Mathematisches Institut der Universität München
Phan Thành Nam, Arnaud Triay

Summer Semester 2022
13.7.2022

Partial Differential Equations II

Homework Sheet 10

(Discussed on 18.7.2022)

E10.1 Let $\varphi \in H^1(\mathbb{R}^d)$ and $J_n = \mathbb{1}(-\Delta \le n)$. Prove that $J_n\varphi \to \varphi$ strongly in $L^p(\mathbb{R}^d)$ for all $2 < p < 2^*$. Here $2^* = 2d/(d-2)$ if $d \ge 3$ and $2^* = \infty$ if $d \le 2$.

E10.2 Let $F(t) = e^{it\Delta}$ on $L^2(\mathbb{R}^d)$. By a modification of the analysis of $\Phi_f(t) = \int_0^t F(t-s)f(s)\mathrm{d}s$ from the lecture, we can also show that if (q, r) is admissible and $q > 2$, then

$$\left\| \int_{\mathbb{R}} F(\cdot - s)f(s)\mathrm{d}s \right\|_{L_t^q L_x^r} \le C\|f\|_{L_t^{q'} L_x^{r'}}.$$

(a) Use this bound to show that

$$\left\| \int_{\mathbb{R}} F(t)f(t)\mathrm{d}t \right\|_{L_x^2} \le C\|f\|_{L_t^{q'} L_x^{r'}}.$$

(b) Deduce the first Strichartz estimate

$$\|F(t)\varphi\|_{L_t^q L_x^r} \le C\|\varphi\|_{L_x^2}.$$

Hint: You can mimic the duality used in Step 2 and Step 3 of the analysis of $\Phi_f(t)$.

E10.3 Let $\varphi \in H^1(\mathbb{R}^d)$ and $u(t) = e^{it\Delta}\varphi$.

(a) Prove that

$$\|u(t) - u(s)\|_{L_x^2} \le C|t-s|^{1/2}, \quad \forall t, s \in \mathbb{R}.$$

(b) Let (q, r) be admissible with $2 < r < 2^*$. Prove that

$$\|u(t) - u(s)\|_{L_x^r} \le C|t-s|^{1/2-1/q}, \quad \forall t, s \in \mathbb{R}.$$

(c) Deduce from (b) and the Strichartz estimate (in E10.1 (b)) that $\|u(t)\|_{L_x^r} \to 0$ as $|t| \to \infty$. Can you prove this convergence directly without using the Strichartz estimate?

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Partial Differential Equations II

Homework Sheet 9

(Discussed on 11.7.2022)

E9.1 Let $u \in H^1(\mathbb{R}^3)$ be a radial, positive solution to the equation

$$-\Delta u(x) + u(x) = u^3(x), \quad x \in \mathbb{R}^3.$$

(a) Prove that u is continuous.

(b) Prove that u decays faster than any polynomial, namely for every $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that

$$u(x) \leq \frac{C_n}{(1+|x|)^n}, \quad \forall x \in \mathbb{R}^3.$$

Hint: $G(x) = e^{-|x|}/(4\pi|x|)$ is the 3D Yukawa potential, namely $\hat{G}(k) = (1 + |2\pi k|^2)^{-1}$.

E9.2 Let $d \geq 1$, $2 < p < \infty$ and $f(u) = |u|^{p-2}u$.

(a) For every $n \in \mathbb{N}$, define the operator $J_n = \mathbb{1}(-\Delta \leq n)$ on $L^2(\mathbb{R}^d)$. Prove that there exists a constant $C = C_{d,p,n} > 0$ such that

$$\|f(J_n u) - f(J_n v)\|_{L^2(\mathbb{R}^d)} \leq C\|u - v\|_{L^2(\mathbb{R}^d)}(\|u\|_{L^2(\mathbb{R}^d)} + \|v\|_{L^2(\mathbb{R}^d)})^{p-1}$$

for all $u, v \in L^2(\mathbb{R}^d)$.

(b) Assume further that $2(p-1) < 2^*$, with $2^* = 2d/(d-2)$ if $d \geq 3$ and $2^* = -\infty$ if $d = 1, 2$. Prove that there exist constants $C = C_{d,p} > 0$ and $\alpha \in (0,1)$ such that

$$\|f(u) - f(v)\|_{L^2(\mathbb{R}^d)} \leq C\|u - v\|^\alpha_{L^2(\mathbb{R}^d)}(\|u\|_{H^1(\mathbb{R}^d)} + \|v\|_{H^1(\mathbb{R}^d)})^{p-\alpha}$$

for all $u, v \in H^1(\mathbb{R}^d)$.

Hint: Sobolev's embedding theorem and the bound in E5.2 are helpful.

E9.3 (Grönwall's lemma) Let $0 < \alpha < 1$ and $T > 0$. Prove or disprove the following: If $f, g : [0,T] \to [0,\infty)$ are continuous and satisfy

$$f(t) \leq \int_0^t g(s) f^\alpha(s) \mathrm{d}s, \quad \forall t \in [0,T],$$

then $f(t) = 0$ for all $t \in [0,T]$.

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Partial Differential Equations II

Homework Sheet 8

(Discussed on 4.7.2022)

We consider a generalization of the method of moving planes to nonlinear PDE.

Let $0 \le u \in H^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be a weak solution of the equation

$$-\Delta u(x) + \mu^2 u(x) = f(u(x)), \quad x \in \mathbb{R}^d.$$

Here $\mu > 0$ is a constant and $f : [0, \infty) \to \mathbb{R}$ satisfies that $f \in C^1$ and

$$0 < f'(b) \le C b^{\alpha}, \quad \forall b > 0$$

with some constants $C > 0$ and $0 < \alpha < 2^* - 2$.

Recall that $2^* = 2d/(d-2)$ if $d \ge 3$ and $2^* = -\infty$ if $d = 1, 2$.

E8.1 Let G be the Yukawa potential, namely

$$\hat{G}(k) = (|2\pi k|^2 + \mu^2)^{-1}.$$

Prove that we can write

$$G(x) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t} - \mu^2 t} \mathrm{d}t.$$

Deduce that G is radially symmetric decreasing and $G \in L^p(\mathbb{R}^d)$ for $1 \le p < 2^*/2$.

E8.2 For every $a \in \mathbb{R}$, denote

$$x_a = (2a - x_1, x_2, ..., x_d), \quad u_a(x) = u(x_a), \quad \forall x = (x_1, ..., x_d),$$

namely x_a is the reflection of x via the hyperplane $\{x_1 = a\}$. Prove that

$$u(x) - u_a(x) = \int_{\{y_1 > a\}} \Big(G(x-y) - G(x_a - y) \Big) \Big(f(u(y)) - f(u_a(y)) \Big) \mathrm{d}y.$$

E8.3 Prove that if a is sufficiently negative, then $u(x) \ge u_a(x), \quad \forall x \in \{x_1 \ge a\}$.

E8.4 Prove that the set $A = \{a \in \mathbb{R} : u(x) \ge u_a(x), \forall x \in \{x_1 \ge a\}\}$ has a maximal value a_0. Deduce that

$$u(x) = u_{a_0}(x), \quad \forall x \in \mathbb{R}^d.$$

E8.5 Conclude that u is radial up to translation, namely there exists $x_0 \in \mathbb{R}^d$ and $U : [0, \infty) \to [0, \infty)$ such that

$$u(x) = U(|x - x_0|), \quad \forall x \in \mathbb{R}^d.$$

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Partial Differential Equations II

Homework Sheet 7

(Discussed on 27.6.2022)

E7.1 Let $F \subset \mathbb{R}^d$, $d \geq 1$, be a compact set such that its Lebesgue measure satisfies

$$|F| > |F \cap F^*|.$$

(a) Let $f = \mathbb{1}_{F^*} - \mathbb{1}_{F\cap F^*}$, $g = \mathbb{1}_F - \mathbb{1}_{F\cap F^*}$. Prove that there exists $x_0 \in \mathbb{R}^d \backslash \{0\}$ s.t.

$$\int_{\mathbb{R}^d} f(x_0 - y) g(-y) \mathrm{d}y > 0.$$

(b) Let $e = x_0/\|x_0\|$ with x_0 being given in (a). Prove that

$$|F^{*e} \cap F^*| > |F \cap F^*|.$$

Here F^* is the radially symmetric decreasing rearrangement of F, and F^{*e} is the Steiner symmetrization of F with respect to the direction e.

E7.2 Let A and B be nonempty open bounded sets in $\mathbb{R}^d$, $d \geq 1$. We denote

$$A + B := \{x + y \,|\, x \in A, y \in B\}.$$

(a) Prove that $A + B = \{x \in \mathbb{R}^d : (\mathbb{1}_A * \mathbb{1}_B)(x) > 0\}$.
(b) Prove that $A^* + B^* \subset (A + B)^*$.
(c) Use Riesz' rearrangement inequality to prove the Brunn–Minkowski inequality

$$|A + B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}.$$

E7.3 Let $A \subset \mathbb{R}^d$ be an open bounded set with C^1 boundary. Use the Brunn–Minkowski inequality to prove the isoperimetric inequality

$$|\partial A| \geq |\partial(A^*)|.$$

Hint: The surface area of A can be computed as

$$|\partial A| = \lim_{r\to 0^+} \frac{|A + rB| - |A|}{r} \quad \text{with } B \text{ the unit ball in } \mathbb{R}^d.$$

E7.4 Use the Hardy–Littlewood–Sobolev inequality to prove Sobolev's inequality:

$$\|u\|_{L^{2^*}} \leq C \|\nabla u\|_{L^2}, \quad \forall u \in C_c^\infty(\mathbb{R}^d).$$

Here $d \geq 3$ and $2^* = 2d/(d-2)$.

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4.6.2022

Partial Differential Equations II

Homework Sheet 6

(Discussed on 13.6.2022)

E6.1 Let $d \geq 3$ and $p = 2^* = 2d/(d-2)$.

(a) Prove that if $0 \leq u \in \dot{H}^1(\mathbb{R}^d)$ be an optimizer of the Sobolev inequality

$$\|\nabla u\|_{L^2(\mathbb{R}^d)} \geq E\|u\|_{L^p(\mathbb{R}^d)}$$

then it solves the Euler-Lagrange equation

$$-\Delta u(x) = cu^{p-1}(x), \quad \text{a.e. } x \in \mathbb{R}^d$$

for some constant $c > 0$.

(b) Prove that $h(x) = (1 + |x|^2)^{-\alpha}$ with $\alpha = \frac{d-2}{2}$ is a solution of the above equation (for some c).

E6.2 Let $d \geq 3$. Prove that there exists a unique number $\lambda > 0$ such that the following (i)-(ii) hold:

(i) We have

$$\int_{\mathbb{R}^d} \left(|\nabla u|^2 + V|u|^2\right) \geq 0, \quad \forall u \in \dot{H}^1(\mathbb{R}^d), \quad \forall \|V\|_{L^{d/2}(\mathbb{R}^d)} = \lambda.$$

(ii) There exists $\|V\|_{L^{d/2}(\mathbb{R}^d)} = \lambda$ and $0 \not\equiv u \in \dot{H}^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \left(|\nabla u|^2 + V|u|^2\right) = 0.$$

Compute λ in terms of E given in E5.1(a).

E6.3 Prove that for all $d \geq 3$ and $u \in \dot{H}^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geq 2d(d-2) \int_{\mathbb{R}^d} \frac{|u(x)|^2}{1 + |x|^2} \mathrm{d}x.$$

Moreover, the equality occurs if and only if $u = zh$ with a constant z and with h given in E5.1(b).

Hint: You can use the Perron–Frobenius principle.

E6.4 Let B be the closed unit ball in $\mathbb{R}^d$ and let $f : B \to B$ be a continuous function. Assume that there exists a constant $\varepsilon \in (0, 1)$ such that

$$x \cdot f(x) \geq \varepsilon, \quad \forall 1 - \varepsilon \leq |x| \leq 1.$$

Prove that the equation $f(y) = 0$ has a solution $y \in B$.

Hint: You can apply Brouwer's fixed-point theorem for $g(x) = x - \delta f(x)$ with some $\delta > 0$.

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Partial Differential Equations II

Homework Sheet 5

(Discussed on 30.5.2022)

E5.1 Prove that for every $\alpha, \varepsilon \in (0,1)$, there exists a constant $\delta = \delta(\alpha, \varepsilon) > 0$ such that

$$a^\alpha + b^\alpha \geq (1+\delta)(a+b)^\alpha, \quad \forall a, b \in [\varepsilon, 1].$$

E5.2 (a) Let $2^* = d/(d-2)$ if $d \geq 3$ and $2^* = \infty$ if $d \leq 2$. Prove that for every $d \geq 1$ and every $2 < p < 2^*$, there exists a unique parameter $\theta \in (0,1)$ such that

$$E := \inf \left\{ \frac{\|\nabla u\|^\theta_{L^2(\mathbb{R}^d)} \|u\|^{1-\theta}_{L^2(\mathbb{R}^d)}}{\|u\|_{L^p(\mathbb{R}^d)}} \mid u \in H^1(\mathbb{R}^d)\backslash\{0\} \right\} > 0.$$

(b) Prove that E has an optimizer.

Hint: For (a) you can use the Sobolev inequality $\|u\|_{H^1} \geq C\|u\|_{L^p}$ and a scaling argument (c.f. E3.1). For (b), you can use the existence of optimizers for $\|u\|_{H^1} \geq C\|u\|_{L^p}$.

E5.3 (Helly's selection theorem) Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions $\mathbb{R} \to [0,1]$ such that $t \mapsto f_n(t)$ is increasing for every n. Prove that there exists a subsequence f_{n_k} and a function $f_\infty : \mathbb{R} \to [0,1]$ such that

$$\lim_{k\to\infty} f_{n_k}(t) = f_\infty(t), \quad \forall t \in \mathbb{R}.$$

E5.4 Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of Borel probability measures on $\mathbb{R}^d$, $d \geq 1$. Let μ be a Borel finite measure on $\mathbb{R}^d$ such that $\mu_n \rightharpoonup \mu$ weakly in $(C_c(\mathbb{R}^d))^*$, namely

$$\int_{\mathbb{R}^d} f \mathrm{d}\mu_n \to \int_{\mathbb{R}^d} f \mathrm{d}\mu, \quad \forall f \in C_c(\mathbb{R}^d)$$

Prove that the following three statements are equivalent:

(a) $\mu(\mathbb{R}^d) = 1$

(b) $\mu_n \rightharpoonup \mu$ weakly in $(C_b(\mathbb{R}^d))^*$, namely

$$\int_{\mathbb{R}^d} f \mathrm{d}\mu_n \to \int_{\mathbb{R}^d} f \mathrm{d}\mu, \quad \forall f \in C_b(\mathbb{R}^d).$$

(c) The sequence $\{\mu_n\}_{n=1}^\infty$ is tight, namely for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\liminf_{n\to\infty} \mu_n(B_R(0)) \geq 1 - \varepsilon.$$

Here $C_c(\mathbb{R}^d)$ is the space of continuous functions $\mathbb{R}^d \to \mathbb{C}$ with compact support, and $C_b(\mathbb{R}^d)$ is the space of bounded continuous functions.

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Phan Thành Nam, Arnaud Triay — 18.5.2022

Partial Differential Equations II

Homework Sheet 4

(Discussed on 23.5.2022)

E4.1 (a) Prove that the constant

$$a^* = \inf \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y} \;|\; u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} |u|^2 = 1 \right\}$$

is positive and finite.

(b) For $a \in \mathbb{R}$, consider the variational problem

$$E = \inf \left\{ \mathcal{E}(u) : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^d} |u|^2 = 1 \right\}$$

where

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|} \mathrm{d}x - \frac{a}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^2} \mathrm{d}x \mathrm{d}y.$$

Prove that $E = -\infty$ if $a \geq a^*$, and that E is finite and has a minimizer if $a < a^*$.

E4.2 Let $\{Q\}_{Q \in \mathcal{F}}$ be a collection of disjoint, unit cubes in $\mathbb{R}^d$, $d \geq 1$. Prove that for every $2 \leq p < 2^*$ we have

$$\sum_{Q \in \mathcal{F}} \|u\|^2_{L^p(Q)} \leq C \|u\|^2_{H^1(\mathbb{R}^d)}, \quad \forall u \in H^1(\mathbb{R}^d)$$

for a constant $C = C_{d,p}$ independent of u. Here 2^* is the critical power in Sobolev's inequality (namely $2d/(d-2)$ if $d \geq 3$, and ∞ if $d \leq 2$).

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Partial Differential Equations II

Homework Sheet 3

(Discussed on 16.5.2022)

E3.1 Prove that there exists a constants $c > 0$ such that

$$\sup_{\|u\|_{H^1(\mathbb{R}^2)}=1} \int_{\mathbb{R}^2} |u(x)|^2 e^{c|u(x)|^2} \mathrm{d}x < \infty.$$

Deduce that for every $2 < q < \infty$, there exist $C = C_q > 0$ and $\theta = \theta_q \in (0,1)$ such that

$$\|u\|_{L^q(\mathbb{R}^2)} \le C \|u\|^{\theta}_{L^2(\mathbb{R}^d)} \|\nabla u\|^{1-\theta}_{L^2(\mathbb{R}^d)}, \quad \forall u \in H^1(\mathbb{R}^d).$$

E3.2 Let $d \ge 1$. Assume that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^d)$ as $n \to \infty$. Prove that there exists a sequence $R_n \to \infty$ as $n \to \infty$ such that

$$\mathbb{1}_{\{|x| \le R_n\}} u_n \to u \quad \text{strongly in } L^2(\mathbb{R}^d).$$

Deduce that

$$\int_{R_n/2 \le |x| \le R_n} |u_n(x)|^2 \mathrm{d}x \to 0.$$

E3.3 Let $d \ge 1$. Let $w \in L^\infty(\mathbb{R}^d) + L^p(\mathbb{R}^d)$ with $p > (d/2, 1)$ and $w(x) \to 0$ as $|x| \to \infty$. Let $\{u_n\}$ and $\{R_n\}$ as in E3.2. Denote

$$a_n = \mathbb{1}_{\{|x| \le R_n/2\}} u_n, \quad b_n = \mathbb{1}_{\{|x| \ge R_n\}} u_n.$$

Prove that

$$D(u_n) - D(a_n) - D(b_n) \to 0 \quad \text{as } n \to \infty$$

where

$$D(f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x)|^2 |f(y)|^2 w(x-y) \mathrm{d}x \mathrm{d}y.$$

E3.4 (IMS formula) Let $\varphi \in C^1(\mathbb{R}^d)$ be non-negative and bounded. Prove that

$$\frac{\varphi^2(-\Delta) + (-\Delta)\varphi^2}{2} = \varphi(-\Delta)\varphi - |\nabla \varphi|^2$$

as quadratic forms in $L^2(\mathbb{R}^d)$.

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Summer Semester 2022
4.5.2022

Partial Differential Equations II

Homework Sheet 2

(Discussed on 9.5.2022)

E2.1 Let $V : \mathbb{R}^2 \to \mathbb{R}$ such that $V \in L^p(\mathbb{R}^2) + L^q(\mathbb{R}^2)$ with $p, q \in (1, \infty)$. Assume that

$$E = \inf \Big\{ \int_{\mathbb{R}^2} |\nabla u|^2 + \int_{\mathbb{R}^2} V|u|^2 \, : \, u \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} |u|^2 = 1 \Big\} < 0.$$

Prove that E has a minimizer.

E2.2 Let $V \in L^{d/2}(\mathbb{R}^d)$, $V \le 0$, $V \not\equiv 0$ and $\lambda > 0$. Consider the variational problem

$$E = \inf \Big\{ \int_{\mathbb{R}^d} |\nabla u|^2 + \lambda \int_{\mathbb{R}^d} V|u|^2 \, : \, u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} |u|^2 = 1 \Big\}.$$

(a) Prove that if $d \ge 3$, then E has a minimizer if λ is large enough, and E has no minimizer if λ is small enough.

(b) What happens if $d = 2$?

E2.3 Let $d \ge 1$. Let $V, w : \mathbb{R}^d \to \mathbb{R}$ such that $V_-, w \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $p > \max(d/2, 1)$ and

$$V(x) \to +\infty \quad \text{as} \quad |x| \to \infty.$$

Prove that the following variational problem

$$E = \inf \Big\{ \mathcal{E}(u) : u \in H^1(\mathbb{R}^d), \|u\|_{L^2(\mathbb{R}^d)} = 1 \Big\}$$

has a minimizer, where

$$\mathcal{E}(u) = \int_{\mathbb{R}^d} |\nabla u|^2 + \int_{\mathbb{R}^d} V|u|^2 + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x)|^2 |u(y)|^2 w(x-y) \mathrm{d}x \mathrm{d}y.$$

Mathematisches Institut der Universität München | Summer Semester 2022
Phan Thành Nam, Arnaud Triay | 27.4.2022

Partial Differential Equations II

Homework Sheet 1

(Discussed on 2.5.2022)

E1.1 (Brezis–Lieb's refinement of Fatou's lemma)
Let Ω be an open subset of $\mathbb{R}^d$ ($d \geq 1$) and let $1 \leq p < \infty$. Let $\{f_n\}_{n=1}^{d}$ be a bounded sequence in $L^p(\Omega)$ such that $f_n(x) \to f(x)$ a.e. Prove that

$$\lim_{n\to\infty} \int_\Omega \Big| |f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p \Big| \mathrm{d}x = 0.$$

Hint: For every $\varepsilon > 0$ you may apply Dominated Convergence Theorem to

$$g_{n,\varepsilon} = \Big(\Big| |f_n|^p - |f|^p - |f_n - f|^p \Big| - \varepsilon |f_n - f|^p \Big)_+ \quad \text{with} \quad t_+ = \max(t, 0).$$

E1.2 (Fourier representation of Sobolev spaces)
Let $d, s \in \mathbb{N}$. Recall the Hilbert space

$$H^s(\mathbb{R}^d) = \{ f \in L^2\mathbb{R}^d) : D^\alpha f \in L^2(\mathbb{R}^d), \quad \forall |\alpha| \leq s \}.$$

Prove that

$$H^s(\mathbb{R}^d) = \{ f \in L^2\mathbb{R}^d) : (1 + |2\pi k|^2)^{s/2} \hat{f}(k) \in L^2(\mathbb{R}^d) \}.$$

E1.3 Recall that $C_c^1(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ for all $d \geq 1$.

(a) Prove that if $d \geq 3$, then $C_c^1(\mathbb{R}^d \backslash \{0\})$ is also dense in $H^1(\mathbb{R}^d)$. Is it true for $d = 1, 2$?

(b) In the lecture, by the Perron–Frobenius principle we proved that for all $u \in C_c^1(\mathbb{R}^3 \backslash \{0\})$

$$\int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} \geq -\frac{1}{4} \int_{\mathbb{R}^3} |u|^2$$

and

$$\int_{\mathbb{R}^3} |\nabla u|^2 \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2}.$$

Deduce the same bounds for all $u \in H^1(\mathbb{R}^3)$ using (a) and the density argument.

E1.4 (Sobolev inequalities in 1D)

(a) Using $\|u\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + |2\pi k|^2) |\hat{u}(k)|^2 \mathrm{d}k$ (see E1.2) to prove that

$$\|u\|_{L^\infty(\mathbb{R})} \leq C \|u\|_{H^1(\mathbb{R})}, \quad \forall u \in H^1(\mathbb{R}).$$

(b) Using the scaling argument to deduce from (a) the following inequality

$$\|u\|_{L^\infty(\mathbb{R})} \leq C \|u\|_{L^2(\mathbb{R})} \|u'\|_{L^2(\mathbb{R})}, \quad \forall u \in H^1(\mathbb{R}).$$

Here $C > 0$ is a universal constant.