LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Partial Differential Equations
Final exam

Family name: . . . . Matriculation no.: . . . .

First name: . . . . Semester: . . . .

Study course: . . . .

Signature: . . . .

You have **3 hours** of official working time and **one additional hour** to prepare and finalize your upload. Solutions must be uploaded until the deadline at 14:00 o'clock on 15.2.2022 via Uni2Work in PDF-format. Make sure to follow these rules:

- Solutions must be **handwritten** (pen on paper & scanned, or digital pen tablet). Do **not** use the colours **red** or **green**.
  If you do not use the official exam preprint (this file), you must follow the official formatting instructions for "plain-paper submissions" given in uni2work.
- Solve each problem on the respective sheet. If you need more space, you can use the extra sheets; in this case please state your name and the problem you refer to.
- All answers and solutions must provide sufficiently detailed arguments. You may refer to all results from the lectures, homeworks and tutorials.
- Solutions must be prepared by yourself. You are not allowed to share information about any of the problems or their solutions of this exam with others before the hand-in deadline.
- With your signature you agree to the rules of the exam.

Before uploading please check whether your pdf-scan is readable and contains all your solutions (in total there are **five problems**). Do not forget to write your name on each sheet. Good luck!

Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Bonus	$\sum$	GRADE
(max 10)	(max 15)	(max 20)	(max 25)	(max 30)			

Problem Overview (you do not have to include this page in your submission).

Problem 1 (10 points). Let $\{u_n\}_{n=0}^{\infty} \subset L^1_{\text{loc}}(\mathbb{R}^d)$ satisfy that

$$-\Delta u_n(x) = |x|^2 e^{-n|x|^2} \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \quad \forall n = 1, 2, ...$$

and $u_n \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ when $n \to \infty$. Prove that u_0 is a harmonic function in $\mathbb{R}^d$.

Problem 2 (15 points). Let $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\}$. Let $g \in C^1_c(\mathbb{R})$ and

$$u(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g(y)}{(x_1 - y)^2 + x_2^2} \mathrm{d}y, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2_+.$$

Prove that $f = \partial_{x_1} u$ is harmonic in $\mathbb{R}^2_+$ and $\lim_{x_2 \to 0^+} f(x_1, x_2) = g'(x_1)$, $\forall x_1 \in \mathbb{R}$.

Problem 3 (10+10 points). Let $B = B(0,1)$ be the unit open ball in $\mathbb{R}^d$ ($d \geq 1$). Let $g \in C(\partial B)$ be an odd function, namely $g(x) = -g(-x)$ for all $x \in \partial B$.

(a) Let $T > 0$ and let $u \in C^2(\overline{B} \times [0, T])$ be a solution of the wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta_x u(x,t) = 0 & \text{in } (x,t) \in B \times (0,T), \\ u(x,t) = \partial_t u(x,t) = 0 & \text{in } (x,t) \in B \times \{t = 0\}, \\ u(x,t) = g(x) & \text{on } \partial B \times [0,T]. \end{cases}$$

Prove that $u(0,t) = 0$, $\forall t \in [0, T]$. (Hint: Uniqueness of the wave equation is helpful.)

(b) Let $v \in C^4(\overline{B})$ satisfy

$$\begin{cases} \Delta(\Delta v) \geq 0 & \text{in } B, \\ \Delta v \leq 0 & \text{on } \partial B, \\ v = g & \text{on } \partial B. \end{cases}$$

Prove that $v(0) \geq 0$. (Hint: You may consider $f = \Delta v$.)

Problem 4 (10+15 points). Let $g \in L^2(\mathbb{R}^d)$ (with $d \geq 1$). Consider the solutions of the heat and Schrödinger equations (with $\mathbf{i}^2 = -1$)

$$u(x,t) = (e^{t\Delta} g)(x), \quad v(x,t) = (e^{\mathbf{i}t\Delta} g)(x), \quad x \in \mathbb{R}^d, \quad t > 0.$$

(a) Prove that if $g \in H^1(\mathbb{R}^d)$, then there exists a constant $C = C(g) > 0$ such that

$$\int_{\mathbb{R}^d} |v(x,t) - g(x)|^2 \mathrm{d}x \leq Ct, \quad \forall t > 0.$$

(b) Let $g \in C_c^\infty(\mathbb{R}^d)$ be an odd function, namely $g(x) = -g(-x)$ for all $x \in \mathbb{R}^d$. Prove that there exists a constant $C = C(g) > 0$ such that

$$\int_{\mathbb{R}^d} |u(x,t)|^2 \mathrm{d}x \leq \frac{C}{t^{1+d/2}}, \quad \forall t > 0.$$

(Hint: You may work on Fourier space. For b), the value of $\widehat{g}(0)$ is important.)

Problem 5 (10+20 points). Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Define

$$\delta_{\mathbb{S}^2}(\varphi) = \int_{\mathbb{S}^2} \varphi(x) \mathrm{d}\omega(x), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3)$$

where ω is the usual Lebesgue measure on $\mathbb{S}^2$ (recall $\int_{\mathbb{S}^2} \mathrm{d}\omega = |\mathbb{S}^2| = 4\pi$).

(a) Prove that $\delta_{\mathbb{S}^2} \in \mathcal{D}'(\mathbb{R}^3)$ but $\delta_{\mathbb{S}^2} \notin L^1_{\text{loc}}(\mathbb{R}^3)$.

(b) Prove that there exists a function $u \in L^1_{\text{loc}}(\mathbb{R}^3)$ such that $-\Delta u = \delta_{\mathbb{S}^2}$ in $\mathcal{D}'(\mathbb{R}^3)$.
(Hint: Guess u by formally using Green's function and Newton's theorem. Then justify.)

Solutions

Problem 1. We have $-\Delta u_n = |x|^2 e^{-n|x|^2} \to 0$ in $L^1(\mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d} |x|^2 e^{-n|x|^2} \mathrm{d}x = \frac{1}{n^{1+d/2}} \int_{\mathbb{R}^d} |y|^2 e^{-|y|^2} \mathrm{d}y = \frac{C}{n^{1+d/2}} \to 0 \quad \text{as } n \to \infty$$

by changing the variables $y = x/\sqrt{n}$. Consequently, $-\Delta u_n \to 0$ in $\mathcal{D}'(\mathbb{R}^d)$. Moreover, since $u_n \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have $u_n \to u_0$ in $\mathcal{D}'(\mathbb{R}^d)$, and hence $-\Delta u_n \to -\Delta u_0$ in $\mathcal{D}'(\mathbb{R}^d)$ (by Homework E4.1). Thus $-\Delta u_0 = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, namely u_0 is a harmonic function in $\mathbb{R}^d$ (by Weyl's lemma).

Remark: Alternatively the argument can be written as follows, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$(\Delta u)(\varphi) = u(\Delta \varphi) = \lim_{n\to\infty} u_n(\Delta\varphi) = \lim_{n\to\infty} (\Delta u_n)(\varphi) = \lim_{n\to\infty} \int_{\mathbb{R}^d} -\varphi(x)|x|^2 e^{-n|x|^2} \mathrm{d}x = 0.$$

where we used $u_n \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for the second equality and used Dominated Convergence for the last equality.

Problem 2. By changing the variables we can write

$$u(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g(y)}{(x_1-y)^2 + x_2^2} \mathrm{d}y = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g(x_1 - y)}{y^2 + x_2^2} \mathrm{d}y.$$

Therefore,

$$\begin{aligned} f(x) = \partial_{x_1} u(x) &= \lim_{h\to 0} \frac{u(x_1+h, x_2) - u(x_1, x_2)}{h} \\ &= \lim_{h\to 0} \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g(x_1+h-y) - g(x_1-y)}{h} \cdot \frac{1}{y^2+x_2^2} \mathrm{d}y. \end{aligned}$$

For every fixed $x = (x_1, x_2) \in \mathbb{R}^2_+$, we have

$$\lim_{h\to 0} \frac{g(x_1+h-y) - g(x_1-y)}{h} \cdot \frac{1}{y^2+x_2^2} = g'(x_1-y)\frac{1}{y^2+x_2^2}, \quad \forall y \in \mathbb{R}$$

and

$$\left| \frac{g(x_1+h-y) - g(x_1-y)}{h} \cdot \frac{1}{y^2+x_2^2} \right| \le \|g'\|_{L^\infty} \frac{1}{y^2+x_2^2} \in L^1(\mathbb{R}, \mathrm{d}y)$$

Thus by Dominated Convergence

$$f(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g'(x_1-y)}{y^2+x_2^2} \mathrm{d}y = \frac{x_2}{\pi} \int_{\mathbb{R}} \frac{g'(y)}{(x_1-y)^2+x_2^2} \mathrm{d}y.$$

Put differently,

$$f(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} g'(y) \Big(\frac{1}{(x_1-y)^2 + x_2^2} \Big) \mathrm{d}y = \int_{\partial\mathbb{R}^2_+} g'(y) K(x,y) \mathrm{d}y$$

where

$$K(x,y) = \frac{x_2}{\pi} \frac{1}{|x-y|}$$

is exactly Poisson's kernel for $\mathbb{R}^2_+$. Here we identify $\mathbb{R}$ and $\partial\mathbb{R}^2_+$. Since $g' \in C_c(\mathbb{R})$, by a theorem on Poisson's equation in $\mathbb{R}^2_+$, we find that $f \in C^2(\mathbb{R}^2_+)$ and it solves

$$\begin{cases} \Delta f = 0 & \text{in } \mathbb{R}^2_+, \\ \lim_{x_2 \to 0^+} f(x_1, x_2) = g'(x_1), & \forall x_1 \in \mathbb{R}. \end{cases}$$

Remark: If at the beginning we do not change the variables, then

$$f(x) = \partial_{x_1} u(x) = \frac{x_2}{\pi} \int_{\mathbb{R}} g(y) \partial_{x_1} \Big(\frac{1}{(x_1-y)^2 + x_2^2} \Big) \mathrm{d}y$$

by Dominated Convergence (need to justify). We can proceed using the identity

$$\partial_{x_1}\Big(\frac{1}{(x_1-y)^2+x_2^2}\Big) = (-\partial_y)\Big(\frac{1}{(x_1-y)^2+x_2^2}\Big)\mathrm{d}y$$

and the integration by parts,

$$f(x) = \frac{x_2}{\pi}\int_{\mathbb{R}} g'(y)\Big(\frac{1}{(x_1-y)^2+x_2^2}\Big)\mathrm{d}y = \int_{\partial\mathbb{R}^2_+} g'(y)K(x,y)\mathrm{d}y$$

Problem 3. a) Denote $\widetilde{u}(x,t) = u(x,t) + u(-x,t)$. Then $\widetilde{u}(x,t)$ satisfies the same equation, but with all 0 boundary conditions:

$$\begin{cases} \partial_t^2\widetilde{u}(x,t) - \Delta_x\widetilde{u}(x,t) = 0 & \text{in } (x,t)\in B\times(0,T), \\ \widetilde{u}(x,t) = \partial_t\widetilde{u}(x,t) = 0 & \text{in } (x,t)\in B\times\{t=0\}, \\ \widetilde{u}(x,t) = 0 & \text{on } \partial B\times[0,T] \end{cases}$$

where we have used $g(x)+g(-x)=0$ on ∂B. By the uniqueness of the wave equation, we have $\widetilde{u}(x,t)=0$ on $\overline{B}\times[0,T]$. In particular, $2u(0,t)=\widetilde{u}(0,t)=0$ for all $t\in[0,T]$.

b) The function $f=\Delta v\in C^2(\overline{B})$ satisfies

$$\begin{cases} \Delta f \ge 0 & \text{in } B, \\ f\le 0 & \text{on } \partial B. \end{cases}$$

Hence, $f\le 0$ in $\overline{B}$ by maximum principle. Thus

$$\begin{cases} \Delta v \le 0 & \text{in } B, \\ v = g & \text{on } \partial B. \end{cases}$$

Similarly to a), we define $\widetilde{v}(x)=v(x)+v(-x)$. Then since $g(x)+g(-x)=0$ on ∂B, we have

$$\begin{cases} \Delta \widetilde{v} \le 0 & \text{in } B, \\ \widetilde{v} = 0 & \text{on } \partial B. \end{cases}$$

Hence, $\widetilde{v}\ge 0$ in $\overline{B}$ by maximum principle. In particular, $2v(0)=\widetilde{v}(0)\ge 0$.

Problem 4. Recall the Fourier transform

$$\widehat{u}(k,t) = e^{-t|2\pi k|^2}\hat{g}(k), \quad \widehat{v}(k,t) = e^{-it|2\pi k|^2}\hat{g}(k).$$

a) This is similar to Homework E11.1 c). By Plancherel theorem,

$$\int_{\mathbb{R}^d}|v(x,t)-g(x)|^2\mathrm{d}x = \int_{\mathbb{R}^d}|\widehat{v}(k,t)-\widehat{g}(k)|^2\mathrm{d}k = \int_{\mathbb{R}^d}|e^{-it|2\pi k|^2}-1|^2|\widehat{g}(k)|^2\mathrm{d}k.$$

Note that

$$|e^{i\theta}-1|^2 = |\cos(\theta)-1|^2+|\sin(\theta)|^2 \le C\min(1,|\theta|^2)\le C|\theta|, \quad \forall\theta\in\mathbb{R}.$$

Therefore,

$$\int_{\mathbb{R}^d}|v(x,t)-g(x)|^2\mathrm{d}x = \int_{\mathbb{R}^d}|e^{-it|2\pi k|^2}-1|^2|\widehat{g}(k)|^2\mathrm{d}k \le \int_{\mathbb{R}^d}Ct|2\pi k||\widehat{g}(k)|^2\mathrm{d}k \le Ct\|g\|_{H^1}^2.$$

b) Since g is odd, we have $\widehat{g}(0)=\int_{\mathbb{R}^d}g=0$. Hence

$$|\widehat{g}(k)| = |\widehat{g}(k)-\widehat{g}(0)| \le |k|\|\nabla_k\widehat{g}\|_{L^\infty} \le |k|\||2\pi x|g\|_{L^1}\le C|k|$$

where we have used $\partial_{k_j}\widehat{g}(k)=\mathcal{F}((-2\pi i x_j)g(x))$ and $\|\widehat{f}\|_{L^\infty}\le\|f\|_{L^1}$. Therefore,

$$\int_{\mathbb{R}^d}|u(x,t)|^2\mathrm{d}x = \int_{\mathbb{R}^d}e^{-2t|2\pi k|^2}|\hat{g}(k)|^2\mathrm{d}k \le C\int_{\mathbb{R}^d}e^{-2t|2\pi k|^2}|k|^2\mathrm{d}k$$

$$= \frac{C}{t^{1+d/2}} \int_{\mathbb{R}^d} e^{-2|2\pi\xi|^2} |\xi|^2 \mathrm{d}\xi \le \frac{C}{t^{1+d/2}}, \quad \forall t > 0$$

where we changed the variables $k = \xi/t^{1/2}$.

Problem 5. a) Let us check that $\delta_{\mathbb{S}^2} \in \mathcal{D}'(\mathbb{R}^3)$. Let $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^3)$ as $n \to \infty$. Then in particular, we have

$$\max_{x \in \mathbb{R}^3} |\varphi_n(x) - \varphi(x)| \to 0.$$

Hence

$$|\delta_{\mathbb{S}^2}(\varphi_n) - \delta_{\mathbb{S}^2}(\varphi)| = \Big| \int_{\mathbb{S}^2} (\varphi_n(y) - \varphi(y)) \mathrm{d}\omega(y) \Big| \le \max_{x \in \mathbb{S}^2} |\varphi_n(x) - \varphi(x)| \int_{\mathbb{S}^2} \mathrm{d}\omega(y) \to 0.$$

Thus $\delta_{\mathbb{S}^2} \in \mathcal{D}'(\mathbb{R}^3)$.

Next, let us show that $\delta_{\mathbb{S}^2} \notin L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Assume by contradiction that $\delta_{\mathbb{S}^2} = g \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Then for every $n > 1$, there exists a function $\varphi_n \in C_c^\infty(\mathbb{R}^3)$ such that

$$\varphi_n(x) = 1 \text{ if } |x| = 1, \quad \varphi_n(x) = 0 \text{ if } ||x| - 1| \ge 1/n.$$

Then for all $\varphi \in C_c^\infty(\mathbb{R}^3)$ we have $(1 - \varphi_n)\varphi \in C_c^\infty(\mathbb{R}^3)$ and $(1 - \varphi_n)\varphi(x) = 0$ if $|x| = 1$, and hence

$$\int_{\mathbb{R}^3} g(1 - \varphi_n)\varphi = \delta_{\mathbb{S}^2}((1 - \varphi_n)\varphi) = \int_{\mathbb{S}^2} (1 - \varphi_n(y))\varphi(y) \mathrm{d}\omega(y) = 0.$$

Thus by the fundamental lemma of calculus of variations, $g(1 - \varphi_n) = 0$ a.e. Consequently, since $1 - \varphi_n(x) = 1$ for $||x| - 1| \ge 1/n$, we find that $g(x) = 0$ for a.e. $||x| - 1| \ge 1/n$. Taking $n \to \infty$, we conclude that $g(x) = 0$ for a.e. $x \in \mathbb{R}^3$. But clearly $\delta_{\mathbb{S}^2} \ne 0$. So this contradiction shows that $\delta_{\mathbb{S}^2} \notin L^1_{\mathrm{loc}}(\mathbb{R}^3)$.

b) Formally using Green's function $G(x) = 1/(4\pi|x|)$ we guess

$$u(x) = (G * \delta_{\mathbb{S}^2})(x) = \delta_{\mathbb{S}^2}(G(x - y)) = \int_{\mathbb{S}^2} G(x - y) \mathrm{d}\omega(y) = \frac{1}{\max(1, |x|)}.$$

Here we used Newton's theorem in the last identify.

It remains to check that u satisfies the desired properties. Clearly $|u| \le 1$, and hence $u \in L^\infty(\mathbb{R}^3) \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)$. Next, from the definition

$$u(x) = \int_{\mathbb{S}^2} G(x - y) \mathrm{d}\omega(y),$$

for every $\varphi \in C_c^\infty(\mathbb{R}^3)$ we can write by Fubini's theorem

$$\begin{aligned} (\Delta u)(\varphi) &= \int_{\mathbb{R}^3} u(x)(\Delta\varphi)(x) \mathrm{d}x = \int_{\mathbb{R}^3} \left(\int_{\mathbb{S}^2} G(x - y) \mathrm{d}\omega(y) \right) (\Delta\varphi)(x) \mathrm{d}x \\ &= \int_{\mathbb{S}^2} \left(\int_{\mathbb{R}^3} G(x - y) \Delta\varphi(x) \mathrm{d}x \right) \mathrm{d}\omega(y) = \int_{\mathbb{S}^2} (G * (\Delta\varphi))(y) \mathrm{d}\omega(y). \end{aligned}$$

Here the use of Fubini's theorem is allowed since $G(x - y)|\Delta\varphi(x)| \in L^1(\mathbb{R}^3 \times \mathbb{S}^2)$, as $\Delta\varphi(x) \in C_c(\mathbb{R}^3)$. We also used $G(x - y) = G(y - x)$ for the convolution form.

We know that $f = G * \varphi$ is the solution to Poisson's equation $-\Delta f = \varphi$. Actually in a theorem in Chapter 3 we proved that $-G * (\Delta\varphi) = -\Delta(G * \varphi) = \varphi$ for all $\varphi \in C_c^\infty$. Thus we conclude that

$$(\Delta u)(\varphi) = \int_{\mathbb{S}^2} (G * (\Delta\varphi))(y) \mathrm{d}\omega(y) = -\int_{\mathbb{S}^2} \varphi(y) \mathrm{d}\omega(y) = -\delta_{\mathbb{S}^2}(\varphi), \quad \forall \varphi \in C_c^\infty(\mathbb{R}^3),$$

namely $\Delta u = -\delta_{\mathbb{S}^2}$ in the distributional sense.

Remark: We can also use $u(x) = 1/\max(1, |x|)$ and compute for every $\varphi \in C_c^\infty(\mathbb{R}^3)$

$$(\Delta u)(\varphi) = \int_{\mathbb{R}^3} u(x)(\Delta\varphi)(x)\mathrm{d}x = \int_{|x|\le 1} (\Delta\varphi)(x)\mathrm{d}x + \int_{|x|>1} \frac{\Delta\varphi(x)}{|x|}\mathrm{d}x.$$

By integration by parts

$$\int_{|x|\le 1} (\Delta\varphi)(x)\mathrm{d}x = \int_{|x|\le 1} \mathrm{div}(\nabla\varphi)(x)\mathrm{d}x = \int_{\mathbb{S}^2} \nabla\varphi(x)\cdot \vec{n}_x \mathrm{d}\omega(x) = \int_{\mathbb{S}^2} \frac{\partial\varphi}{\partial n}(x)\mathrm{d}\omega(x)$$

and

$$\begin{aligned}\int_{|x|>1} \frac{\Delta\varphi(x)}{|x|}\mathrm{d}x &= -\int_{|x|>1} \nabla\varphi(x)\cdot\nabla(|x|^{-1})\mathrm{d}x - \int_{\mathbb{S}^2} \frac{\partial\varphi}{\partial n}(x)|x|^{-1}\mathrm{d}\omega(x)\\ &= \int_{|x|>1} \varphi(x)(\Delta|x|^{-1})\mathrm{d}x + \int_{\mathbb{S}^2} \varphi(x)\frac{\partial}{\partial n}(|x|^{-1})\mathrm{d}\omega(x) - \int_{\mathbb{S}^2} \frac{\partial\varphi}{\partial n}(x)|x|^{-1}\mathrm{d}\omega(x)\\ &= 0 - \int_{\mathbb{S}^2} \varphi(x)\mathrm{d}\omega(x) - \int_{\mathbb{S}^2} \frac{\partial\varphi}{\partial n}(x)\mathrm{d}\omega(x).\end{aligned}$$

Here in the last line we used $-\Delta|x|^{-1} = 0$ in $\{|x| > 1\}$ and $\frac{\partial}{\partial n}(|x|^{-1}) = -1$ on $\mathbb{S}^2$. Thus we conclude that

$$(\Delta u)(\varphi) = -\int_{\mathbb{S}^2} \varphi(x)\mathrm{d}\omega(x) = -\delta_{\mathbb{S}^2}(\varphi), \quad \forall\varphi \in C_c^\infty(\mathbb{R}^3),$$

namely $\Delta u = -\delta_{\mathbb{S}^2}$ in the distributional sense.