = \{0\}$, then we can identify $Y \times Z$ to $Y + Z$, i.e.

$$\begin{aligned} (y, z) &\longmapsto y + z \quad \text{is a bijective} \\ Y \times Z &\longrightarrow Y + Z \end{aligned}$$

(Indeed, the mapping $Y \times Z \to Y + Z$ is clearly surjective, and this mapping is injective since $Y \cap Z = \{0\}$, i.e. if $y + z = y' + z'$ with $y, y' \in Y$, $z, z' \in Z$, then: $\underbrace{y - y'}_{\in Y} = \underbrace{z' - z}_{\in Z} \in Y \cap Z = \{0\}$

$\Rightarrow y = y'$, $z = z'$.)

If $X = Y + Z$, then we say that $X = Y \oplus Z$, the direct sum in algebraic way.

<u>Def.</u> We say that $X = Y \oplus Z$, the direct sum in a Banach space X, if Y, Z are subspace of X, $Y \cap Z = \{0\}$, and the mapping

$$X = Y \oplus Z \longrightarrow Y \times Z$$

is continuous.

Remark: If Y and Z are closed subspaces of X (Banach)
then Y, Z are Banach spaces $\Rightarrow$ $Y \times Z$ is
a Banach space. Thus the linear, bijective map

$$X = Y \oplus Z \longrightarrow Y \times Z$$

is continuous $\iff$ its inverse is continuous, by
the Open mapping theorem.

We can define the projection $X = Y \oplus Z \to Y$ by $x = y + z \mapsto Px = y$. Then
P is continuous $X \to Y$ and $Y = PX$.

Moreover, $P^2 = P$ since

$$P^2 x = P(\underbrace{Px}_{\in Y}) = Px$$

$$(\iff Py = y, \ \forall y \in Y)$$

Def: Let Y be a closed subspace of a
Banach space X. Then we say that Y is
complemented in X if $\exists$ a closed subspace Z
in X s.t. $X = Y \oplus Z$.

Theorem: Let Y be a closed subspace of X. Then Y is complemented in X $\Leftrightarrow$ $Y = PX$ with a projection $P: X \to X$, i.e. $P \in \mathcal{L}(X,X)$ and $P^2 = P$.

Proof: "$\Rightarrow$" Assume that Y is complemented in X, i.e. $X = Y \oplus Z$ with Y, Z are closed subspaces and $Y \cap Z = \{0\}$. Then

$$Px = y \quad \text{where } x = y + z, \; y \in Y, \; z \in Z$$

Then clearly P is linear, $Y = PX$ and $P^2 = P$ (as $Py = y, \forall y \in Y$).

Why P is continuous in X: We prove that using the Closed Graph Theorem. We will prove that $\{(x, Px) : x \in X\}$ is closed in $X \times X$, i.e. $x_n \to x$ in X and $Px_n \to g$ in X, then $g = Px$.

First, $g \in Y$ since $Px_n \in Y$ and $Px_n \to g$ and Y is closed. Thus $Pg = g$.

Thus the equation $Px = g$

$$\Leftrightarrow Px = Pg \Leftrightarrow P(x-g) = 0$$

$$\Leftrightarrow x - g \in Z \quad \Big(x - y = P(x-y) + \underbrace{\square}_{\in Z} \Big)$$

Indeed,

$$x - y = \lim_{n \to \infty} \big(\underbrace{x_n - Px_n}_{\in Z,\ \forall n} \big) \in Z$$

as Z is closed

Thus we conclude that $P : X \to X$ is continuous and it is a projection.

"$\Leftarrow$" If Y is a closed subspace of a Banach space X and $Y = PX$ for a projection $P \in \mathcal{L}(X, X)$ and $P^2 = P$, then we prove that $X = Y \oplus Z$ for a closed subspace Z s.t. $Y \cap Z = \{0\}$.

Indeed:
$$x = \underbrace{Px}_{\in Y} + \underbrace{(1-P)x}_{\in Z}$$

$\rightsquigarrow Z = \operatorname{Ker} P = \{ x \in X : Px = 0 \}$ is closed and $Y \cap Z = \{0\}$ since if $x \in Y \cap Z \Rightarrow$

$$\left.\begin{array}{l} Px = 0 \quad \text{as } x \in Z \\ \| \\ x \quad \text{as } x \in Y \end{array}\right\} \Rightarrow x = 0.$$

<u>Remark:</u> Given a Banach space X and a closed subspace $Y \subset X$, then in general, it might happen that Y is <u>not</u> complemented.

<u>Example:</u> $Y = c_0(\mathbb{N}) = \{ x = (x_n)_{n=1}^{\infty}, x_n \in \mathbb{C}, x_n \to 0$ as $n \to \infty \}$ is a closed subspace of $X = \ell^\infty(\mathbb{N})$. However, Y is <u>not</u> complemented in X. (It is a Theorem of Phillip 1940, we will come to that later).

Actually, there is a deep result:

<u>Thm:</u> (Lindenstrauss - Tzafriri) Let X be a Banach space. Then TFAE:

(1) $\forall Y$ closed subspace of X, then Y is complemented in X.

(2) X is isomorphic to a Hilbert space.

We will come to that later. Here

X is isomorphic to M if

$\exists$ bijective $f \in \mathcal{L}(X, M)$

$\Longleftrightarrow$ f is continuous & f^{-1} is continuous.

$\Longleftrightarrow \frac{1}{C}\|x\|_X \leq \|f(x)\|_M \leq C\|x\|_X \,, \forall x \in X$

<u>Remark:</u> A stronger concept is

X is isometric to M if

$\exists$ a bijective $f \in \mathcal{L}(X, M)$ <u>and</u>

$\|f(x)\|_M = \|x\|_X \,, \forall x \in X.$

<u>Exercise:</u> Let X be a Banach space and Y be a subspace s.t. $\dim Y < \infty$. Then Y is complemented.

(Hint: You can use Hahn-Banach theorem, i.e. $\forall x \in X, \exists f \in X^*$ s.t. $f(x) = \|x\|$ and $\|f\| = 1$)

Remark: If $X = Y \oplus Z$ then

$$X \simeq Y \times Z$$

This is similar to the "factorization" in number theory. For that reason, a Banach space X is called a prime space if $\forall Y$ complemented subspace of X, then either $\dim Y < \infty$ or $X \simeq Y$ (isomorphic).

Example: $\ell^p(\mathbb{N})$ is prime for $1 \leq p < \infty$. (we will come to that later)

Quotient spaces: let X be a Banach space and Y be a closed subspace. We define the quotient space

$$X/Y := \{ q(x) ; x \in X \}$$

where $q(x) = x + Y$ as a equivalent class i.e. $q(x) = q(y)$ if $x - y \in Y$.

Define the norm:

$$\| q(x) \|_{X/Y} := \text{dist}(x, Y) = \inf_{y \in Y} \|x - y\|$$

<u>Theorem:</u> If X is a Banach space and Y is a closed subspace, then X/Y is a Banach space.

<u>Proof:</u> First, we need to check that X/Y is a normed space (exercise).

Now we prove that X/Y is complete.

<u>Lemma:</u> Let X be a normed space. Then X is complete $\Leftrightarrow$ $\forall \{x_n\}_{n=1}^{\infty} \subset X$ and

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \text{ , then } \sum_{n=1}^{\infty} x_n \text{ converges}$$

$$\left(\text{i.e. } \sum_{n=1}^{m} x_n \text{ converges when } m \to \infty \right)$$

<u>Proof</u> (exercise)

<u>Proof of the theorem.</u> Assume $\{x_n\}_{n=1}^{\infty} \subset X$;

$$\sum_{n=1}^{\infty} \|q(x_n)\|_{X/Y} < \infty$$

Then we prove that $\sum_{n=1}^{\infty} q(x_n)$ converges.

By definition

$$\|q(x_n)\|_{X/Y} = \operatorname{dist}(x_n, Y) = \inf_{y \in Y} \|x_n - y\|$$

Then $\exists y_n \in Y$ s.t.

$$\|q(x_n)\|_{X/Y} \leq \|x_n - y_n\| \leq 2\,\|q(x_n)\|_{X/Y}$$

(we also used if $\|q(x_n)\|_{X/Y} = 0 \Rightarrow x_n \in Y$)

Thus:

$$\sum_{n=1}^{\infty} \|x_n - y_n\| \leq 2 \sum_{n=1}^{\infty} \|q(x_n)\|_{X/Y} < \infty$$

Since X is a Banach space, we conclude that

$$\sum_{n=1}^{\infty} (x_n - y_n) = z \quad \text{in } X$$

i.e.

$$\Big\| \sum_{n=1}^{m} (x_n - y_n) - z \Big\| \to 0 \text{ as } m \to \infty$$

<u>Claim:</u> $\displaystyle\sum_{n=1}^{\infty} q(x_n) = q(z)$ in X/Y

<u>Proof:</u>

$$\Big\| \sum_{n=1}^{m} q(x_n) - q(z) \Big\|_{X/Y}$$

$$= \Big\| q\Big(\sum_{n=1}^{m} x_n - z\Big)\Big\|_{X/Y} = \Big\| q\Big(\sum_{n=1}^{m} (x_n - y_n) - z\Big)\Big\|_{X/Y}$$

$$\leq \Big\| \sum_{n=1}^{m} (x_n - y_n) - z \Big\| \to 0 \text{ as } m \to \infty. \quad \square$$

<u>Remark:</u> Here we used the obvious inequality

$$\|q(x)\|_{X/Y} \leq \|x\| \quad , \forall x \in X .$$

In particular, this implies that the "quotient map"

$$q: \quad X \longrightarrow X/Y$$

is linear and continuous! Moreover, q is surjective, hence it is an open map by the Open Mapping theorem. Actually:

$$q\left(B_X(0,1)\right) = B_{X/Y}(0,1) \qquad (\text{why?})$$

<u>Remark:</u> The quotient space X/Y is well-defined for any closed subspace Y of X. However, when Y is complemented, then X/Y is simply the complement of Y.

<u>Theorem:</u> Let X be a Banach space and let Y be a complemented subspace, i.e. $X = Y \oplus Z$ with Y, Z closed subspace. Then:

$$X/Y \text{ is isometric to } Z .$$

__Proof:__ Because Y is complemented, $\exists$ a projection $P \in \mathcal{L}(X,X)$ s.t. $Y = PX$ and $Z = (1-P)X$.

Thus:
$$x = \underbrace{Px}_{\in Y} + \underbrace{(1-P)x}_{\in Z}, \quad \forall x \in X$$

Define $T: X/Y \to Z$ by
$$q(x) \mapsto Tq(x) = (1-P)x, \quad \forall x \in X.$$

__T is well-defined:__ If $q(x) = q(y)$, then:
$$(1-P)x = (1-P)y \iff x - Px = y - Py$$
$$\iff x - y = P(x-y)$$
$$\iff x - y \in Y \quad \checkmark$$

__T is linear and bijective:__

$$\begin{tikzcd}
X \arrow[rr, "T \circ q = 1-P"] \arrow[dr, "q"'] & & Z \\
& X/Y \arrow[ur, "T"'] &
\end{tikzcd}$$

__T & T^{-1} are continuous:__ We prove that T^{-1} is continuous. Here: $T^{-1}: Z \to X/Y$
$$z \mapsto q(z)$$

$$\|T^{-1}z\|_{X/Y} = \|q(z)\|_{X/Y} \leq \|z\|_X = \|z\|_Z$$

$\Rightarrow$ T^{-1} is continuous (actually $\|T^{-1}\| \leq 1$).

By the open mapping theorem, T is continuous.

Conclusion:

$$X/Y \simeq Z.$$ $\square$

<u>Remark</u>: In particular, if Y is complemented in X, then X/Y "is" a subspace of X. But in general, if Y is <u>not</u> complemented, then X/Y might be not a subspace of X.

<u>Remark</u>: Let X, Y be Banach spaces. Let $f \in \mathcal{L}(X,Y)$ and $f(X) = Y$. By Open mapping theorem, $f(B_X(0,1)) \supset B_Y(0,r)$.

Moreover, define $M = \operatorname{Ker} f = \{x \in X : f(x) = 0 \text{ in } Y\}$

$= f^{-1}(\{0\})$ closed in X

Then:

$$\tilde{f} : X/M \to Y$$
$$q(x) \to f(x)$$

is an isomorphic map (bijective, linear, cont.)

Thus: $X/M \simeq Y$. (isomorphic)

In particular, $\tilde{f}$ is both open and closed map.

<u>Theorem</u> (Banach - Mazur theorem) If X is a separable Banach space, then:

$$X \simeq \ell^1(\mathbb{N}) / M$$

with a closed subspace M of $\ell^1(\mathbb{N})$.

<u>Proof</u>: Recall $\ell^1(\mathbb{N}) = \{ (a_n)_{n=1}^{\infty}, a_n \in \mathbb{C}, \sum_n |a_n| < \infty \}$

$$= \left\{ \sum_{n=1}^{\infty} a_n e_n \ ; \ \sum_n |a_n| < \infty \right\}$$

where $e_n = (0, 0, \dots, \underset{\uparrow\ n\text{-th position}}{1}, 0, 0, \dots)$.

Because X is separable, $\exists \{x_n\}_{n=1}^{\infty} \subset X$ s.t. it is dense in $\overline{B(0,1)}$ of X. Define

$$f : \ell^1(\mathbb{N}) \to X \quad \text{by}$$

$$e_n \mapsto x_n \quad \text{and by linearity}$$

$$\sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n=1}^{\infty} a_n x_n$$

① f is well-defined:

If $(a_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} a_n e_n \in \ell^1(\mathbb{N})$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| = \| (a_n)_{n=1}^{\infty} \|_{\ell^1} < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \|a_n x_n\| = \sum_{n=1}^{\infty} |a_n| \underbrace{\|x_n\|}_{\leq 1} \leq \sum_{n=1}^{\infty} |a_n| < \infty$$

$\Rightarrow \sum_{n=1}^{\infty} a_n x_n$ converges since X is Banach Space

② f is continuous:

$$\| f(\sum_n a_n e_n) \| = \| \sum_n a_n x_n \|$$

$$\leq \sum_n \|a_n x_n\| \leq \sum_n |a_n| \underbrace{\|x_n\|}_{\leq 1}$$

$$\leq \sum_n |a_n| = \| \sum_n a_n e_n \|_{\ell^1(\mathbb{N})}$$

Thus $\|f\| \leq 1$.

③ f is surjective: We need to prove that $f(B(0,1))$ contains an open ball.

Recall: $f(\overline{B(0,1)}) \subset \overline{B(0,1)}$ as $\|f\| \le 1$

<u>Claim:</u>

$$\overline{f(B(0,1))} = \overline{B(0,1)}.$$

We have the obvious direction (why)

$$\overline{f(B(0,1))} \subset \overline{B(0,1)}.$$

Thus the main point here is the other direction

$$\overline{f(B(0,1))} \supset \overline{B(0,1)}.$$

This follows: $f(e_n) = x_n$ and $\{x_n\}_{n=1}^{\infty}$ is dense in $\overline{B(0,1)}$.

<u>Conclusion:</u> From $\overline{f(B(0,1))} \supset \overline{B(0,1)}$, by following the proof of the Open Mapping theorem (or an exercise), then

$$f(B(0,1)) \text{ contains a ball.}$$

Thus f is surjective. Then by Open Mapping theorem, $f(B(0,1))$ is open and we have

$$f(B(0,1)) = B(0,1).$$

Thus: $\quad f : \ell^1(\mathbb{N}) \to X \quad$ surjective

$\to \quad \tilde{f} : \ell^1(\mathbb{N})/M \to X \quad$ bijective

where $\quad M = \ker f$

$$\tilde{f}(q(x)) = f(x), \forall x \in \ell^1(\mathbb{N})$$

Thus

$$\ell^1(\mathbb{N})/M \simeq X \quad \text{(isomorphic)}$$

From the proof, you can see that $\tilde{f}$ is an isometry.

<u>Exercise</u>: Let X, Y be Banach spaces, let $f \in \mathcal{L}(X, Y)$ s.t.

$$\begin{cases} f(B(0,1)) \subset B(0,1) \\ \overline{f(B(0,1))} \supset \overline{B(0,1)} \end{cases}$$

Then: $\quad f(B(0,1)) = B(0,1)$

and $\quad \overline{f(B(0,1))} = f\left(\overline{B(0,1)}\right) = \overline{B(0,1)}$.

<u>Def:</u> let X be a Banach space and let Y be a closed subspace. Then

$\dim(X/Y)$ is called the co-dimension of Y.

<u>Remark.</u> If $X = Y \oplus Z$, then

$$X/Y \simeq Z$$

$$\Rightarrow \dim(X/Y) = \dim(Z)$$

<u>Exercise:</u> let X be a Banach space and let Y be a closed subspace. Assume co-dim of Y is finite. Prove that Y is complemented.