An approximate solution for nonlinear backward parabolic equations

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Abstract

We consider the backward parabolic equation

$$\begin{cases} u_t + Au = f(t, u(t)), & 0 < t < T, \\ u(T) = g, \end{cases}$$

where A is a positive unbounded operator and f is a nonlinear function satisfying a Lipschitz condition, with an approximate datum g. The problem is severely ill-posed. Using the truncation method we propose a regularized solution which is the solution of a system of differential equations in finite dimensional subspaces. According to some a priori assumptions on the regularity of the exact solution we obtain several explicit error estimates including an error estimate of Hölder type for all $t \in [0, T]$. An example on heat equations and numerical experiments are given.

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1 Introduction

Let H be a real or complex Hilbert space with the inner product $(.,.)$ and the norm $||.||$. Let $A : D(A) \to H$ be a positive self-adjoint unbounded operator and let $f : [0, T] \times H \to H$. We consider the problem of finding a function $u : [0, T] \to H$ such that

$$\begin{cases} u_t + Au = f(t, u(t)), & 0 < t < T, \\ u(T) = g, \end{cases} \tag{1.1}$$

where the datum $g \in H$ is given with an error of order ε. We shall always assume that A admits an orthonormal eigenbasis $\{\phi_n\}_{n=1}^{\infty}$ corresponding to eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, where

$$0 < \lambda_1 \le \lambda_2 \le ... \text{ and } \lim_{n\to\infty} \lambda_n = \infty,$$

and that f satisfies the Lipschitz condition

$$\|f(t,w_1)-f(t,w_2)\| \le k\,\|w_1-w_2\|\,, \tag{1.2}$$

where $k \ge 0$ is a constant independent of t, w_1, w_2.

In spite of the uniqueness (see Theorem 2), Problem (1.1) is severely ill-posed, e.g. a small error of datum g may cause a large error of the corresponding solution (if exists). Indeed, from the formal form

$$u(t)=\sum_{n=1}^{\infty}\left(e^{(T-t)\lambda_n}\,(\phi_n,g)-\int\limits_t^T e^{(s-t)\lambda_n}\,(\phi_n,f(s,u(s)))\,ds\right)\phi_n$$

we can see that the instability is due to the fast growth of $e^{(T-t)\lambda_n}$ as $\lambda_n \to \infty$. Therefore, regularization methods are necessary to make the numerical computation possible.

Let us first review some results on the homogeneous problem, i.e. Problem (1.1) with $f=0$. In this case there are several regularization methods in the literature such as the quasi-reversibility method of Lattès and Lions [8], the Tikhonov regularization method [13], the Gajewski and Zacharias' method based on eigenfunctions expansion [3], the method of semi-group and Sobolev equation [2, 12]. In the pioneering work in 1967, Lattès and Lions [8] introduced the quasi-reversibility method in which they added a "corrector" into the main equation to get the well-posed problem

$$\begin{cases} u_t + Au + \varepsilon A^* A u = 0, \\ u(T) = g. \end{cases}$$

However, the stability magnitude of the approximating problem is of order $e^{T/\varepsilon}$ which is very large for $\varepsilon > 0$ small. In 1984, Showalter [11] proposed the method of quasi-boundary value problem in which he added a corrector into the final value to get the well-posed problem

$$\begin{cases} u_t + Au = 0, \\ \varepsilon u(0) + u(T) = g. \end{cases}$$

The stability magnitude of the approximating problem in this case is of order ε^{-1}. However, while this method may give approximation for any fixed $t > 0$, it is still difficult to derive an explicit error estimate at the original time $t = 0$. In 2005, Denche and Bessila [1] used a variant of this method to give an error estimate of logarithmic type at $t = 0$ provided that $u(0) \in D(A)$. Recently, Hao et al. [5] employed the original method in [11] to improve the approximation. More precisely, they considered three assumptions on the exact solution

$$||u(0)|| \le E_0, \tag{1.3}$$

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta'}\,|(\phi_n,u(0))|^2 \le E_1^2, \tag{1.4}$$

$$\sum_{n=1}^{\infty} e^{2\beta\lambda_n}\,|(\phi_n,u(0))|^2 \le E_2^2, \tag{1.5}$$

where β, β' stand for positive constants. Under the very weak condition (1.3) they obtained an error estimate of Hölder type at any fixed $t \in (0,T)$. If (1.4) holds then they had an error estimate of logarithmic type at $t=0$ and if (1.5) holds then they even had an error estimate of Hölder type at $t=0$. Note that the assumption $u(0) \in D(A)$ in [1] is a special case of (1.4) with $\beta = 1$.

Although there are many works on the homogeneous problem, the literature on inhomogeneous cases, and in particular on the nonlinear case, is quite scarce. In 1994, Long and Dinh [9] used the semi-group method of Ewing [2] to treat the nonlinear case and attained an error estimate of order $t^{-2}(\ln(1/\varepsilon))^{-1}$ for each $t > 0$. This estimate is of logarithmic type at any fixed $t > 0$ but useless at $t = 0$. More recently, in 2008, Trong and Tuan [15] improved the quasi-reversibility method to give an approximation of order $\varepsilon^{t/T}$ for $t > 0$ and $(\ln(1/\varepsilon))^{-1/2}$ at $t = 0$. However they required a condition somehow similar to $u(t) \in D(e^{TA})$ for all $t \in [0,T]$ which is equivalent to (1.8) below with $\beta = T$.

The aim of the present paper is to generalize the results in [5] and improve the existing results for nonlinear case (although our approach is different from [5]). We consider three conditions

$$\sum_{n=1}^{\infty} e^{2\lambda_n \min\{t,\beta\}} \left|(\phi_n, u(t))\right|^2 \leq E_0^2, \tag{1.6}$$

$$\sum_{n=1}^{\infty} \lambda_n^{2\beta'} e^{2\lambda_n \min\{t,\beta\}} \left|(\phi_n, u(t))\right|^2 \leq E_1^2, \tag{1.7}$$

$$\sum_{n=1}^{\infty} e^{2\beta\lambda_n} \left|(\phi_n, u(t))\right|^2 \leq E_2^2 \tag{1.8}$$

for all $t \in [0,T]$, where β, β' stand for positive constants. We shall show that (1.6) is sufficient to get an approximation with error estimate of Hölder type at any fixed $t \in (0,T]$. Moreover if either (1.7) or (1.8) holds then we obtain an error estimate of logarithmic type or Hölder type for all $t \in [0,T]$, respectively.

Let us briefly discuss on the motivation of the conditions (1.6)-(1.8). Technically, they require that the exact solution $u(t)$ of Problem (1.1) must be very smooth, especially for small time $t \in [0,\beta]$. To make a comparison, we note that in the homogeneous case, namely $f = 0$, (1.6)-(1.8) reduces to the conditions (1.3)-(1.5) above, which are used in [5], due to the identity

$$e^{\lambda_n t} (\phi_n, u(t)) = (\phi_n, u(0)) \quad \text{for all } n \in \mathbb{N}.$$

Moreover, many earlier works on the nonlinear case, for example [15, 16], needed that (1.8) holds for $\beta = T$. In this case, our assumptions (1.6)-(1.8) seems a litte slighter since, for instance, we do not demand the following condition on the final value

$$\sum_{n=1}^{\infty} e^{2T\lambda_n} \left|(\phi_n, u(T))\right|^2 < \infty.$$

We mention that while such a condition is reasonable for the homogeneous problem, it is not necessarily true for inhomogeneous cases. In our opinion, the open problem on relaxing the assumptions on the exponential growth in (1.6)-(1.8) is very interesting, but also really difficult.

Let us sketch our method. As we discussed above, the fast growth of the term $e^{(T-t)\lambda_n}$ is the source of the instability of Problem (1.1). A natural way to treat it is to restrict the problem in a finite dimensional subspace, an idea from the truncation method. More precisely, we shall use the following well-posed problem

$$\begin{cases} u_t + Au = \mathbb{P}_M f(t, u(t)), \quad 0 \le t < T, \\ u(T) = \mathbb{P}_M g, \end{cases} \tag{1.9}$$

where $\mathbb{P}_M$ is the orthogonal projection onto the eigenspace $\mathrm{span}\{\phi_n | \lambda_n \le M\}$, i.e.

$$\mathbb{P}_M w = \sum_{\lambda_n \le M} (\phi_n, w)\phi_n \quad \text{for all } w \in H.$$

As we shall see later, Problem (1.9) is well-posed and its solution is a local approximation (namely for $t > T - \beta$) of the exact solution of the original problem (1.1). Our method is first to compute the solution for $t \in [T - t_1, T)$ for some $0 < t_1 < \beta$, then use the resulting value at $T - t_1$ to calculate the solution for $t \in [T - 2t_1, T - t_1)$, and so on.

The rest of the paper is organized as follows. In Section 2 we shall consider the well-posed problem (1.9) and its relation to the original problem (1.1). In Section 3, we construct a regularized solution and give error estimates. A heat equation is considered in Section 4 as an example for our construction and a numerical test is implemented in Section 5 to verify the effect of our method. We finish the paper with some concluding remarks in Section 6.

2 Well-posed problem

In this section we consider the well-posed problem (1.9) and error estimates between its solution and the solution of the original problem (1.1).

Theorem 1 (Well-posed problem)**.** *For each $g \in H$ Problem (1.9) has a unique solution $u \in C^1([0,T], \mathbb{P}_M(H))$. Moreover, the solution depends continuously on the datum in the sense that if u_i is the solution with respect to g_i, $i = 1, 2$, then*

$$\|u_1(t) - u_2(t)\| \le e^{(k+M)(T-t)} \|g_1 - g_2\| .$$

Proof. Note that if u is a solution of (1.9) then $u(t) \in \mathbb{P}_M(H)$ for all $t \in [0, T]$. Define $G_M(t, w) = -Aw + \mathbb{P}_M f(t, w)$. Thus Problem (1.9) is a system of nonlinear differential equations

$$\begin{cases} u_t = G_M(t, u(t)), 0 < t < T \\ u(T) = \mathbb{P}_M g \end{cases}$$

on the finite dimensional subspace $\mathbb{P}_M(H)$. Using the fact $\|Aw\| \le M\|w\|$ for $w \in \mathbb{P}_M(H)$ and the Lipschitz condition (1.2) we deduce that

$$\|G_M(t, w_1) - G_M(t, w_2)\| \le (k + M) \|w_1 - w_2\| \quad \text{for all } w_1, w_2 \in \mathbb{P}_M(H).$$

The well-posedness of the above system thus follows from the Picard-Lindelöf theorem, a basis result in ODEs (see, e.g., [7]). $\square$

We call $u \in C([0,T],H)$ a (weak) solution of Problem (1.1) if

$$(\phi_n, u(t)) = e^{\lambda_n (T-t)}(\phi_n, g) - \int\limits_t^T e^{\lambda_n(s-t)}(\phi_n, f(s,u(s)))ds \tag{2.1}$$

for all $n = 1, 2, \ldots$ Note that u is a solution of Problem (1.9) if and only if (2.1) holds for all n such that $\lambda_n \le M$ and $(\phi_n, u(t)) = 0$ otherwise.

A simple analysis shows that Problem (1.9) approximates Problem (1.1) in the sense that if u_j is the solution of Problem (1.9) with $(g, M) = (g_j, M_j)$ and

$$\lim_{j\to\infty} M_j = \infty, \quad \lim_{j\to\infty} u_j = u \ \text{ in } C([0,T],H),$$

then u is a (weak) solution of Problem (1.1) with $g := \lim_{j\to\infty} g_j$. However it is still unknown if the convergence of solutions of Problem (1.9) occurs, and even if it does then we still know nothing about the convergence rate. At this point some a priori assumptions on the regularity of the exact solution of Problem (1.1) are necessary. The following lemma gives some error estimates between the solutions of two problems (1.9) and (1.1).

Lemma 1. *Assume that Problem (1.1) with $g = g_0 \in H$ has a weak solution $u_0 \in C([0,T],H)$. For any $\varepsilon > 0$, let $g_\varepsilon \in H$ such that $||g_\varepsilon - g_0|| \le E\varepsilon$, where E is a constant independent of ε. Denote by u_ε the solution of Problem (1.9) with $g = g_\varepsilon$ and $M = \log(1/\varepsilon)/\tau$, for some $\tau \ge T$.*
(i) If u_0 satisfies (1.6) with $\beta \ge T$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C\varepsilon^{t/\tau} \quad \textit{for all } t \in [0,T].$$

(ii) If u_0 satisfies (1.7) with $\beta \ge T$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C \max\{(\log(1/\varepsilon)^{-\beta'}, \varepsilon^{(\tau-T)/\tau}\}\varepsilon^{t/\tau} \quad \textit{for all } t \in [0,T].$$

(iii) If u_0 satisfies (1.8) then

$$\|u_\varepsilon(t) - u_0(t)\| \le C \max\{\varepsilon^{(\beta-T)/\tau}, \varepsilon^{(\tau-T)/\tau}\}\varepsilon^{t/\tau} \quad \textit{for all } t \in [0,T].$$

Here $C = C(E,k,T,u_0)$ stands for a positive constant independent of t and ε.

Remark 1. *(1) In the homogeneous case ($f = 0$) the conditions (1.3)-(1.4) imply (1.6)-(1.7), respectively, with $\beta = T$. In this case u_ε is a good approximation for u_0 for all $t \in (0,T)$.*

(2) In (ii) if $\tau > T$ then we get an error estimate of logarithm type at $t = 0$.

(3) In (iii) if $\beta > T$ and $\tau > T$ then we get an error estimate of Hölder type for all $t \in [0,T]$. However if $\beta < T$ then the estimate in (iii) is just useful if t is near T, namely $t > T - \beta$.

Proof. (i) Using the Parseval equality, the representation (2.1) and the Lipschitz condition (1.2) we have

$$\|u_\varepsilon(t) - P_M u(t)\|^2 = \sum_{\lambda_n \le M} |(\phi_n, u(t) - u_\varepsilon(t))|^2$$

$$= \sum_{\lambda_n \le M} \left| e^{\lambda_n (T-t)}(\phi_n, g_\varepsilon - g_0) - \int_t^T e^{\lambda_n (s-t)}(\phi_n, f(s, u_\varepsilon(s)) - f(s, u_0(s)))ds \right|^2$$

$$\le \sum_{\lambda_n \le M} \left(2e^{2M(T-t)} |(\phi_n, g_\varepsilon - g_0)|^2 + 2T \int_t^T e^{2M(s-t)} |(\phi_n, f(s, u_\varepsilon(s)) - f(s, u_0(s)))|^2 ds \right)$$

$$\le 2e^{2M(T-t)} \|g_\varepsilon - g_0\|^2 + 2T \int_t^T e^{2M(s-t)} \|f(s, u_\varepsilon(s)) - f(s, u_0(s))\|^2 ds$$

$$\le 2e^{2M(T-t)} \varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2M(s-t)} \|u_\varepsilon(t) - u_0(t)\|^2 ds.$$

On the other hand from (1.6) with $\beta = T$ one has

$$\begin{aligned} \|u(t) - P_M u(t)\|^2 &= \sum_{\lambda_n > M} |(\phi_n, u_0(t))|^2 \\ &\le e^{-2Mt} \sum_{\lambda_n > M} e^{2\lambda_n t} |(\phi_n, u_0(t))|^2 \le e^{-2Mt} E_0^2. \end{aligned} \tag{2.2}$$

From the above estimates using Parseval equality again we get

$$\|u_\varepsilon(t) - u_0(t)\|^2 = \|u(t) - P_M u(t)\|^2 + \|u_\varepsilon(t) - P_M u(t)\|^2$$
$$\le e^{-2Mt} E_0^2 + 2e^{2M(T-t)} \varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2M(s-t)} \|u_\varepsilon(t) - u_0(t)\|^2 ds.$$

The latter inequality can be rewritten as

$$e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 \le E_0^2 + 2e^{2MT} \varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2Ms} \|u_\varepsilon(t) - u_0(t)\|^2 ds.$$

The Gronwall's inequality implies

$$e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 \le \left(E_0^2 + 2e^{2MT} \varepsilon^2 E^2\right) e^{2k^2 T}.$$

Replacing $M = \log(1/\varepsilon)/\tau$ with $\tau \ge T$ we obtain

$$\|u_\varepsilon(t) - u_0(t)\|^2 \le C^2 e^{-2Mt} = C^2 \varepsilon^{t/\tau} \quad \text{for all } t \in [0, T]$$

where $C = \sqrt{E_0^2 + 2E^2} e^{k^2 T}$.

(ii) If (1.7) holds with $\beta = T$ then we can process as in the above proof where the only change is to replace (2.2) by

$$\|u(t) - P_M u(t)\|^2 \le M^{-2\beta'} e^{-2Mt} \sum_{\lambda_n > M} \lambda_2^{2\beta'} e^{2\lambda_n t} |(\phi_n, u_0(t))|^2 \le M^{-2\beta'} e^{-2Mt} E_1^2.$$

We thus obtain

$$e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 \le M^{-2\beta'} E_1^2 + 2e^{2MT}\varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2Ms} \|u_\varepsilon(t) - u_0(t)\|^2 \, ds.$$

Using the Gronwall's inequality we find

$$\begin{aligned} e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 &\le \left(M^{-2\beta'} E_2^2 + 2e^{2MT}\varepsilon^2 E^2 \right) e^{2k^2 T} \\ &\le C^2 \max\{(M\tau)^{-2\beta'}, e^{2MT}\varepsilon^2\} \end{aligned}$$

where $C = \sqrt{\tau^{2\beta'} E_2^2 + 2E^2} e^{k^2 T}$. Replacing $M = \log(1/\varepsilon)/\tau$ we obtain

$$\|u_\varepsilon(t) - u_0(t)\|^2 \le C^2 \max\{(\log(1/\varepsilon)^{-2\beta'} \varepsilon^{2t/\tau}, \varepsilon^{2(t+\tau-T)/\tau}\} \text{ for all } t \in [0, T].$$

(iii) If u_0 satisfies (1.8) then we may replace (2.2) in the proof of part (i) by

$$\|u(t) - P_M u(t)\|^2 \le e^{-2M\beta} \sum_{\lambda_n > M} e^{2\lambda_n \beta} |(\phi_n, u_0(t))|^2 \le e^{-2M\beta} E_2^2.$$

Thus

$$\begin{aligned} e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 &\le e^{2M(t-\beta)} E_2^2 + 2e^{2MT}\varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2Ms} \|u_\varepsilon(t) - u_0(t)\|^2 \, ds \\ &\le e^{2M(T-\beta)} E_2^2 + 2e^{2MT}\varepsilon^2 E^2 + 2k^2 T \int_t^T e^{2Ms} \|u_\varepsilon(t) - u_0(t)\|^2 \, ds. \end{aligned}$$

It follows from the Gronwall's inequality that

$$e^{2Mt} \|u_\varepsilon(t) - u_0(t)\|^2 \le \left(e^{2M(T-\beta)} E_2^2 + 2e^{2MT}\varepsilon^2 E^2 \right) e^{2k^2 T}.$$

We conclude that

$$\|u_\varepsilon(t) - u_0(t)\|^2 \le C^2 \max\{e^{2M(T-t-\beta)}, e^{2M(T-t)}\varepsilon^2\}$$

where $C = \sqrt{E_2^2 + 2E^2} e^{k^2 T}$. Replacing $M = \log(1/\varepsilon)/\tau$ we get the desired result. $\square$

3 Regularized solution and error estimates

We first prove the uniqueness for Problem (1.1) before considering the regularization.

Theorem 2 (Uniqueness). *For any $g \in H$ Problem (1.1) has at most one solution $u \in C^1((0,T),H) \cap C([0,T],D(A))$.*

Here the requirement $u \in C([0,T],D(A))$ means $Au \in C([0,T],H)$.

Proof. Assume that u_1 and u_2 are two solutions for (1.1). Put $w = u_1 - u_2$. Then $w(T) = 0$ and due to Lipschitz condition (1.2)

$$\|w_t + Aw\| = \|f(t,u_1(t)) - f(t,u_2(t))\| \le k\,\|w\|\,, \quad 0 < t < T.$$

This implies that $w = 0$ due to the theorem of Ghidaglia (see [4], Theorem 1.1). Thus $u_1 = u_2$. □

We now employ the well-posed problem (1.9) to construct a regularized solution for Problem (1.1). Assume that Problem (1.1) has an exact solution u_0 satisfying a priori condition (1.6). If $\beta \ge T$ then Lemma 1 (i) allows us to approximate $u_0(t)$ for any $t > 0$. However in general $\beta > 0$ may be small and Lemma 1 (iii) gives an approximation for $t > T - \beta$. Our method is first to compute the solution for $t \in [T - t_1, T)$ for some $0 < t_1 < \beta$, then use the resulting solution at $T - t_1$ to calculate the solution for $t \in [T - 2t_1, T - t_1)$ and so on. By this way after finite steps we return to the case $\beta \ge T$ and then we may solve the problem completely.

Theorem 3 (Regularized solutions). *Assume that Problem (1.1) with $g = g_0 \in H$ has a (weak) solution $u_0 \in C([0,T],H)$. For any $\varepsilon > 0$, let $g_\varepsilon \in H$ such that $||g_\varepsilon - g_0|| \le \varepsilon$. Let $n_0 \in \mathbb{N}$ and $t_1 = T/n_0$. We construct a function $u_\varepsilon : [0,T] \to H$ from g_ε as follows. For $n = 1, 2, ..., n_0$ put*

$$T_n = T - (n-1)t_1, \quad M_n = \frac{\log(1/\varepsilon)t_1^n}{(1+\delta_{n,n_0})T_1 ... T_{n-1}T_n^2},$$

where $\delta_{n,n_0} = 1$ if $n = n_0$ and $\delta_{n,n_0} = 0$ otherwise. Define $u_\varepsilon(t) := w_n(t)$ on $t \in [T_{n+1}, T_n)$ where $w_n : [T_{n+1}, T_n] \to H$ $(n = 1, 2, ..., n_0)$ solve the system

$$\begin{cases} \partial_t w_n + A w_n = \mathbb{P}_{M_n} f(t, w(t)), & T_{n+1} \le t < T_n, \\ w_n(T_n) = \mathbb{P}_{M_n}(w_{n-1}(T_n)), \end{cases} \tag{3.1}$$

with $w_0(T_1) = g_\varepsilon$ and $T_{n_0+1} = 0$.
(i) If u_0 satisfies (1.6) with $\beta = 2t_1 = 2T/n_0$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C\varepsilon^{\delta t} \quad \textit{for all } t \in [0,T].$$

(ii) If u_0 satisfies (1.7) with $\beta = 2t_1 = 2T/n_0$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C(\log(1/\varepsilon))^{-\beta'}\varepsilon^{\delta t} \quad \textit{for all } t \in [0,T].$$

(iii) If u_0 satisfies (1.8) with $\beta = 2t_1 = 2T/n_0$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C\varepsilon^\delta \quad \textit{for all } t \in [0,T].$$

Here C and δ stand for positive constants independent of t and ε.

Proof. (i) Put $\varepsilon_1 = \varepsilon$, $\varepsilon_{n+1} = \varepsilon^{\frac{t_1^n}{T_1\cdots T_n}}$ then

$$\varepsilon_{n+1} = \varepsilon_n^{t_1/T_n} \quad \text{and } M_n = \frac{\log(1/\varepsilon_n)}{(1+\delta_{n,n_0})T_n} \quad \text{for } n = 1, 2, ..., n_0.$$

We shall prove that for any $n = 1, 2..., n_0 - 1$ we have

$$\|w_n(t) - u_0(t)\| \le C_n \varepsilon_{n+1}, \quad t \in [T_{n+1}, T_n),$$

where $C_n > 0$ always stands for a constant independent of t and ε. Indeed, recall that w_n is the solution of the system

$$\begin{cases} \partial_t w_n + A w_n = \mathbb{P}_{M_n} f(t, w(t)), \quad T_{n+1} \le t < T_n, \\ w_n(T_n) = \mathbb{P}_{M_n}(w_{n-1}(T_n)) \end{cases}$$

with $||w_{n-1}(T_n) - u_0(T_n)|| \le C_{n-1}\varepsilon_n$. If $n \le n_0 - 2$ then since u_0 satisfies (1.8) with $\beta = 2t_1$ for all $t \in [T_{n+1}, T_n]$, Lemma 1 (iii) with $\tau = T_n$ implies

$$\|w_n(t) - u_0(t)\| \le C_n \varepsilon_n^{(t+2t_1-T_n)/T_n} \le C_n \varepsilon_n^{t_1/T_n} = C_n \varepsilon_{n+1}$$

for all $t \in [T_{n+1}, T_n]$. If $n = n_0 - 1$ then since u_0 satisfies (1.6) with $\beta = 2t_1$ for all $t \in [T_{n_0}, T_{n_0-1}] = [t_1, 2t_1]$, Lemma 1 (i) with $\tau = T_{n_0-1}$ implies

$$\|w_{n_0-1}(t) - u_0(t)\| \le C_{n_0-1}\varepsilon_{n_0-1}^{t/T_{n_0-1}} \le C_{n_0-1}\varepsilon_{n_0-1}^{t_1/T_{n_0-1}} = C_{n_0-1}\varepsilon_{n_0}$$

for all $t \in [t_1, 2t_1]$.

It remains to consider the final equation

$$\begin{cases} \partial_t w_{n_0} + A w_{n_0} = \mathbb{P}_{M_{n_0}} f(t, w(t)), \quad 0 \le t < t_1, \\ w_{n_0}(t_1) = \mathbb{P}_{M_{n_0}}(w_{n_0-1}(t_1)) \end{cases} \tag{3.2}$$

with $||w_{n_0-1}(t_1) - u_0(t_1)|| \le C_{n_0-1}\varepsilon_{n_0}$. Applying Lemma 1 (i) (with $\tau = 2T_{n_0} = 2t_1$) we get

$$\|w_{n_0}(t) - u_0(t)\| \le C_{n_0}\varepsilon_{n_0}^{t/(2t_1)}$$

for all $t \in [0, t_1]$. This gives the desired result.

(ii) If u_0 satisfies (1.7) for $\beta = 2t_1$ then for the final equation (3.2) we may apply Lemma 1 (ii) with $\tau = 2T_{n_0} = 2t_1$ to get

$$\begin{aligned} \|w_{n_0}(t) - u_0(t)\| &\le C_{n_0+1} \max\{\log(1/\varepsilon_{n_0})^{-2\beta}, \varepsilon_{n_0}^{1/2}\}\varepsilon_{n_0}^{-t/(2t_1)} \\ &\le \text{const.} C_{n_0+1} \log(1/\varepsilon)^{-2\beta}\varepsilon_{n_0}^{-t/(2t_1)} \end{aligned}$$

for all $t \in [0,t_1]$ since $(\varepsilon_{n_0})^{1/2} \geq \text{const.}(\log(1/\varepsilon_{n_0}))^{-2\beta'} = \text{const.}(\log(1/\varepsilon))^{-\beta'}$.

(iii) If u_0 satisfies (1.8) for $\beta = 2t_1$ then for the final equation (3.2) we may apply Lemma 1 (iii) with $\tau = 2T_{n_0} = 2t_1$ to obtain

$$\|w_{n_0}(t) - u_0(t)\| \leq C_{n_0+1}\varepsilon_{n_0}^{(t+t_1)/(2t_1)} \leq C_{n_0+1}\varepsilon_{n_0}^{1/2}$$

for all $t \in [0,t_1]$. This completes the proof. □

Remark 2. *In the final equation the choice $M = \log(1/\varepsilon_{n_0})/(2T_{n_0})$ instead of $M = \log(1/\varepsilon_{n_0})/T_{n_0}$ is crucial to get the error estimate at $t = 0$ in (ii) and (iii). See Remark 1 and the discussion before the statement of Theorem 3.*

Remark 3. *Since $M_n \leq \ln(1/\varepsilon)/(2t_1)$ for all n, we conclude from Theorem 1 that the stability magnitude of each equation in the system (3.1) does not exceed*

$$\exp(M_n(T_n - T_{n+1})) = \exp(M_n t_1) \leq \varepsilon^{-1/2}.$$

It is smaller than the stability magnitude of the approximating problem in the quasi-reversibility method, which is of order $e^{T/\varepsilon}$, and the one in the quasi-boundary value method, which is of order ε^{-1} (see the discussion in the introduction).

Remark 4. *In this method (as we see in the examples below), the larger number dividing step n_0 corresponds to the worse error estimate. Therefore, the readers may argue that why we do not choose, for example, $n_0 = 1$. We mention here that in order to have these error estimate, we need the conditions (1.6)-(1.8) to be valid for $\beta = 2T/n_0$. Thus by increasing the number dividing step n_0, we may weaken the assumptions (1.6)-(1.8) but the cost is, of course, to obtain worse error estimates.*

Let us consider some examples. If we know that (1.6)-(1.8) holds for $\beta = T$, as in [15, 16], we may simply choose $n_0 = 2$ as in Corollary 1 below. But our method works on even weaker condition, for example Corollary 2 below.

Corollary 1 ($n_0 = 2$,$\beta = T$)**.** *Let $g_0, g_\varepsilon \in H$ such that $\|g_\varepsilon - g_0\| \leq \varepsilon$. Assume that Problem (1.1) with $g = g_0$ has a (weak) solution $u_0 \in C([0,T],H)$. Let*

$$M_1 = \frac{\ln(1/\varepsilon)}{T}, \quad M_2 = \frac{\ln(1/\varepsilon)}{2T}$$

and let $u_\varepsilon = (w_1, w_2)$ be the solution of the following system

$$\begin{cases} \partial_t w_1 + Aw_1 = \mathbb{P}_{M_1} f(t, w_1(t)), & T/2 \leq t < T, \\ w_1(T) = \mathbb{P}_{M_1}(g_\varepsilon), \end{cases}$$
$$\begin{cases} \partial_t w_2 + Aw_2 = \mathbb{P}_{M_2} f(t, w_2(t)), & 0 \leq t < T/2, \\ w_2\,(T/2) = \mathbb{P}_{M_2}(w_1\,(T/2)). \end{cases}$$

(i) If u_0 satisfies (1.6) with $\beta = T$, i.e. $\sum_{n=1}^{\infty} e^{2\lambda_n t}|(\phi_n, u_0(t))|^2 \leq E_0^2$, then

$$\|u_\varepsilon(t) - u_0(t)\| \leq \begin{cases} C\varepsilon^{t/T}, & T/2 \leq t \leq T, \\ C\varepsilon^{t/(2T)}, & 0 \leq t < T/2. \end{cases}$$

(ii) Moreover, if u_0 satisfies $\sum_{n=1}^{\infty} \lambda_n^{2\beta'} e^{2\lambda_n t} |(\phi_n, u_0(t))|^2 \le E_1^2$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C(\log(1/\varepsilon))^{-\beta'} \varepsilon^{t/(2T)}, \quad 0 \le t < T/2.$$

In particular

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - u_0(t)\| \le C(\log(1/\varepsilon))^{-\beta'}.$$

(iii) Finally if u_0 satisfies $\sum_{n=1}^{\infty} e^{2\lambda_n T} |(\phi_n, u_0(t))|^2 \le E_2^2$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C\varepsilon^{(2t+T)/(4T)}, \quad 0 \le t < T/2.$$

In particular,

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - u_0(t)\| \le C\sqrt[4]{\varepsilon}.$$

Here C stands for a constant independent of t and ε.

Corollary 2 ($n_0 = 3$,$\beta = 2T/3$)**.** *Let $g_0, g_\varepsilon \in H$ such that $||g_\varepsilon - g_0|| \le \varepsilon$. Assume that Problem (1.1) with $g = g_0$ has a (weak) solution $u_0 \in C([0,T], H)$. Let*

$$M_1 = \frac{\ln(1/\varepsilon)}{T} \quad , M_2 = \frac{\ln(1/\varepsilon)}{2T}, \quad M_3 = \frac{\ln(1/\varepsilon)}{4T}$$

and let $u_\varepsilon = (w_1, w_2, w_3)$ be the solution of the following system

$$\begin{cases} \partial_t w_1 + Aw_1 = \mathbb{P}_{M_1} f(t, w_1(t)), & 2T/3 \le t < T, \\ w_1(T) = \mathbb{P}_{M_1}(g_\varepsilon), \end{cases}$$
$$\begin{cases} \partial_t w_2 + Aw_2 = \mathbb{P}_{M_2} f(t, w_2(t)), & T/3 \le t < 2T/3, \\ w_2\,(2T/3) = \mathbb{P}_{M_2}(w_1\,(2T/3)), \end{cases}$$
$$\begin{cases} \partial_t w_3 + Aw_3 = \mathbb{P}_{M_3} f(t, w_3(t)), & 0 \le t < T/3, \\ w_3\,(T/3) = \mathbb{P}_{M_3}(w_2\,(T/3)). \end{cases}$$

(i) If u_0 satisfies $\sum_{n=1}^{\infty} e^{2\lambda_n \min\{t, 2T/3\}} |(\phi_n, u_0(t))|^2 \le E_0^2$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le \begin{cases} C\varepsilon^{t/T - 1/3}, & 2T/3 \le t \le T, \\ C\varepsilon^{t/(2T)}, & T/3 \le t < 2T/3, \\ C\varepsilon^{t/(4T)}, & 0 \le t < T/3. \end{cases}$$

(ii) Moreover, if u_0 satisfies $\sum_{n=1}^{\infty} \lambda_n^{2\beta'} e^{2\lambda_n \min\{t, 2T/3\}} |(\phi_n, u_0(t))|^2 \le E_1^2$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C(\log(1/\varepsilon))^{-\beta'} \varepsilon^{t/(4T)}, \quad 0 \le t < T/3.$$

(iii) Finally if u_0 satisfies $\sum_{n=1}^{\infty} e^{\lambda_n 4T/3} |(\phi_n, u_0(t))|^2 \le E_2^2$ then

$$\|u_\varepsilon(t) - u_0(t)\| \le C\varepsilon^{t/(4T) + 1/12}, \quad 0 \le t < T/3.$$

Here C stands for a constant independent of t and ε.

4 Application to a heat equation

In this section we give an explicit example for Problem (1.1). Let us consider the backward heat equation

$$\begin{cases} u_t - \Delta u = f(x,t,u(x,t)), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(T) = g, & x \in \Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0,\pi)^N \subset \mathbb{R}^N$ and f satisfies the Lipschitz condition

$$|f(x,t,w_1) - f(x,t,w_2)| \le k\,|w_1 - w_2| \tag{4.2}$$

for some constant $k \ge 0$ independent of $(x,t,w_1,w_2) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}$. This is a particular case of (1.1) where $H = L^2(\Omega)$ and $A = -\Delta$, which associates with the homogeneous Dirichlet boundary condition. This operator admits an eigenbasis $\phi_\ell(x) = (2/\pi)^{N/2}\sin(\ell_1 x_1)...\sin(\ell_N x_N)$ for $L^2(\Omega)$ corresponding to the eigenvalues $\lambda_\ell = |\ell|^2$. Here we denote $x = (x_1,...,x_N) \in \mathbb{R}^N$, $\ell = (\ell_1,...,\ell_N) \in \mathbb{N}^N$ and $|\ell| = \sqrt{\ell_1^2 + ... + \ell_N^2}$. The pointwise Lipschitz condition (4.2) ensures the functional Lipschitz condition (1.2).

The heat equation (4.1) in one dimension has been considered by many authors, e.g. [10, 14, 16]. In 2005, Quan and Dung [10] offered a regularized solution by semi-group method. However, they were able to give error estimate only in a very special case that the exact solution has a finite Fourier series expansion and the Lipschitz constant $k > 0$ is small enough. In 2007, Trong et al. [14] used the quasi-boundary value method to construct a regularized solution which gives an approximation of order $\epsilon^{\frac{t}{T}}$ for $t > 0$ and $(\ln(1/\varepsilon))^{1/4}$ at $t = 0$. Very recently, Trong and Tuan [16] improved this method to give an error estimate of order $\varepsilon^{t/T}(\ln(1/\varepsilon))^{t/T-1}$ for all $t \in [0,T]$. However they required a very strong condition

$$\sup_{t\in[0,T]} \sum_{\ell} |\ell|^4 e^{2T|\ell|^2} \left|(\phi_\ell, u(t))_{L^2}\right|^2 < \infty.$$

Moreover, the approximation at $t = 0$ was still of logarithm type.

We now apply our construction of the regularized solution in Section 3 to the heat equation (4.1). Of course we have all regularization results in Theorem 3. Moreover we have the following estimate in higher Sobolev spaces $H^p(\Omega)$. We shall use the usual norm

$$\|w\|_{H^p(\Omega)} = \sqrt{\sum \left\| \frac{\partial^m w}{\partial x_1^{m_1} ... \partial x_N^{m_N}} \right\|_{L^2}^2}$$

where the sum is computed in the set

$$\{m = (m_1,...,m_N) | m_i = 0,1,2,...; m_1 + ... + m_N \le p\}.$$

Theorem 4 (Error estimate in higher Sobolev spaces). *Assume that f satisfies the Lipschitz condition (4.2) and that Problem (4.1) with $g = g_0$ has a (weak) solution*

$u_0 \in C([0,T], L^2(\Omega))$. *Let* $n_0 \in \mathbb{N}$ *and* $t_1 = T/n_0$. *For any* $g_\varepsilon \in L^2(\Omega)$ *such that* $||g_\varepsilon - g_0||_{L^2} \le \varepsilon$ *we construct* u_ε *as in Theorem 3. If* u_0 *satisfies (1.8) with* $\beta = 2t_1$ *then*

$$\|u_\varepsilon(t) - u_0(t)\|_{H^p(\Omega)} \le C_p \varepsilon^\delta \left(\ln(1/\varepsilon)\right)^{p/2} \quad \text{for all } t \in [0,T].$$

Here C_p *and* δ *stand for positive constants independent of* t *and* ε.

Proof. It is straightforward to check that for any $w \in H^p(\Omega)$

$$\|w\|^2_{H^p(\Omega)} \le \sum_\ell (1 + |\ell|^2 + |\ell|^4 + ... + |\ell|^{2p}) \left|(\phi_\ell, w)\right|^2 \le 2 \sum_\ell |\ell|^{2p} \left|(\phi_\ell, w)\right|^2. \quad (4.3)$$

Note that the regularized solution $u_\varepsilon(t)$ always belongs to the finite dimensional subspace $\mathbb{P}_{M_0}(H)$ where $M_0 := \ln(1/\varepsilon)/T$ since $M_0 \ge M_1 \ge ... \ge M_{n_0}$ (here we use the notations in Theorem 3). Thus employing (4.3) one has

$$\begin{aligned}
\|u_\varepsilon(t) - \mathbb{P}_{M_0} u_0(t)\|^2_{H^p(\Omega)} &\le 2 \sum_{|\ell|^2 \le M_0} |\ell|^{2p} \left|(\phi_\ell, u_\varepsilon(t) - u_0(t))\right|^2 \\
&\le 2M_0^p \sum_{|\ell|^2 \le M_0} \left|(\phi_\ell, u_\varepsilon(t) - u_0(t))\right|^2 \\
&\le 2M_0^p \|u_\varepsilon(t) - u_0(t)\|^2_{L^2(\Omega)} \\
&\le C^2 \varepsilon^{2\delta} (\ln(1/\varepsilon))^p. \qquad (4.4)
\end{aligned}$$

In the last inequality we have used the estimate in Theorem 3 (i).

On the other hand, note that the function $\xi \mapsto \xi^p e^{-\xi}$ is strictly decreasing when $\xi \ge p$. Thus if $\varepsilon \le \exp(-pT/(2\beta))$, i.e. $2\beta M_0 \ge p$, then we have

$$|\ell|^{2p} e^{-2\beta|\ell|^2} \le M_0^p e^{-2\beta M_0} \quad \text{provided } M_0 \le |\ell|^2.$$

It implies that

$$\begin{aligned}
\|\mathbb{P}_{M_0} u_0(t) - u_0(t)\|^2_{H^p(\Omega)} &\le 2 \sum_{|\ell|^2 > M_0} |\ell|^{2p} \left|(\phi_\ell, u_0(t))\right|^2 \\
&\le 2M_0^p e^{-2\beta M_0} \sum_{|\ell|^2 > M_0} e^{2\beta|\ell|^2} e \left|(\phi_\ell, u_0(t))\right|^2 \\
&\le C_p^2 \varepsilon^{2\beta/T} (\ln(1/\varepsilon))^p.
\end{aligned}$$

In the case $\varepsilon > \exp(-pT/(2\beta))$ then we may simply used

$$\|\mathbb{P}_{M_0} u_0(t) - u_0(t)\|_{H^p(\Omega)} \le ||u_0||_{H^p(\Omega)} \le C_p \varepsilon^\delta (\ln(1/\varepsilon))^p.$$

Thus we always have

$$\|\mathbb{P}_{M_0} u_0(t) - u_0(t)\|^2_{H^p(\Omega)} \le C_p^2 \varepsilon^{2\delta} (\ln(1/\varepsilon))^p. \quad (4.5)$$

The desired result thus follows from (4.4)-(4.5) and the triangle inequality. $\square$

For example if (1.8) holds for $\beta = T$ then we have the following error estimate.

Corollary 3 ($n_0 = 2, \beta = T$)**.** *Assume that (4.2) holds and that Problem (4.1) with $g = g_0$ has a (weak) solution $u_0 \in C([0,T], L^2(\Omega))$ satisfying*

$$\sum_{\ell} e^{2T|\ell|^2} |(\phi_\ell, u_0(t))|^2 \leq E_2^2.$$

Let $g_\varepsilon \in L^2(\Omega)$ such that $||g_\varepsilon - g_0||_L^2 \leq \varepsilon$. Let

$$M_1 = \frac{\ln(1/\varepsilon)}{T}, \quad M_2 = \frac{\ln(1/\varepsilon)}{2T}.$$

and let $u_\varepsilon = (w_1, w_2)$ be the solution of the following system

$$\begin{cases} \partial_t w_1 - \Delta w_1 = \mathbb{P}_{M_1} f(t, w_1(t)), & T/2 \leq t < T, \\ w_1(T) = \mathbb{P}_{M_1}(g_\varepsilon), \end{cases}$$
$$\begin{cases} \partial_t w_2 - \Delta w_2 = \mathbb{P}_{M_2} f(t, w_2(t)), & 0 \leq t < T/2, \\ w_2\,(T/2) = \mathbb{P}_{M_2}(w_1\,(T/2)). \end{cases}$$

Then for any $p = 0, 1, 2, \ldots$ we have

$$\|u_\varepsilon(t) - u_0(t)\|_{H^p(\Omega)} \leq \begin{cases} C_p \varepsilon^{\frac{t}{T}} (\ln(1/\varepsilon))^{p/2}, & T/2 \leq t < T, \\ C_p \varepsilon^{\frac{2t+T}{4T}} (\ln(1/\varepsilon))^{p/2}, & 0 \leq t < T/2, \end{cases}$$

where C_p stands for a positive constant independent of t, ε. In particular

$$\sup_{t \in [0,T]} \|u_\varepsilon(t) - u_0(t)\|_{H^p(\Omega)} \leq C_p \varepsilon^{1/4} (\ln(1/\varepsilon))^{p/2}.$$

Remark 5. *The condition of Corollary 3 is similar to the ones in [14, 16] where the error estimates at $t = 0$ are given in L^2 and of logarithm type only.*

Although the stability estimate for any higher Sobolev space is quite unusual in the regularization theory for ill-posed problems, it did appear in some earlier papers, for example in a recent paper [6] where the homogeneous heat equation was treated.

5 Numerical experiment

In this section we give a numerical implementation for our method. First we recall that the well-posed problem (1.9) is just an ordinary differential equation in finite dimension subspace. To solve this problem we may apply the standard Euler's method to discrete it into the form

$$\begin{cases} \dfrac{u(t_m) - u(t_{m+1})}{\Delta t} = -Au(t_m) + \mathbb{P}_M f(t_m, u(t_m)), \\ u(t_0) = \mathbb{P}_M g. \end{cases}$$

Here we use a uniform mesh $t_m = T - m\Delta t$ $(m = 0, 1, 2, ...)$ with the meshsize Δt. More clearly, we shall find $u(t_m)$ under the form

$$u(t_m) = \sum_{\lambda_n \le M} U_{m,n} \phi_n$$

where the scalar matrix $U_{m,n}$ is computed by induction with m as follows

$$\begin{cases} U_{0,n} = (\phi_n, g), \\ U_{m+1,n} = (1 + \lambda_n \Delta t)\, U_{m,n} - (\phi_n, f(t_m, u(t_m)))_H \,\Delta t. \end{cases}$$

To make a comparison, we shall work on a numerical example given in [14, 16]. Let us consider the backward heat problem

$$\begin{cases} u_t - u_{xx} = f(u) + 2e^t \sin(x) - e^{4t}(\sin(x))^4, \quad (x,t) \in (0,\pi) \times (0,1), \\ u(0,t) = u(\pi,t) = 0, \quad t \in (0,1), \\ u(x,1) = e\sin(x), \quad x \in (0,\pi), \end{cases}$$

where

$$f(u) = \begin{cases} u^4 & \text{if } u \in [-e^{10}, e^{10}], \\ -\dfrac{e^{10}}{e-1}u + \dfrac{e^{41}}{e-1} & \text{if } u \in (-e^{10}, e^{11}], \\ \dfrac{e^{10}}{e-1}u + \dfrac{e^{41}}{e-1} & \text{if } u \in (-e^{11}, -e^{10}], \\ 0 & \text{if } |u| > e^{11}. \end{cases}$$

It is easy to see that the Lipschitz condition (4.2) holds (e.g. for $k = 4e^{30}$) and the exact solution is $u_0(x,t) = e^t \sin(x)$. Similarly to [14, 16], we choose the approximate datum $g_\varepsilon(x) = (\varepsilon + 1)e\sin(x)$ with the error

$$\|g_\varepsilon - g\|_{L^2} = \left(\int_0^\pi \varepsilon^2 e^2 (\sin(x))^2 dx \right)^{1/2} = \varepsilon e \sqrt{\frac{\pi}{2}}.$$

We now compute the regularized solution with respect to datum $g_\varepsilon(x)$. For simplicity we shall use the scheme given in Corollary 3 (this is the case $n_0 = 2$, $\beta = T$), i.e. we solve a system of two equations

$$\begin{cases} \partial_t w_1 - (w_1)_{xx} = \mathbb{P}_{M_1} f(t, w_1(t)), \quad 1/2 \le t < 1, \\ w_1(1) = \mathbb{P}_{M_1}(g_\varepsilon), \end{cases}$$

$$\begin{cases} \partial_t w_2 - (w_2)_{xx} = \mathbb{P}_{M_2} f(t, w_2(t)), \quad 0 \le t < 1/2, \\ w_2(1/2) = \mathbb{P}_{M_2}(w_1(1/2)) \end{cases}$$

with $M_1 = \log(1/\varepsilon)$, $M_2 = \log(1/\varepsilon)/2$.

We first compute the numerical solution at T' very near T, says $T' = 0.999$. The exact solution at this time is

$$u_0\left(x, T'\right) = 2.715564905 \sin(x).$$

The numerical solution produced by our scheme with $\Delta t = 1/5000$ is given in Table 1. We can see that the error is nearly of order ε, which agrees with the theoretical result that the convergence is of order $\varepsilon^{T'/T}$. The corresponding results of [14] and [16], where the same meshsize Δt were used, are given in Table 2 and Table 3, respectively.

Table 1.

ε	$u_\varepsilon(t_1)$	$\|\|u_\varepsilon(t_1) - u_0(t_1)\|\|_{L^2}$
$\varepsilon = 10^{-1}$	$2.970952310 \sin(x)$	0.3200806448
$\varepsilon = 10^{-2}$	$2.741303217 \sin(x)$	0.03225818991
$\varepsilon = 10^{-3}$	$2.718140386 \sin(x)$	0.003227886349
$\varepsilon = 10^{-4}$	$2.715822290 \sin(x) + 0.6150593 \times 10^{-5} \sin(3x)$	0.0003226759514
$\varepsilon = 10^{-5}$	$2.715590416 \sin(x) + 0.1788815 \times 10^{-5} \sin(3x)$	0.00003205140426
$\varepsilon = 10^{-7}$	$2.715564898 \sin(x) + 0.1058266 \times 10^{-7} \sin(3x)$	0.1612612×10^{-7}
$\varepsilon = 10^{-11}$	$2.715564639 \sin(x) - 0.2484836 \times 10^{-8} \sin(3x) +$ $+ 0.2668597 \times 10^{-9} \sin(5x)$	0.3337955×10^{-6}

Table 2.

ε	$u_\varepsilon(t_1)$	$\|\|u_\varepsilon(t_1) - u_0(t_1)\|\|_{L^2}$
$\varepsilon = 10^{-5}$	$2.430605996 \sin(x) - 0.0001718460902 \sin(3x)$	0.3266494251
$\varepsilon = 10^{-7}$	$2.646937077 \sin(x) - 0.002178680692 \sin(3x)$	0.05558566020
$\varepsilon = 10^{-11}$	$2.649052245 \sin(x) - 0.004495263004 \sin(3x)$	0.05316693437

Table 3.

ε	$u_\varepsilon(t_1)$	$\|\|u_\varepsilon(t_1) - u_0(t_1)\|\|_{L^2}$
$\varepsilon = 10^{-5}$	$2.718264487 \sin(x) - 0.005466473792 \sin(6x)$	0.002729464336
$\varepsilon = 10^{-7}$	$2.715833791 \sin(x) - 0.005461493459 \sin(6x)$	0.0002987139108
$\varepsilon = 10^{-11}$	$2.715552177 \sin(x) - 0.005518178192 \sin(6x)$	0.0000431782905

Remark 6. *In Table 1 the error corresponding* $\varepsilon = 10^{-11}$ *is not better than the one corresponding* $\varepsilon = 10^{-7}$. *In our opinion, this is due to the limit of the discrete process rather than a theoretical reason. For example, by choosing a finer meshsize, namely* $\Delta t = 10^{-5}$, *we obtain a better error* $0.1544632662 \times 10^{-7}$ *for numerical solution corresponding to* $\varepsilon = 10^{-11}$.

We now compute the regularized solution for all t, and in particular at $t = 0$ (these works were not given in [14, 16]). Our regularized solution corresponding to $\varepsilon = 10^{-3}$, which is computed in the meshsize $\Delta t = 1/100$, is displayed in Figure 1 while the exact solution is plotted in Figure 2 in order to give a visual comparison.

Figure 1. Exact solution

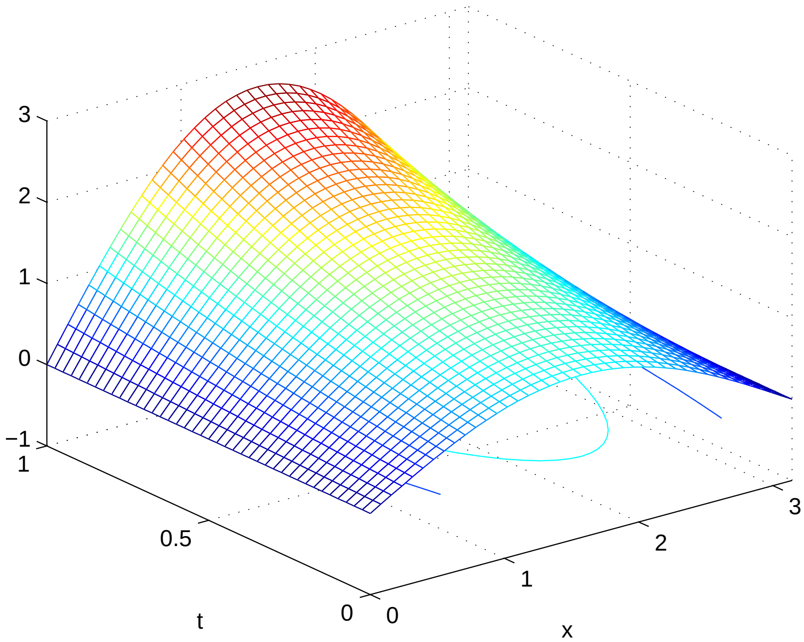
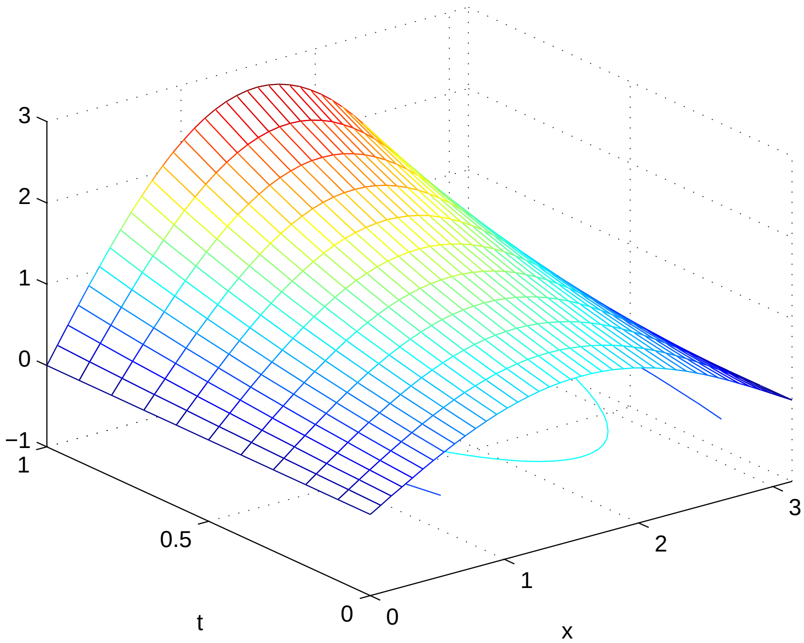


Figure 2. Regularized solution



In particular, the regularized solution at $t = 0$ with $\varepsilon = 10^{-3}$ is $0.9970179573\sin(x)$ and its error to the exact solution $u_0(x, 0) = \sin(x)$ is

$$||u_\varepsilon(0) - u_0(0)||_{L^2} = 0.003737436274,$$

which is very reasonable.

6 Conclusion

The paper considers the regularization problem for a class of nonlinear backward parabolic equations in abstract Hilbert spaces, namely Problem (1.1). In many earlier works on the nonlinear problem, e.g. [14, 15, 16], while ones may obtain an Hölder-type error estimate at any fixed time $t > 0$, an explicit error estimate at $t = 0$ is still difficult and was given in logarithm type only. The present paper proposes a regularized solution with several error estimates which includes an error estimate of Hölder type for all $t \in [0, T]$. In the homogeneous case our results are comparable to [5] while in the nonlinear case they improve the results in many earlier works, e.g. [9, 10, 14, 15, 16]. Moreover, our regularization is simple enough for a numerical setting and the numerical results seems satisfactory.

However our method is still a little theoretical since in general the power β in conditions (1.6)-(1.8) is unknown in practice. We mention that while such conditions are reasonable for the homogeneous problem (even for $\beta = T$), they are not necessarily true for inhomogeneous cases. However, up to my knowledge, such assumptions on the exponential growth of the exact solution are crucial in various works on the regularization theory for the nonlinear ill-posed problem. Finding a way to relax these assumptions is an interesting, but difficult, problem for future works.

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