13.04.2021 : Motivation

Finite dim:

$$A : \mathbb{R}^d \longrightarrow \mathbb{R}$$

$$A \text{ linear} \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R})$$

$$= \{ A : \mathbb{R}^d \to \mathbb{R} \text{ linear} \}$$

finite dim (dim = d)

Take a basis $\{e_i\}_{i=1}^d$ of $\mathbb{R}^d$, then:

$$x = \sum_{i=1}^d x_i e_i \quad , \forall x \in \mathbb{R}^d$$

$$\Rightarrow f(x) = \sum_{i=1}^d x_i \underline{f(e_i)}$$

$$f \longmapsto (f(e_i))_{i=1}^d$$

Infinite dim:

$$f : \underset{\dim E = \infty}{E} \longrightarrow \mathbb{R}, \mathbb{C}$$

$$E^* = \mathcal{L}(E, \mathbb{C}) = \{ f : E \to \mathbb{C} \text{ linear continuous} \}$$

Example. $E = C([0,1], \mathbb{R})$

$= \{ f : [0,1] \to \mathbb{R},\ f \text{ continuous} \}$

$E^* = \{ A : E \to \mathbb{R} \text{ or } \mathbb{C} \text{ linear \& continuous} \}$

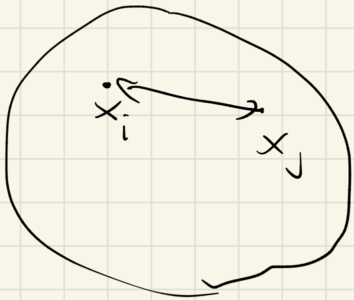
$E = L^1(\mathbb{R}^d, \mathbb{C})$

$= \{ f : \mathbb{R}^d \to \mathbb{C},\ \int_{\mathbb{R}^d} |f(x)|\,dx < \infty \}$

Finite dim spaces	∞ dim spaces
. All convergences are the same	. Convergences not the same
i.e. all norms are equivalent	$\to$ topology
Eg. $\mathbb{R}^d$, $\lvert x\rvert = \sqrt{x_1^2 + \dots + x_d^2}$	Eg: $L^1(\mathbb{R}^d, \mathbb{C})$
$\lvert x\rvert_\infty = \max(\lvert x_i\rvert)$	$f_n \xrightarrow{L^1} 0 \iff \int_{\mathbb{R}^d} \lvert f_n(x)\rvert\,dx \xrightarrow[n\to\infty]{} 0$
$\Rightarrow \lvert x\rvert_\infty \le \lvert x\rvert \le \sqrt{d}\,\lvert x\rvert_\infty$	$L^\infty(\mathbb{R}^d, \mathbb{C}) = \{ f : \sup \lvert f(x)\rvert < \infty \}$
	$f_n \xrightarrow{L^\infty} 0 \iff \sup_x \lvert f_n(x)\rvert \xrightarrow[n\to\infty]{} 0$

Finite dim

• Take a unit ball in $\mathbb{R}^d$



$$\forall i\neq j: \ |x_i - x_j| \geq \frac{1}{2}$$

$$\#(x_i) \leq C_d$$

• Compactness.

If $\{x_n\}_{n=1}^{\infty}$ bounded

$\Rightarrow \exists$ subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ s.t.

$$x_{n_k} \xrightarrow[k\to\infty]{} x_\infty$$

∞ dim

• Take a unit ball

$$B = \{x \in E : \|x\| \leq 1\}$$

$\exists \{x_i\}_{i=1}^{\infty}$ s.t.

$$|x_i - x_j| \geq \frac{1}{2}, \ \forall i \neq j$$

- No compactness

Example.

• Weak convergence & weak compactness

Eg. $E \rightsquigarrow E^*$

Let $\{f_n\}_{n=1}^{\infty} \subset E^*$

$f_n \to f$ weakly

if $f_n(x) \xrightarrow[n\to\infty]{} f(x), \ \forall x \in E$

Banach–Alaoglu Thm: Ball in E^* $\underset{\text{weak}}{-}$ compact set.

Functional spaces.

Topological spaces

$\downarrow$

Topological vector spaces

$\downarrow$

Norm spaces

$\downarrow$

Banach spaces

$\downarrow$

Hilbert spaces

Example: L^p spaces

operators on spaces

general

$\mathcal{L}(E, F)$

$= \{ f : E \to F$

f linear

& continuous$\}$

particular

$\mathcal{L}(E, \mathbb{C}) = E^*$

$\mathcal{L}(E, E)$

= bounded operators

Chapter 1. Topological spaces

Def: Let X be a set. Define

$$N(X) = \{ Y : Y \subset X \}$$

Define $\mathcal{O}(X) = \{ Y \in N(X) : Y \text{ is "open"} \}$

1. $X, \emptyset \in \mathcal{O}(X)$
2. $\forall \{Y_i\}_{i \in I} \subset \mathcal{O}(X)$

   $\Rightarrow Y = \bigcup_{i \in I} Y_i \in \mathcal{O}(X)$
3. If $Y_1, Y_2 \in \mathcal{O}(X)$

   $\Rightarrow Y_1 \cap Y_2 \in \mathcal{O}(X)$

Then: $(X, \mathcal{O}(X))$ a topological space

Example: ① $X = \mathbb{R}$, $\mathcal{O}(X) =$ "normal" open sets

② $X = \mathbb{Z}$ if $\mathcal{O}(X) =$ all finite subsets of X

$\rightsquigarrow$ not satisfy the assumption

if $\mathcal{O}(X) =$ all subsets of X

$\rightarrow (X, \mathcal{O}(X))$ topological space

Def. Let $(X, \mathcal{O}(X))$ be topological space.

Then: $Y \subset X$ is a closed set

if and only if $X \setminus Y$ is an open set.

Exercise: Let $(X, \mathcal{O}(X))$ be topological space.

① Let $\{X_i\}_{i \in I}$ be a collection of closed set.

Then:

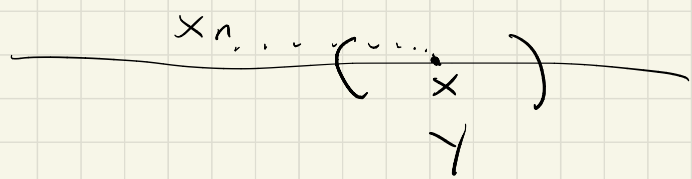
$$\bigcap_{i \in I} X_i \text{ is also a closed set.}$$

② Let X_1 and X_2 be two closed set. Then:

$X_1 \cup X_2$ is also a closed set.

Def. Take a sequence $\{x_n\}_{n=1}^{\infty} \subset (X, \mathcal{O}(X))$. Then we say that $x_n \to x$ in the topological space $(X, \mathcal{O}(X))$ if $\forall Y \in \mathcal{O}(X)$ and $x \in Y$ then: $\exists N \in \mathbb{N}$: $x_n \in Y$ if $n \geq N$.

Ex.



$X = \mathbb{R}$

normal topo.

Remark: To characterise the topology, then in general we need the concept "net" instead of "sequence"

$$\{x_\alpha\}_{\alpha\in I}, \quad I \text{ partial order}$$

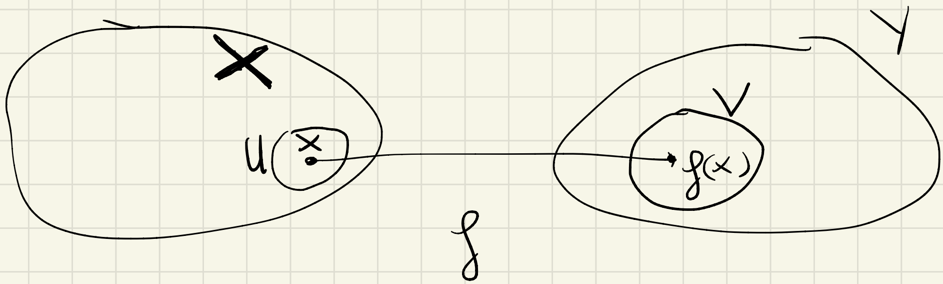
i.e. $\alpha > \beta$ & $\beta > \gamma$
$\Rightarrow \alpha > \gamma$.

We will not use this in our course.

Later we will focus on the situations where the topology is completely characterised by the convergence of sequences.

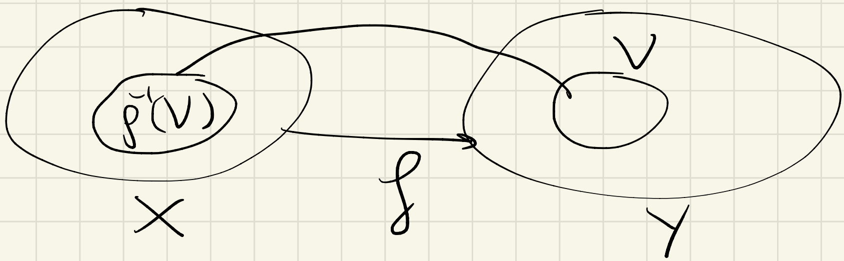
Def. Let $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(Y))$ two topological spaces. Then: $f: X \to Y$ is a continuous function if:

$\forall x \in X$, $\forall V \in \mathcal{O}(Y)$ s.t. $f(x) \in V$
then $\exists U \in \mathcal{O}(X)$ s.t. $x \in U$ & $f(U) \subset V$



Exercise: Let $(X, \mathcal{O}(X))$ & $(Y, \mathcal{O}(Y))$ be two topological spaces. Then $f: X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for any V open in Y

$$f^{-1}(V) = \{ x \in X : f(x) \in V \}$$



Remark: Let $\mathbb{R}$ be a topological space with the normal topology. Then:

$f: \mathbb{R} \to \mathbb{R}$ is continuous if & only if

$$\boxed{x_n \to x \implies f(x_n) \to f(x)}$$

Vector spaces (linear spaces)

Def. Let X be a vector space with field $\underbrace{\mathbb{R} \text{ or } \mathbb{C}}_{\mathbb{F}}$

namely it satisfies:

1. $\forall x, y \in X \Rightarrow x + y \in X$

and "+" has the 3 properties

- $x + y = y + x$
- $(x+y)+z = x+(y+z)$
- $\exists\, 0 \in X$ s.t. $x + 0 = x, \ \forall x \in X$

2. $x \in X, \ \forall \alpha \in \mathbb{F}$ then: $\alpha x \in X$

and $(\alpha, x) \mapsto \alpha x$ satisfies

$$\begin{cases} (\alpha+\beta)x = \alpha x + \beta x, & \forall \alpha, \beta \in \mathbb{F}, \forall x \in X \\ \alpha(x+y) = \alpha x + \alpha y, & \forall \alpha \in \mathbb{F}, \forall x, y \in X \\ \alpha(\beta x) = (\alpha\beta)x, & \forall \alpha, \beta \in \mathbb{F}, \forall x \in X \\ 0x = 0, & \forall x \in X \end{cases}$$

Remark: We can justify the last property as:

$$0x = (1-1)x = 1.x + (-1)x = x - x = 0$$

16.04

<u>Thm</u>: If X is a vector space, $\exists$ a basis $\{x_i\}_{i\in I}$

<u>Recall</u>: $A=\{x_i\}_{i\in I}$ is a basis if

·) The vectors in A are linearly independent

namely, we can not write any $a\in A$ as a nontrivial linear combination of finite elements in A $\longrightarrow \sum \alpha_i x_i,\ \alpha_i\in\mathbb{C}$

·) $\mathrm{Span}(A)=X$, i.e. $\forall x\in X$, we can write $x=\sum \alpha_i x_i$, $x_i\in A$, $\alpha_i\in\mathbb{C}$

finite sum

<u>Idea</u>: If X is finite dim $\rightarrow$ easy

- Take $x_1\in X$
- If $\mathrm{Span}(x_1)=X \rightarrow$ stop
- If $\mathrm{Span}(x_1)\subsetneq X \Rightarrow \exists x_2\in X\setminus \mathrm{Span}(x_1)$
- If $\mathrm{Span}(x_1,x_2)=X \rightarrow$ stop

  If not $\exists x_3\in X\setminus \mathrm{Span}(x_1,x_2)$.

" Optimal way to do induction"

Zorn Lemma: If $\emptyset \neq P$ is a ordered set, and inductive then $\exists$ a maximal element.

•) P is ordered $\Leftrightarrow$ $\exists$ $<$ ("$\leq$") on P

s.t. if) $a < b$ and $b < c \Rightarrow a < c$

$a < b$ and $b < a \Rightarrow a = b$

• Q is totally ordered $\Leftrightarrow$ Q ordered and $\forall a, b \in Q \Rightarrow a < b$ or $b < a$.

• P is inductive if $\forall Q \subset P$, Q totally ordered $\Rightarrow \exists$ a maximal element of Q.

• Maximal element $a \in P \Leftrightarrow \nexists\, b \in P$ s.t. $a < b$ and $b \neq a$.

Prog of " any vector space X has a basis "

Define $P_0 = \{ A \overset{\text{subset}}{\subset} X ,\ A$ is linearly independent$\}$

$\Rightarrow \quad P = \{ Span(A) : A \in P_0 \}$

Define $X_1 < X_2$ if $X_1 \subset X_2$

- Then $(P, <)$ is ordered
- P is inductive because if $Q \subset P$ is totally ordered, $Q = \{ X_i , i \in I \}$

$\Rightarrow \forall i \& j, \ X_i \subset X_j$ or $X_j \subset X_i$.

The maximal element of Q is

$$X_Q = \bigcup_{X_i \in Q} X_i .$$

Zorn lemma
$\Rightarrow \quad \exists$ maximal element of P

$$Span(A).$$

We claim that $Span(A) = X$

by contradiction. If $Span(A) \subsetneq X$, then

$\exists\, x_A \in X \setminus \mathrm{Span}(A)$

$$\Rightarrow \quad \underbrace{\mathrm{Span}(A)}_{\text{maximal element of } P} \subsetneq \underbrace{\mathrm{Span}(A \cup \{x_A\})}_{\in P} \subseteq X$$

<u>Def</u> (Topological vectorspaces)

Let X be a vector space. Assume $(X, \mathcal{O}(X))$ is a topological space. Then X is called a topological vector space if $(x, y) \to x+y$ and $(\alpha, x) \to \alpha x$ are continuous, where $x, y \in X$, $\alpha \in \mathbb{C}$

<u>Example</u>. (Normed spaces) Let X be a vector space and assume that $\exists \|\cdot\| : X \to [0, \infty)$ s.t.

- $\|x\| \geq 0$, and $\|x\| = 0 \iff x = 0$
- $\|\alpha x\| = |\alpha| \, \|x\|$, $\forall x \in X$, $\forall \alpha \in \mathbb{C}$
- $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$.

Example: Assume $\{X_i\}_{i\in I}$ a collection of normed spaces. Assume $X := \bigcap_{i\in I} X_i \neq \emptyset$

We want to define on X a topology. In general, the topology on X is not necessarily induced by a norm.

Think of $X_p = L^p(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C}, \int_{\mathbb{R}^d} |f|^p < \infty \}$

$$\Rightarrow \underset{\text{OK}}{X} = \bigcap_{p\geq 1} L^p(\mathbb{R}^d)$$

non empty since $\underbrace{C_c^\infty(\mathbb{R}^d)}_{\text{topology of test functions}}$

$$C_c^\infty(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d), \text{ compactly supp.} \}$$

Take $\{f_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^d)$. We call

$f_n \to f$ on $C_c^\infty(\mathbb{R}^d)$ if

$$\begin{cases} \bigcup_n \operatorname{supp} f_n \text{ is bounded} \\ \sup |f_n - f| \to 0, \quad \sup |D^\alpha f_n - D^\alpha f| \to 0 \ \forall \alpha \end{cases}$$

Distributions

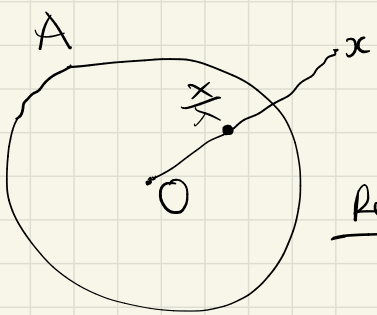
$$\mathcal{D}(\mathbb{R}^d) = \left(C_c^\infty(\mathbb{R}^d)\right)^*$$

$$= \{ \mathcal{L} : C_c^\infty(\mathbb{R}^d) \to \mathbb{C}$$

linear & continuous $\}$

$\hookrightarrow$ topological vector space.

<u>Def</u>: (Minkowski function) Let X be a topological vector space. Let A be a convex, open set in X, $0 \in A$. Define:



$$p_A(x) = \inf_{\lambda > 0} \left\{ \frac{x}{\lambda} \in A \right\}$$

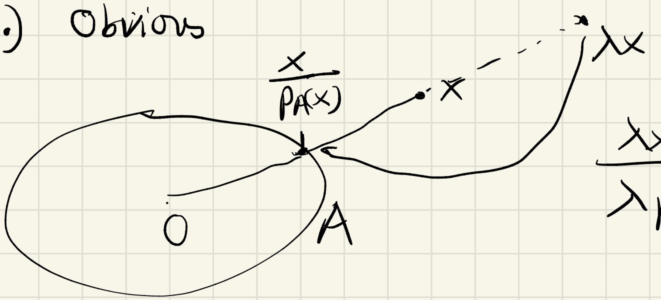
<u>Remark</u>:

- $p_A(x) = \infty$ if $\nexists \lambda$ s.t. $\frac{x}{\lambda} \in A$
- In most of applications, $p_A(x) < \infty$, $\forall x$.

Thm: If A is convex, open, $0 \in A \subset X$ normed space

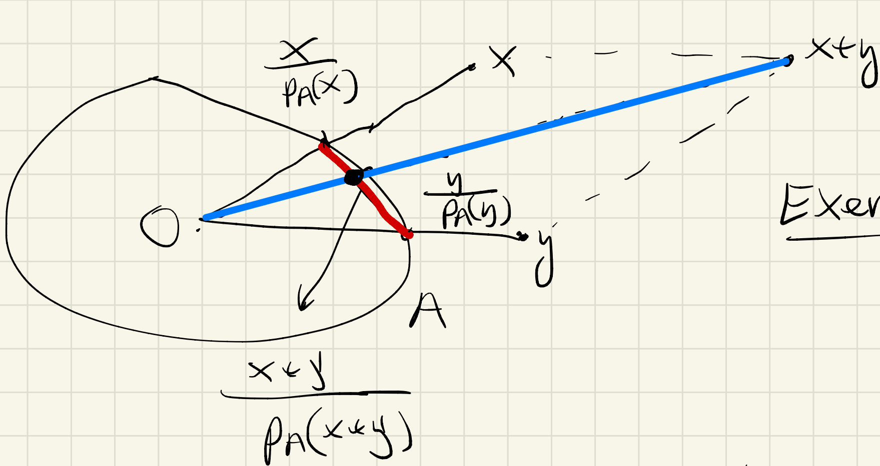
- $p_A(\lambda x) = \lambda\, p_A(x)$, $\forall \lambda > 0,\ \forall x \in X$
- $p_A(x+y) \leq p_A(x) + p_A(y)$
- $A = \{x : \ p_A(x) < 1\}$.

Proof. ·) Obvious



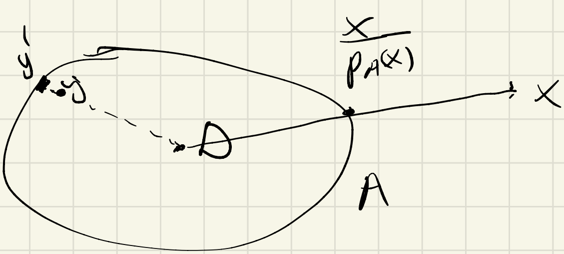
$$\frac{\lambda x}{\lambda\, p_A(x)} = \frac{x}{p_A(x)}$$

·)



<u>Exercise</u>

·)



$x \notin A$ & $\frac{x}{\lambda} \in A$

$\Rightarrow \lambda < 1$

<u>Exercise</u>

<u>Theorem</u> (Hahn - Banach Theorem , Helly version)

let X be (topological) vector space and let Y be a subspace of X.

given a linear functional $f: Y \to \mathbb{R}$ s.t.

$$|f(y)| \leq p(y), \ \forall y \in Y.$$

Then, $\exists$ a linear functional $\tilde{f} : X \to \mathbb{R}$ s.t.

$$\begin{cases} \tilde{f}|_Y = f \\ |\tilde{f}(x)| \leq p(x), \ \forall x \in X. \end{cases}$$

This holds for any function $p : X \to \mathbb{R}$ satisfying

$$\begin{cases} p(\lambda x) = \lambda p(x), \ \forall \lambda > 0, \ \forall x \in X \\ p(x+y) \leq p(x) + p(y), \ \forall x, y \in X \end{cases}$$

<u>Proof.</u> We use Zorn lemma

$$\mathcal{P} = \{ g : D(g) \xrightarrow{\text{linear}} \mathbb{R} ; \ Y \subset \underset{\text{subspace}}{D(g)} \subset X$$

$$\text{and } g|_Y = f, \text{ and } |g(x)| \leq p(x) \}$$

On $\mathcal{P}$ define $g_1 < g_2$ if g_2 is an extension of g_1

i.e. $D(g_1) \subset D(g_2)$ and $g_2|_{D(g_1)} = g_1$.

<u>Claim:</u> P is inductive.

Take $Q \subset P$, Q totally ordered

$$\begin{cases} D(g_Q) = \bigcup_{g \in Q} D(g) \\ \forall x: g_Q(x) = g(x) \text{ for some } g \in Q \end{cases}$$

$\rightsquigarrow$ g_Q is the maximal element for Q

By Zorn lemma, $\exists$ maximal element of P called g. We prove that

$$D(g) = X.$$

Assume by contradiction that $D(g) \subsetneq X$

then $\exists x_0 \in X \setminus D(g)$.

<u>Define</u>

$$\tilde{X} = D(g) + \mathbb{R}x_0$$

chosen later

$$\tilde{g}: \tilde{X} \to \mathbb{R}, \quad \tilde{g}(x + \lambda x_0) = g(x) + \lambda a$$

linear

$\forall x \in D(g_p), \forall \lambda \in \mathbb{R}$

Wants

$$|\tilde{g}(x+\lambda x_0)| \leq p(x+\lambda x_0)$$

$$\Leftrightarrow -p(x+\lambda x_0) \leq g(x) + \lambda a \leq p(x+\lambda x_0)$$

$\forall x \in D(g)$

$\forall \lambda \in \mathbb{R}$

<u>Question:</u> How to prove that $\exists a$?

Need: $(\lambda = 1)$ $\forall x \in D(g)$

$$-p(x+x_0) - g(x) \leq \overset{\nearrow \text{independent of } x}{a} \leq p(x+x_0) - g(x)$$

$$\Leftrightarrow -p(y+x_0) - g(y) \leq a \leq p(x+x_0) - g(x)$$

$\forall x, y \in D(g)$

$$\Leftrightarrow \sup_{y \in D(g)} [-p(y+x_0) - g(y)]$$

$$\leq a \leq \inf_{x \in D(g)} [p(x+x_0) - g(x)]$$

$\exists a$ is ensured if

$$-p(y+x_0) - g(y) \leq p(x+x_0) - g(x)$$

$$\Leftrightarrow -g(y) + g(x) \leq p(x+x_0) + p(y+x_0)$$

$$\Leftrightarrow -g(y) + g(x) \leq p(x+x_0) + p(y+x_0)$$

$$\text{“}\Leftrightarrow\text{”} \quad g(x+x_0) - g(y+x_0) \leq p(x+x_0) + p(y+x_0)$$

heuristic

and “use” $|g(x+x_0)| \leq p(x+x_0)$

$|g(y+x_0)| \leq p(y+x_0)$ $\rightarrow$ “done”

$$\Leftarrow \quad g(x-y) \leq p(x+x_0) + p(y+x_0)$$

Use: $|g(x-y)| \leq p(x-y) \quad \forall x, y \in D(g)$

$$= p(x+x_0 - (y+x_0))$$

$$\leq p(x+x_0) + p(y+x_0)$$

(p satisfies triangle ineq on X)

Come back to the general statement

$$-P(x+\lambda x_0) - g(x) \leq \lambda a \leq P(x+\lambda x_0) - g(x)$$

$$\forall \lambda \in \mathbb{R},\ \forall x \in D(g)$$

$$\Leftrightarrow\ -P(y+\lambda x_0) - g(y) \leq \lambda a \leq P(x+\lambda x_0) - g(x)$$

$$\forall \lambda \in \mathbb{R},\ \forall x, y \in D(g)$$

$$\Leftrightarrow \sup_{y \in D(g)} \left(\frac{-P(y+\lambda x_0) - g(y)}{\lambda} \right) \leq a \leq \inf_{x \in D(g)} \left(\frac{P(x+\lambda x_0) - g(x)}{\lambda} \right)$$

if $\lambda > 0$

and $$\inf_{y \in D(g)} \left(\frac{-P(y-\lambda_1 x_0) - g(y)}{-\lambda_1} \right) \geq a \geq \sup_{x \in D(g)} \left(\frac{P(x-\lambda_1 x_0) - g(x)}{-\lambda_1} \right)$$

if $\lambda = -\lambda_1 < 0$

<u>Conclusion</u>: we need

$$\max\left[\sup_y \left(\frac{-P(y+\lambda x_0) - g(y)}{\lambda} \right), \sup \left(\frac{P(y-\lambda_1 x_0) - g(y)}{-\lambda_1} \right) \right]$$

$$\leq a \leq \min\left[\inf_x \frac{P(x+\lambda x_0) - g(x)}{\lambda}, \inf \frac{-P(x-\lambda_1 x_0) - g(x)}{-\lambda_1} \right]$$

The existence of a requires that $\forall \lambda, \lambda_1 > 0$

$$\max\left[\sup_y \left(\frac{-P(y+\lambda x_0) - g(y)}{\lambda}\right), \sup\left(\frac{P(y-\lambda_1 x_0) - g(y)}{-\lambda_1}\right)\right]$$

$$\leq \min\left[\inf_x \frac{P(x+\lambda x_0) - g(x)}{\lambda}, \inf \frac{-P(x-\lambda_1 x_0) - g(x)}{-\lambda_1}\right]$$

① $-P(y+\lambda x_0) - g(y) \leq P(x+\lambda x_0) - g(x), \forall x, y \in D(g)$

$$\iff g(x-y) \leq P(x+\lambda x_0) + P(y+\lambda x_0)$$

This follows from $|g(x-y)| \leq P(x-y)$

$$= P((x+\lambda x_0) - (y+\lambda x_0))$$

$$\leq P(x+\lambda x_0) + P(y+\lambda x_0) \text{ by triangle ineq}$$

② $$\frac{P(x+\lambda x_0) - g(x)}{\lambda} \geq \frac{-P(y-\lambda_1 x_0) + g(y)}{+\lambda_1}$$

$$\iff \lambda_1 P(x+\lambda x_0) - \lambda_1 g(x) \geq -\lambda P(y-\lambda_1 x_0) + \lambda g(y)$$

$$\iff P(\lambda_1 x + \lambda\lambda_1 x_0) + P(\lambda y - \lambda\lambda_1 x_0)$$

$$\geq g(\lambda_1 x + \lambda y)$$

Use: $|g(\lambda_1 x + \lambda y)| \leq P(\lambda_1 x + \lambda y)$

$$= P[(\lambda_1 x + \lambda\lambda_1 x_0) + (\lambda y - \lambda\lambda_1 x_0)] \leq \dots$$

$\leq P(\lambda_1 x + \lambda\lambda_1 x_0) + P(\lambda y - \lambda\lambda_1 x_0)$ as desired by the triangle inequality.

<u>Remark.</u> Here the proof works if $f : Y \to \mathbb{R}$.

More generally, if $f : Y \to \mathbb{C}$

$$\Rightarrow f = f_1 + i f_2 \quad \text{where } f_1, f_2 : Y \to \mathbb{R}$$

and $|f(x)| = \sqrt{|f_1(x)|^2 + |f_2(x)|^2} \leq p(x)$

Use the real case, we can extend f_1 and f_2 to $\tilde{f}_1, \tilde{f}_2 : X \to \mathbb{R}$ and $\tilde{f} := \tilde{f}_1 + i\tilde{f}_2$ satisfies $|\tilde{f}(x)| = \sqrt{|\tilde{f}_1(x)|^2 + |\tilde{f}_2(x)|^2} \leq p(x)$.

|| We need to require $p(\lambda x) = |\lambda|\, p(x)$, $\forall \lambda \in \mathbb{C}$

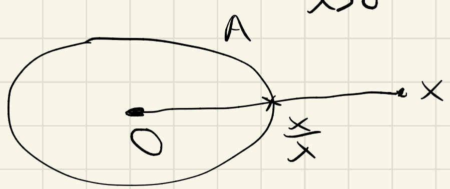
Geometric version of Hahn-Banach Theorem

Consider X a normed space.

Thm: If A is convex, open, $0 \in A \subset X$ normed space

- $p_A(\lambda x) = \lambda\, p_A(x)$, $\forall \lambda > 0$, $\forall x \in X$
- $p_A(x+y) \leq p_A(x) + p_A(y)$
- $A = \{x : p_A(x) < 1\}$.

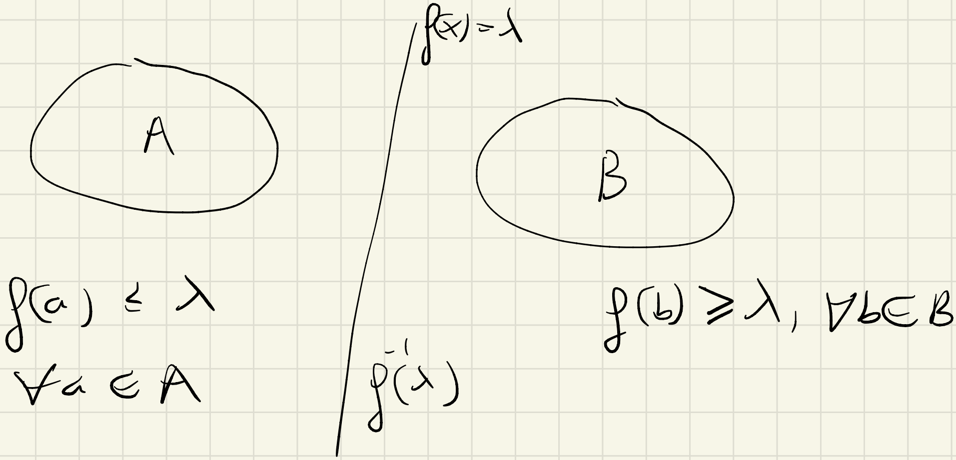
where $\quad p_A(x) = \inf\limits_{\lambda > 0} \{ \frac{x}{\lambda} \in A \}$



Def. For X a normed space, let $f : X \to \mathbb{R}$ be a linear functional.

Exercise. f is continuous $\iff$ $f^{-1}(\lambda) = \{x \in X : f(x) = \lambda\}$ is a closed set in X

The set $f^{-1}(\lambda)$ is called a hyperplane.



<u>Theorem:</u> (Hahn-Banach, geometric version)
Let X be a normed space. Let $A, B \subset X$ such that A, B convex, non-empty, $A \cap B = \emptyset$. Assume that A is <u>open</u>. Then $\exists$ a closed hyperplane that separates A and B, namely $\exists f: X \to \mathbb{R}$ linear & continuous s.t. $0 \not\equiv f$

$$f(a) \le \lambda \le f(b), \quad \forall a \in A, b \in B.$$

<u>Proof.</u> Recall A convex, open.

<u>Step 1</u> Think of the case when $B = \{x_0\}$. Then $x_0 \notin A$. We prove that $\exists f: X \to \mathbb{R}$ linear & continuous s.t. $f(a) < f(x_0)$, $\forall a \in A$.

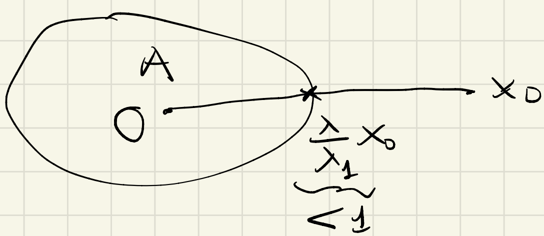
By translation, we can assume $0 \in A \not\ni \{x_0\}$. Then we can define the Minkowski function $p_A : X \to \mathbb{R}_+$

$$p_A(x) = \inf_{\lambda > 0} \left\{ \frac{x}{\lambda} \in A \right\}.$$

Define the linear functional $g : \mathrm{Span}\{x_0\} \to \mathbb{R}$

$$g(\lambda x_0) = \lambda \quad , \forall \lambda \in \mathbb{R}$$

Then:
$$|g(\lambda x_0)| \leq p_A(\lambda x_0) \, , \forall \lambda \in \mathbb{R}$$
$$\Leftarrow \quad \lambda \leq p_A(\lambda x_0)$$
$$= \inf_{\lambda_1 > 0} \left\{ \frac{\lambda x_0}{\lambda_1} \in A \right\} \quad \checkmark$$



By Hahn-Banach theorem, we can find an extension $f : X \to \mathbb{R}$ linear and

$$|f(x)| \leq p_A(x) \quad , \forall x \in X$$

$\Rightarrow$ f is continuous as $\sup_{x \in A} |f(x)| \leq 1$

and A is open, $0 \in A$.

<u>Ex</u>: Let X be a normed space. Let $f : X \to \mathbb{C}$ be a linear functional. Then

$$f \text{ is continuous} \iff \sup_{x \in A} |f(x)| < \infty \text{ for some open set } A.$$

<u>Conclusion</u>:

$$f(x) \leq p_A(x) < \underset{f(x_0) = g(x_0)}{\overset{\shortparallel}{1}}, \quad \forall x \in A$$

<u>Step 2.</u> Consider the general case A convex, open and B convex, and $A \cap B = \emptyset$.

Define $C = A - B = \{a - b : a \in A, b \in B\}$.

<u>Claim</u>: C is open and convex (check!)

‖ Write $C = \bigcup_{b \in B} \underbrace{(A - b)}_{\text{open because } A \text{ open}}$ ‖

Using Step 1 with C open, convex, and $\underset{}{\overset{x_0}{\overset{\shortparallel}{0}}} \notin C$

Then: $\exists f : X \to \mathbb{R}$ linear continuous s.t.

$$f(c) < f(x_0) = 0, \; c \in C$$

$$\iff f(a - b) < 0, \quad \forall a \in A, b \in B$$

$$\iff f(a) < f(b), \quad \forall a \in A, b \in B.$$

Take $\sup_{a \in A} f(a) \leq \lambda \leq \inf_{b \in B} f(b)$

$\Rightarrow \quad f(a) \leq \lambda \leq f(b) \quad , \forall a \in A, b \in B.$

$\|$ From the proof, $f(a) < f(b)$, $\forall a \in A, b \in B$.

$\boxed{\text{April 23}}$

<u>Corollary</u> Let X be a normed space. Let A be convex, closed in X. Let $x_0 \notin A$. Then $\exists f \in X^*$ (i.e. f is linear, continuous) and f is real-valued s.t.

$$f(a) < f(x_0), \quad \forall a \in A.$$

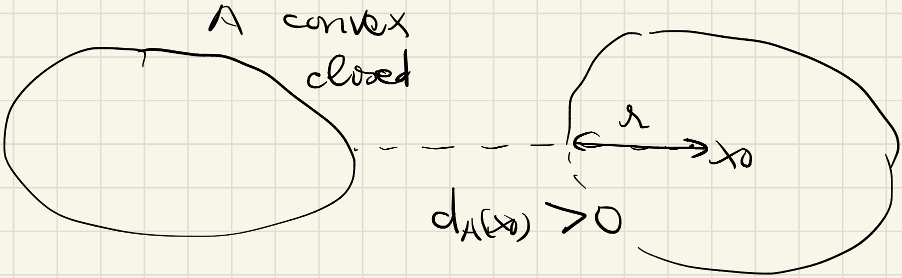
<u>Proof.</u> <u>Claim.</u> A is closed & $x_0 \notin A \Rightarrow$ $\exists r > 0$ s.t. $B(x_0, r) \cap A = \phi$.

In fact, if we define the distance function

$$d_A(x) = \inf_{a \in A} \| x - a \|$$

then $d_A(x) = 0 \Leftrightarrow x \in A$ (exercise)

We choose $r = d_A(x_0)/2 \Rightarrow B(x_0, r) \cap A = \phi$.



$B = B(x_0, r)$

open, convex

By Hahn-Banach Theorem (geometric version)

$\Rightarrow \exists f : X \to \mathbb{R}$ linear, continuous s.t.

$$f(a) < f(x_0), \quad \forall a \in A. \quad \square$$

<u>Remark</u>: In general, if A is convex, $x_0 \notin A$, then there are situations that we cannot find any hyperplane to separate A and x_0, namely $\nexists f : X \to \mathbb{R}$ linear, continuous s.t.

$$f(a) < f(x_0), \quad \forall a \in A.$$

<u>Example</u>: $X = \ell^1(\mathbb{N})$

$$= \{ x = (x_n)_{n=1}^{\infty}, \ x_n \in \mathbb{R}, \ \sum_{n=1}^{\infty} |x_n| < \infty \}$$

$$\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n|, \quad x = (x_n)_{n=1}^{\infty}$$

Choose $A = \{ x = (x_n)_{n=1}^{\infty} \in X, \sum_{n=1}^{\infty} 2^n |x_{2n}| < \infty \}$

Then A is a subspace of $X \to A$ is convex.

Then take $a_0 = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \dots)$

$$= (x_n)_{n=1}^{\infty} \text{ where } \begin{cases} x_{2n-1} = 0 \\ x_{2n} = \frac{1}{2^n} \end{cases}$$

$\Rightarrow a_0 \in \ell^1(\mathbb{N}) = X$ but $a_0 \notin A$

There are no $f : X \to \mathbb{R}$ linear & continuous s.t.

$$f(a) < f(a_0), \forall a \in A.$$

Indeed, assume $\exists$ such a map f. Then:

$$\lambda f(a) = f(\lambda a) < f(a_0), \forall a \in A, \forall \lambda \in \mathbb{R}$$

$$\Rightarrow \begin{cases} f(a) = 0, \forall a \in A \\ \text{and } f(a_0) > 0 \end{cases}$$

<u>Key observation</u>: A is <u>dense</u> in X, in particular

$\exists \{a_n\}_{n=1}^{\infty} \subset A$ and $a_n \to a_0$ in X as $n \to \infty$

actually $a_n = (0, \frac{1}{2}, 0, \frac{1}{4}, \dots, 0, \frac{1}{2^n}, 0, 0, 0, \dots)$

Thus:

$$0 = f(a_n) \quad , \forall n$$

$$\Big\downarrow n \to \infty \quad \text{because } f \text{ is continuous } X \to \mathbb{R}$$

$$f(a_0) > 0$$

$\rightsquigarrow$ contradiction! $\square$

<u>Corollary</u>: Take X be a normed space. Let A be a subspace of X s.t. $\overline{A} \neq X$.

Then $\exists\, 0 \not\equiv f : X \to \mathbb{R}$ linear & continuous s.t.

$$f(a) = 0, \ \forall a \in \overline{A}$$

<u>Proof</u>. Because $\overline{A} \neq X \Rightarrow \exists\, x_0 \in X \setminus \overline{A}$.

Using the Hahn-Banach Theorem for $(\overline{A}, x_0)$

$\Rightarrow \exists\, f : X \to \mathbb{R}$ linear & continuous s.t.

$$f(a) < f(x_0), \ \forall a \in \overline{A}$$

(convex, closed)

$$\lambda f(a) = f(\lambda a) < f(x_0) \quad , \forall a \in \overline{A}, \forall \lambda \in \mathbb{R}$$

$\Rightarrow f(a) = 0, \ \forall a \in \overline{A}$. $\square$

Chapter 2. Metric spaces

Def Let X be a set and let $d : X \times X \to \mathbb{R}_+$

s.t.

- $d(x,y) = d(y,x) \geq 0 \ , \forall x,y \in X$
- $d(x,y) = 0 \iff x = y$
- $d(x,y) \leq d(x,z) + d(y,z) \ , \forall x,y,z \in X$

Then (X, d) is called a metric space.

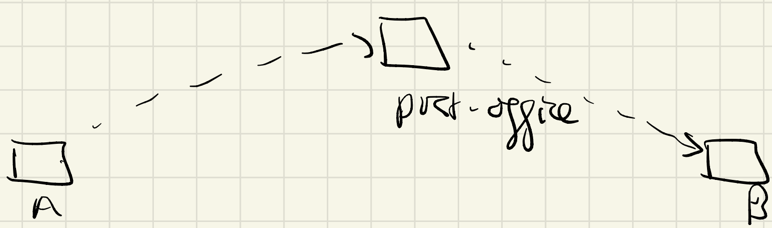
Examples.

- Normed space $(X, \|\cdot\|)$ then $d(x,y) = \|x-y\|$ is the standard metric.

- Normed space $(X, \|\cdot\|)$, then define

$$d(x,y) = \begin{cases} \|x\| + \|y\| & , \forall x \neq y \\ 0 & , x = y \end{cases},$$

This metric is called "post-office metric"



•) Discrete metric space: Let X be a set, define
$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Def Let (X,d) be a metric space. Define the open ball
$$B(x,r) = \{ y \in X : d(x,y) < r \}$$

and closed ball
$$\tilde{B}(x,r) = \{ y \in X : d(x,y) \leq r \}.$$

Let $A \subset X$ be a subset of X. Take $x_0 \in A$. Then $x_0 \in \mathring{A}$ (interior of A) $\iff$
$\exists r > 0 : \quad B(x_0, r) \subset A$.

Remark: In general,
$$\tilde{B}(x_0,r) \text{ might be diff } \overline{B(x_0,r)}^{\text{closure}}$$

Remark: If (X,d) is a metric space, then X is a topological space where the open sets are the unions of open balls.

Def. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (X, d). Then $x_n \to x \iff d(x_n, x) \to 0$ $n \to \infty$.

Remind. In general, if X is a topological space, $x_n \to x \iff \forall U^{open}, x \in U$, then

$x_n \in U$ if n large enough.

The two definitions are equivalent! (exercise)

Def. Let (X, d) be a metric space and let $A \subset X$. Then define $\overline{A} = \{x \in X ; x = \lim_{n \to \infty} x_n$ for $(x_n)_{n=1}^{\infty} \subset A\}$.

Def. Let (X, d) be a metric space and $A \subset X$. Then A is <u>dense</u> $\iff \overline{A} = X$.

<u>Exercise:</u> Let (X, d) be a metric space. Let $A \subset X$.

(a) Prove that A is closed $\iff$ if $\{x_n\}_{n=1}^{\infty} \subset A$ and $x_n \to x$ in X, then $x \in A$.

(b) Prove that A is open $\iff \forall x \in A, \exists r > 0$ s.t. $B(x, r) \subset A$.

Complete metric space.

Def. Let (X, d) be a metric space. We say that X is complete if any Cauchy sequence in X has a limit in X.

Namely, if $(x_n)_{n=1}^{\infty} \subset X$ and

$$\lim_{\min(n,m)\to\infty} d(x_n, x_m) = 0$$

then $\exists\, x \in X$ s.t. $x_n \to x$, i.e. $d(x_n, x) \xrightarrow[n\to\infty]{} 0$

Example. •) $\mathbb{Q} = \{$ rational numbers, $d(x,y) = |x-y|\}$

$\to$ not complete

•) $\mathbb{R} = \{$ real numbers, $d(x,y) = |x-y|\}$

$\to$ complete

•) Take $X = \{ (x_n)_{n=1}^{\infty},\ x_n = 0$ if n large $\}$

$$\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n| \quad , \quad x = (x_n)_{n=1}^{\infty}$$

$\Rightarrow (X, \|\cdot\|_{\ell^1})$ is not complete.

However, $\overline{X} = \ell^1(\mathbb{N})$ and $\ell^1(\mathbb{N})$ is complete

·) $X = C_c(\mathbb{R}^d, \mathbb{C}) = \{ f : \mathbb{R}^d \to \mathbb{C}$ cont. compactly supported$\}$

$$\|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)|\, dx$$

Then $(X, \|\cdot\|_{L^1})$ is <u>not</u> complete.

but $\overline{X} = L^1(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$ is complete.

<u>Theorem:</u> Let (X, d) be a complete metric space. Assume that

$$B(x_1, r_1) \supset B(x_2, r_2) \supset \dots, \forall n$$

Then if $\lim_{n \to \infty} r_n = 0$, then $\exists\, x \in X$ s.t.

$$\lim_{n \to \infty} x_n = x. \quad \text{Equivalently:}$$

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} = \{x\}. \checkmark$$

<u>Proof.</u> We know that if $m > n \Rightarrow x_m \in B(x_n, r_n)$

$\Rightarrow d(x_n, x_m) < r_n \to 0$ as $n \to \infty$

$\Rightarrow \{x_n\}$ is a Cauchy sequence

Since (X, d) is a complete metric space,

$\exists$ limit x s.t. $\lim_{n\to\infty} x_n = x$.

It's not difficult to check that

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} = \{x\}. \quad \text{(exercise)}.$$

<u>Q:</u> If (X, d) is a metric space and

$$B(x_1, r_1) \supset B(x_2, r_2) \supset \dots$$

Can we claim that $\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \neq \emptyset$?

(We will come back next time)

<u>Answers</u>: 1 No, e.g. if (X, d) is <u>not</u> complete

like $X = \mathbb{Q}$ and $x_n \to \sqrt{2}$

2 An example for complete metric spaces: (?)

Hint: $X = \mathbb{N}$, find $r_n > 0$ s.t.

$$B(n, r_n) = \{n, n+1, n+2, \dots\}$$

3 If X is complete <u>normed space</u>, then

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \neq \emptyset \quad \text{(why?)}$$

<u>Theorem:</u> (Baire Category Theorem)

Let (X, d) be a complete metric space. Let $(X_n)_{n=1}^{\infty}$ be a sequence of subset of X, $X_n \subset X$ s.t. X_n is open and dense in X, $\forall n$.

<u>Then:</u> $A = \bigcap_{n=1}^{\infty} X_n$ is dense in X.

<u>Proof.</u> To prove that A is dense, we need to show that $\forall$ ball $B(x_0, r_0) \subset X$, then:

$$A \cap B(x_0, r_0) \neq \emptyset. \quad \text{(why?)}$$

We can write

$$A \cap B(x_0, r_0) = \bigcap_{n=1}^{\infty} \left(X_n \cap B(x_0, r_0) \right)$$

Since X_1 is open and dense, $X_1 \cap B(x_0, r_0)$ is also open, & non-empty Then $\exists\, B(x_1, r_1) \subset \left(X_1 \cap B(x_0, r_0) \right)$, $r_1 < 1$

Then $X_2 \cap B(x_1, r_1)$ open & non-empty

$\Rightarrow \exists\, B(x_2, r_2) \subset X_2 \cap B(x_1, r_1)$, $r_2 < 1/2$

$\vdots$

By induction, $\exists\, B(x_n, r_n) \subset X_n \cap B(x_{n-1}, r_{n-1})$, $r_n < 1/n$

Thus we get

$$B(x_1, r_1) \supset B(x_2, r_2) \supset \dots$$

and $r_n < \frac{1}{n} \to 0$. Thus since (X,d) is complete, we conclude that

$$\bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} = \{x\} \neq \emptyset.$$

In particular, $x \in B(x_n, r_n) \subset X_n \cap B(x_0, r_0)$
$\forall n$

$$\Rightarrow \quad x \in \bigcap_{n=1}^{\infty} \left(X_n \cap B(x_0, r_0) \right)$$

$$= A \cap B(x_0, r_0) \qquad \square$$

<u>Corollary:</u> Let (X,d) be a complete metric space. Let $(Y_n)_{n=1}^{\infty}$ be s.t. $Y_n \subset X$, closed $\forall n$ and

$$\bigcup_{n=1}^{\infty} Y_n = X.$$

Then $\exists n$ and $x \in X$, $r > 0$ s.t.

$$B(x, r) \subset Y_n.$$

Proof: let Y_n be closed, $\bigcup_{n=1}^{\infty} Y_n = X$.

We claim that $\exists n, \exists$ open ball $B(x,r) \subset Y_n$.

Assume by contradiction that

$\forall n, \forall$ open ball $B(x,r) \not\subset Y_n$.

$\Rightarrow \forall n, \quad X \setminus Y_n$ is dense in X (why?)

and it is also open.

Then: $$\bigcap_{n=1}^{\infty} (X \setminus Y_n) = X \setminus \left(\bigcup_{n=1}^{\infty} Y_n\right) = \emptyset$$

this is a contradiction to Baire Category Thm since we know that

$$\bigcap_{n=1}^{\infty} (X \setminus Y_n) \text{ is dense in } X.$$

Exercise (*hard) let (X,d) be a complete metric space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X s.t. it has no subsequence which is convergent. Prove that $\exists \varepsilon > 0$ and a subsequence $\{x_{n_k}\}$ s.t. $d(x_{n_k}, x_{n_\ell}) \geq \varepsilon, \forall k \neq \ell$.

Compactness:

Def. Let X be topological space. Then X is compact if the following holds:

$$\text{if } X = \bigcup_{i \in I} A_i \,, \quad A_i \text{ open } \forall i$$

$$\text{then } X = \bigcup_{i \in I'} A_i \,, \quad I' \text{ finite}, \, I' \subset I.$$

Theorem (Borel-Lebesgue) Let (X, d) be a metric space. Then X is compact if and only if $\forall$ sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and $x \in X$ s.t.

$$x_{n_k} \to x \quad \text{as} \quad k \to \infty.$$

Proof. **Part 1** "$\Rightarrow$" Assume that X is compact (in the topological sense). Let $\{x_n\}_{n=1}^{\infty} \subset X$.

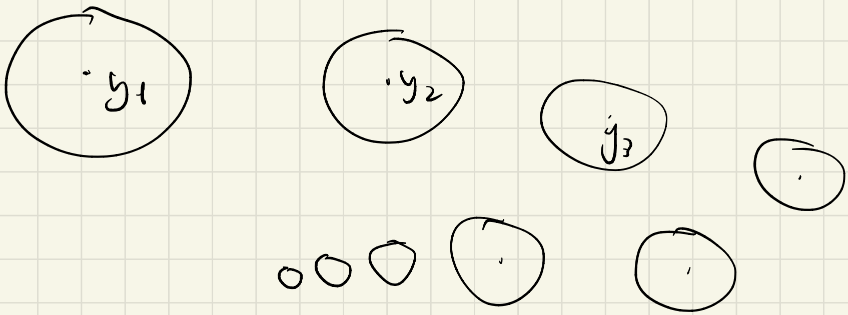
Then we prove that $\exists$ subsequence $x_{n_k} \to x$ in X.

Assume by contradiction that $\nexists$ subsequence convergent.

Step 1. Claim. $\exists$ a subsequence $\{y_i\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $r_i > 0$

s.t.

$$B(y_i, r_i) \not\ni y_j \quad \forall j \neq i.$$



We construct the balls $B(y_i, r_i)$ by induction.

$\cdot)$ $\{x_n\}_{n=1}^{\infty}$ does not converge to x_1. This implies

$\exists$ a subsequence $\{x_n^{(1)}\}_{n=1}^{\infty}$ s.t. $x_1^{(1)} = x_1$ and

$$B(x_1^{(1)}, r_1) \not\ni x_n^{(1)}, \forall n \neq 1.$$

($\{x_n^{(1)}\} = \{x_{f_1(n)}\}$ with $f_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing)

$\cdot)$ Since $\{x_n^{(1)}\}$ does not converge to $x_2^{(1)}$

$\Rightarrow$ $\exists$ a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ s.t.

$$B(x_2^{(1)}, r_2) \not\ni x_n^{(2)}, \forall n \neq 2.$$

By induction, we can find subsequences $\{x_n^{(m)}\}$

s.t. $B(x_n^{(n)}, r_n) \not\ni x_k^{(m)} \quad \forall k \neq n$

We take final subsequence

$$y_n = x_n^{(n)}$$

$$\Rightarrow \quad B(y_n, r_n) \not\ni y_m \quad \text{with } m \neq n$$

This is called Cantor diagonal argument.

<u>Remind.</u> To prove that $\mathbb{R}$ is uncountable

we can consider $\sum_{n=1}^{\infty} a_n \left(\frac{1}{3}\right)^n, \; a_n \in \{0,1\}$

Then $(a_n)_{n=1}^{\infty} \longrightarrow \sum_{n=1}^{\infty} a_n \left(\frac{1}{3}\right)^n \in [0,1]$

The set $\{(a_n)_{n=1}^{\infty} : a_n \in \{0,1\}\}$ is uncountable

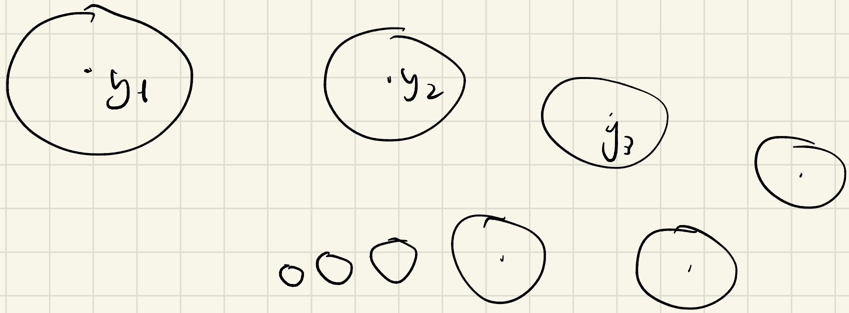
because if you have a sequence

$$x_m = \left(a_n^{(m)}\right)_{n=1}^{\infty} \quad \text{in this set}$$

then $\tilde{x} = \left(\tilde{a}_n^{(n)}\right)_{n=1}^{\infty}$ where

$$\left.\tilde{a}_n^{(n)} = \begin{cases} 0 & \text{if } a_n^{(n)} = 1 \\ 1 & \text{if } a_n^{(n)} = 0 \end{cases} \right\} \Rightarrow \tilde{x} \neq x_m \quad \forall m$$

Step 2. Claim $Y = \bigcup_{i=1}^{\infty} \{y_i\}$ is closed.



let $\{z_n\}_{n=1}^{\infty}$ be a sequence in Y s.t. $z_n \to z$ in X. Then we prove that $z \in Y$.

There are two possibilities:

① If $\exists i$; $\{n : z_n = y_i\}$ is infinite

$\Rightarrow \quad z = y_i \in Y$.

② $\forall i$; $\{n : z_n = y_i\}$ is finite, then $\exists$ a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ s.t.

$$z_{n_k} = y_{m_\ell} \quad \text{for} \quad m_\ell \geq k$$

$\Rightarrow$ $\exists$ subsequence $\{y_{m_\ell}\}$ which converges to z

but $\{y_{m_\ell}\}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$

$\Rightarrow$ we get a contradiction as

$\nexists$ no subsequence convergent of $\{x_n\}$.

$\rightarrow$ the second possibility is ruled out.

Step 3. Write

$$X = \bigcup_{i=1}^{\infty} \underbrace{B(y_i, r_i)}_{\text{open}} \cup \underbrace{(X \setminus Y)}_{\text{open}}, \quad Y = \bigcup_{i=1}^{\infty} \{y_i\}$$

$\Rightarrow$ by the topological compactness, $\exists$ finite covering of X, namely

$$X = \bigcup_{i=1}^{N} B(y_i, r_i) \cup (X \setminus Y)$$

but this is a contradiction with the fact that $B(y_i, r_i) \not\ni y_j$ with $j \neq i$.

(in particular, y_{N+1} is not in the right side)

Thus we get a contradiction with the assumption on the topological compactness of X.

This completes "$\Rightarrow$".

Part 2 "$\Leftarrow$" Assume that $\forall \{x_n\}_{n=1}^{\infty} \subset X$, $\exists$ a subsequence convergent. We need to prove that X is topologically compact.

Namely, if $X = \bigcup_{i \in I} U_i$ with U_i open, $\forall i$,

then we find a finite covering

$$X = \bigcup_{i \in I'} U_i \quad , \quad I' \text{ finite} \subset I.$$

Step 1. $\exists r > 0$ s.t. $\forall x \in X$, then

$$B(x,r) \subset U_i \text{ for some } i \in I.$$

Assume by contradiction: $\forall r > 0$, $\exists x \in X$, $\forall i \in I$:

$$B(x,r) \not\subset U_i.$$

In particular, $r = \frac{1}{n}$, $n = 1, 2, 3, \ldots$

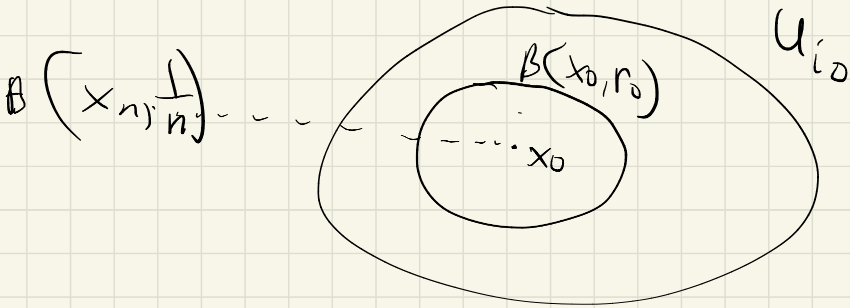
$\forall n \geq 1$, $\exists x_n \in X$, $\forall i \in I$, $B(x_n, \frac{1}{n}) \not\subset U_i$.

Because $\{x_n\}_{n=1}^{\infty} \subset X \Rightarrow \exists$ subsequence

$$x_{n_k} \longrightarrow x_0 \text{ in } X.$$

Because $x_0 \in X = \bigcup_{i \in I} U_i \Rightarrow x_0 \in U_{i_0}$ for some i_0

$\Rightarrow B(x_0, r_0) \subset U_{i_0}$ (since U_{i_0} is open)



Then since $x_n \to x_0 \Rightarrow x_n \in B(x_0, \frac{r_0}{2})$ if n is large enough

$$\Rightarrow B(x_n, \frac{r_0}{2}) \subset B(x_0, r_0)$$

by the triangle inequality

$$\Rightarrow B(x_n, \frac{1}{n}) \subset B(x_0, r_0) \subset U_{i_0}$$

if n is large enough

But this is a contradiction to the fact that $B(x_n, \frac{1}{n}) \not\subset U_i, \forall i \in I$.

$\to$ the claim is correct.

Step 2. $X = \bigcup_{i \in I} U_i = \bigcup_{x \in X} B(x, r)$

Claim: $\exists$ finite set $Y \subset X$ s.t.

$$X = \bigcup_{x \in Y} B(x,r) \subset \bigcup_{i \in I' \text{ finite}} U_i$$

$\left(B(x,r) \subset U_i \text{ for } x \in Y, i \in I' \right)$

By contradiction, assume $\nexists$ finite set Y s.t.

$$X = \bigcup_{x \in Y} B(x,r).$$

This means that $\exists \{x_n\}_{n=1}^{\infty}$ s.t.

$$x_m \notin \bigcup_{n=1}^{M} B(x_n, r), \forall m > M$$

$$\Rightarrow \quad d(x_m, x_n) \geq r > 0, \forall m \neq n.$$

Then $\nexists$ subsequence of $\{x_n\}$ which is convergent $\to$ a contradiction.

This completes the proof. $\square$

Exercise. Let (X,d) be a compact metric space. Prove that X is complete.

Remark. If $\{x_n\}$ is a Cauchy sequence and $\exists$ subsequence $x_{n_k} \to x$, then $x_n \to x$.

Def. Let (X,d) be a metric space and $A \subset X$. Then A is compact if (A,d) is a compact space.

Exercise. Let (X,d) be a metric space and let $A \subset X$ be a compact set.

1. Prove that A is closed.
2. Prove that $A \subset B(x,r)$ for some $x \in A$ and $r > 0$.

Exercise. Let (X,d) be a metric space and $x_n \to x$. Then $A = \bigcup_{n=1}^{\infty} \{x_n\} \cup \{x\}$ is a compact set.

Continuity.

Let (X, d_X) and (Y, d_Y) be two metric space. Take $f: X \to Y$. Then f is continuous

(by definition) $\Longleftrightarrow$ $f^{-1}(A)$ is open in X for all set A open in Y.

Exercise. The function $f: X \to Y$ (two metric spaces) is continuous $\Longleftrightarrow$

$$f(x_n) \to f(x) \text{ in } Y, \; \forall \; x_n \to x \text{ in } X.$$

Thm: Let X, Y be two metric spaces and $f: X \to Y$ be continuous. Then: $\forall$ K compact set in X, $f(K) = \{ f(x) : x \in K \}$ is a compact set in Y.

Proof. Take $\{y_n\}_{n=1}^{\infty} \subset f(K)$. We prove that $\exists$ a subsequence $y_{n_k} \to y \in f(K)$.

By definition, $y_n = f(x_n)$ for $x_n \in K$.

Since K is compact, from $\{x_n\}_{n=1}^{\infty} \subset K$, $\exists$ a subsequence $x_{n_k} \to x$ in K. Because f is continuous,

$$y_{n_k} = f(x_{n_k}) \longrightarrow f(x) \in f(K) . \qquad \square$$

Corollary: If (X, d) is a compact metric space and $f: X \to \mathbb{R}$ is continuous, then

$$\exists x_0 \in X : \quad f(x_0) = \min_{x \in X} f(x) .$$

Proof. By the previous theorem, $f(X)$ is a compact set in $\mathbb{R}$ $\Rightarrow$ $\exists$ a minimum

$$y_0 = \min f(X)$$

$$\Rightarrow \quad y_0 = f(x_0) \qquad \square$$

Corollary: If (X, d) is a compact metric space and $f: X \to \mathbb{R}$ is continuous and $f(x) > 0$, $\forall x \in X$. Then $\exists\, \varepsilon > 0$ s.t. $f(x) \geq \varepsilon$, $\forall x \in X$.

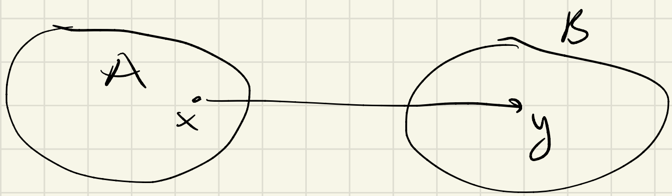
Proof $\exists x_0$: $\underset{\substack{\vee \\ 0}}{f(x_0)} = \min_{x \in X} f(x)$

Then we can take $\varepsilon = f(x_0) > 0$ $\square$

<u>Example</u>: Let (X,d) be a metric space. Let $A, B \subset X$, A is closed and B is compact. If $A \cap B = \emptyset$, then $d(A,B) > 0$.

Here

$$d(A,B) = \inf \{ d(x,y) : x \in A, y \in B \}$$



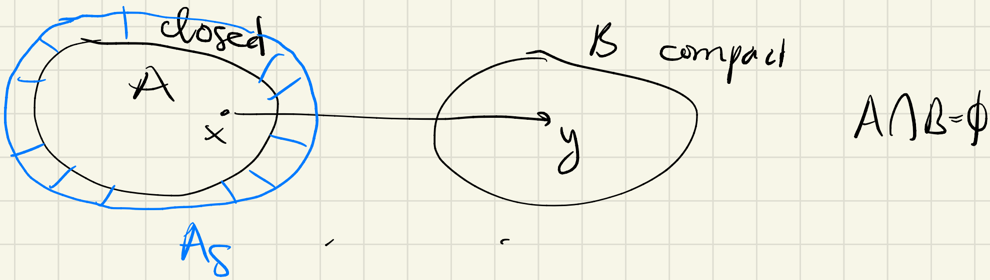
<u>Proof</u>: $$d(A,B) = \inf_{x \in A} \inf_{y \in B} d(x,y)$$

$$= \inf_{y \in B} \Big(\underbrace{\inf_{x \in A} d(x,y)}_{=: d(A,y)} \Big)$$

The function $f : B \to \mathbb{R}$, $f(y) = d(A,y)$ is continuous and $f(y) > 0$, $\forall y \in B$ since $y \notin A^{\text{closed}}$

$\Rightarrow \exists \varepsilon > 0$ s.t. $f(y) = d(A, y) \geq \varepsilon$, $\forall y \in B$

$$\Rightarrow \quad d(A, B) = \inf_{y \in B} d(A, y) \geq \varepsilon > 0.$$



Moreover, if we define

$$A_\delta = \{ x \in X : d(x, A) < \delta \}$$

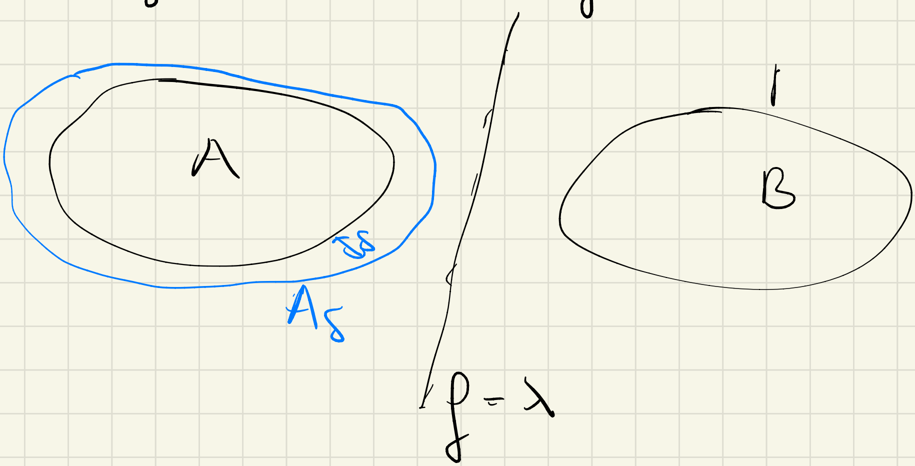
then $A_\delta \cap B = \emptyset$ and

$$d(A_\delta, B) \geq \frac{\varepsilon}{2} > 0$$

provided that $\delta < \frac{\varepsilon}{2}$. $\square$

Thm: (Hahn - Banach, geometric version II)
Let X be a normed space, $A, B \subset X$
s.t. A, B convex, $A \cap B = \emptyset$, and
A is closed & B is compact. Then:
$\exists\, f : X \to \mathbb{R}$ linear continuous and $\lambda \in \mathbb{R}$ s.t.

$$f(a) < \lambda < f(b), \ \forall a \in A, b \in B.$$



Proof. $\exists\, \delta > 0$ s.t.

$$A_\delta = \{ x \in X : d(A, x) < \delta \}$$

$\Rightarrow$ A_δ is convex, open, $A_\delta \cap B = \emptyset$

By the usual geometric version of the Hahn - Banach theorem, $\exists\, f : X \to \mathbb{R}$ linear and $f \not\equiv 0$ s.t. $f(a) \le f(b), \forall a \in A_\delta, b \in B$.

$\forall a \in A, \forall y \in X$ s.t. $\|y\| < \frac{\delta}{2}$ then

$$a + y \in A_\delta$$

$$\Rightarrow \quad f(a+y) \le f(b)$$

$$\Rightarrow \quad f(a) + f(y) \le f(b), \quad \forall a \in A, b \in B$$
$$\forall \|y\| < \frac{\delta}{2}$$

Since $f \not\equiv 0$ and f is linear, then

$$\sup_{\|y\| < \frac{\delta}{2}} f(y) = c_0 > 0$$

Thus, $$f(a) + c_0 \le f(b), \forall a \in A, b \in B$$

$$\Rightarrow \sup_{a \in A} f(a) + c_0 \le \inf_{b \in B} f(b)$$

$$\Rightarrow f(a') < \lambda = \sup_{a \in A} f(a) + \frac{c_0}{2} < f(b')$$

$\forall a' \in A, b' \in B$. $\square$

2 last remarks on metric spaces.

Thm: (Banach Fixed point theorem)

If (X,d) is a complete metric space and $f: X \longrightarrow X$ satisfy: $\exists\, \alpha \in (0,1)$

$$d(f(x), f(y)) \leq \alpha\, d(x,y), \forall x,y \in X$$

Then:
$$\exists!\, x_0 \in X \text{ s.t. } f(x_0) = x_0.$$

Proof. Uniqueness. If $f(x_0) = x_0$ and $f(x_1) = x_1$ then

$$d(x_0, x_1) = d(f(x_0), f(x_1))$$
$$\leq \alpha\, d(x_0, x_1) \quad , \quad \alpha < 1$$
$$\Rightarrow d(x_0, x_1) \leq 0 \Rightarrow x_0 = x_1.$$

Existence: Take $a_0 \in X$, define

$$a_n = f(a_{n-1}) \quad , \forall n = 1, 2, \dots$$

Then:
$$d(a_n, a_{n+1}) = d(f(a_{n-1}), f(a_n))$$
$$\leq \alpha\, d(a_{n-1}, a_n) \leq \dots \leq \alpha^n d(a_0, a_1)$$

Thus: $\quad d(a_n, a_{n+1}) \leq C\alpha^n, \ \forall n = 0,1,2,\dots$

$\Rightarrow \{a_n\}$ is a Cauchy sequence since

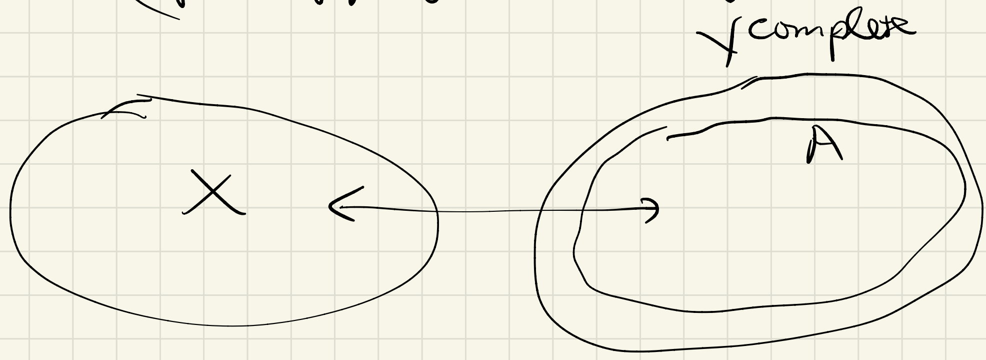
$$\sum_{n\geq 0} d(a_n, a_{n+1}) \leq C \sum_{n\geq 0} \alpha^n < \infty$$

$\Rightarrow \quad a_n \to a_\infty$ $\square$

$\Rightarrow \quad a_\infty = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(a_{n-1}) = f(a_\infty)$

<u>Thm</u> (Completion of a metric space)
Let (X, d) be a metric space. Then $\exists$ a complete metric space (Y, d_Y) s.t. $\exists A$ dense in Y and a bijection $f: X \to A$ which is isometric:

$$d_Y(f(x), f(y)) = d(x,y), \ \forall x, y \in X.$$



<u>Proof</u>. Define $Y = \{ x : x = (x_1, x_2, \dots)$ Cauchy sequence in $X \}$

Define $d_Y(x^{(1)}, x^{(2)}) = \lim\limits_{k\to\infty} d(x^{(1)}_k, x^{(2)}_k)$.

$x^{(1)} = (x^{(1)}_1, x^{(1)}_2, \dots)$

Note that $\{ d(x^{(1)}_k, x^{(2)}_k) \}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, because of the triangle inequality

$$\left| d(x^{(1)}_k, x^{(2)}_k) - d(x^{(1)}_{k'}, x^{(2)}_{k'}) \right|$$

$$\leq d(x^{(1)}_k, x^{(1)}_{k'}) + d(x^{(2)}_k, x^{(2)}_{k'}) \to 0$$

as $k, k' \to \infty$

$\Rightarrow$ $\exists$ limit $\lim\limits_{k\to\infty} d(x^{(1)}_k, x^{(2)}_k)$.

① $A \subset Y$ defined by

$A = \{ x : x = (a, a, a, \dots)$ with $a \in X \}$

$\Rightarrow$ $A \overset{f}{\longleftrightarrow} X$ is a isometry

$$d_Y(f(a), f(b)) = d(a,b), \ \forall a, b \in X$$

$f(a) = (a, a, a, \dots) \in A \subset Y$

② Why A is dense in Y?

given $x = (x_1, x_2, x_3, \dots) \in Y$.

Define $y^{(n)} = (x_n, x_n, x_n, \dots) \in A$

$$d_Y(x, y^{(n)}) = \lim_{k\to\infty} d(x_k, x_n) \to 0$$

as $n \to \infty$

because $d(x_k, x_n) \to 0$ as $k, n \to \infty$

as $(x_1, x_2, \dots)$ is a Cauchy sequence in X.

③ Why is Y complete?

Take a sequence $\{y^{(n)}\}_{n=1}^{\infty}$ of Y, which is a Cauchy sequence, i.e.

$$d_Y(y^{(n)}, y^{(m)}) \to 0 \text{ as } m, n \to \infty.$$

By the definition of Y, $\forall n$:

$y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \dots)$ is a Cauchy sequence in X.

Since $\forall n$, $d(y_k^{(n)}, y_{k'}^{(n)}) \to 0$ as $k, k' \to \infty$

$\Rightarrow \exists k_n$ s.t.

$$d(y_k^{(n)}, y_{k'}^{(n)}) \leq \frac{1}{n}, \forall k, k' \geq k_n.$$

Then

$$\tilde{y} = (\underbrace{y_{k_n}^{(n)}}_{\in X})_{n=1}^{\infty} \in Y$$

Here $\tilde{y}$ is a Cauchy sequence in X because

$$d(y_{k_n}^{(n)}, y_{k_m}^{(m)}) \leq d(y_{k_n}^{(n)}, y_\ell^{(n)})$$

$$+ d(y_\ell^{(n)}, y_\ell^{(m)}) + d(y_\ell^{(m)}, y_{k_m}^{(m)})$$

$$\leq \frac{1}{n} + \underbrace{d(y_\ell^{(n)}, y_\ell^{(m)})}_{(\ell\to\infty)} + \frac{1}{m}, \text{ if } \ell \geq k_n \text{ and } k_m$$

$$\Rightarrow d(y_{k_n}^{(n)}, y_{k_m}^{(m)}) \leq \frac{1}{n} + \frac{1}{m} + d_Y(y^{(n)}, y^{(m)})$$

$$\to 0 \text{ as } m, n \to \infty$$

Thus $(y_{k_n}^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in X

Finally, we prove $y^{(n)} \to \tilde{y}$ in Y. We have

$$d_Y(y^{(n)}, \tilde{y}) = \lim_{m\to\infty} d(y_m^{(n)}, \tilde{y}_m)$$

Here:

$$d(y_m^{(n)}, \tilde{y}_m) = d(y_m^{(n)}, y_{k_m}^{(m)})$$

$$\leq d(y_m^{(n)}, y_\ell^{(n)}) + d(y_\ell^{(n)}, y_\ell^{(m)}) + d(y_\ell^{(m)}, y_{k_m}^{(m)})$$

$$\leq \frac{1}{n} + d(y_\ell^{(n)}, y_\ell^{(m)}) + \frac{1}{m}$$

if $m, \ell \geq k_n$ and $\ell \geq k_m$

$$\overset{(\ell\to\infty)}{\Longrightarrow} d(y_m^{(n)}, \tilde{y}_m) \leq \frac{1}{n} + \frac{1}{m} + d_Y(y^{(n)}, y^{(m)})$$

$$\Rightarrow \limsup_{m\to\infty} d(y_m^{(n)}, \tilde{y}_m) \leq \frac{1}{n} + \limsup_{m\to\infty} d_Y(y^{(n)}, y^{(m)})$$

$$\Rightarrow \limsup_{n\to\infty} d_Y(y^{(n)}, \tilde{y}) \leq \limsup_{n\to\infty} \limsup_{m\to\infty} d_Y(y^{(n)}, y^{(m)})$$

$= 0$ as $(y^{(n)})$ Cauchy in Y

Thus $y^{(n)} \to \tilde{y}$ in $Y \Rightarrow Y$ is complete. $\square$

Comments.

.) In Y, $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots)$ is the same with

$$x^{(2)} = (x_1^{(2)}, x_2^{(2)}, \dots)$$

if $d(x^{(1)}, x^{(2)}) = 0$

$$\quad = \lim_{k\to\infty} (x_k^{(1)}, x_k^{(2)})$$

Eg. $x = (a, a, a, \dots) \in A$ if $a \in X$

this is the same

$x = (a_1, a_2, \dots)$ if $a_n \to a$ in X.

Thus Y contains "equivalent classes" of sequences!

.) In the proof, we used the completeness of $\mathbb{R}$. [A key property of $\mathbb{R}$ is that it is totally ordered] $\square$