Chapter 6: Wave equation.

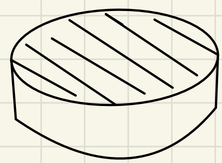
$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & , x \in \mathbb{R}^d, t > 0 \\ u = g, \ \partial_t u = h & , x \in \mathbb{R}^d, t = 0 \end{cases}$$

initial displacement   initial velocity

Motivation:

$d=1$: vibrating string

$d=2$: membrane



$d=3$: elasticity

Solution of wave equation:

$\boxed{d=1}$
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & , (x,t) \in \mathbb{R} \times (0,\infty) \\ u = g, \ \partial_t u = h & , x \in \mathbb{R}, t = 0. \end{cases}$$

Key idea: Factorization

$$\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x)$$

Denote $v = (\partial_t - \partial_x) u$

$\Rightarrow \quad (\partial_t + \partial_x) v = 0 \quad$ (transport eq)

$\Rightarrow \quad v(x,t) = a(x-t) \, , \quad a(x) = v(x,0)$

$\Rightarrow \quad (\partial_t - \partial_x) u = a(x-t) \quad$ (inho. transport eq)

We decompose

$$u = u_1 + u_2$$

where $\begin{cases} (\partial_t - \partial_x) u_1 = 0 \\ (\partial_t - \partial_x) u_2 = a(x-t) \end{cases}$

Like above,

$$u_1 = b(x+t)$$

and an explicit choice of u_2 is

$$u_2 = \frac{1}{2} \int_{x-t}^{x+t} a(y)\, dy$$

Thus: $\quad u = b(x+t) + \frac{1}{2} \int_{x-t}^{x+t} a(y)\, dy$

Compute a & b: $\quad b(x) = u(x,0) = g$

$\quad a(x) = v(x,0) = (\partial_t u - \partial_x u)_{t=0} = h - g'.$

$\Rightarrow$ d'Alembert formula:

$$\boxed{\begin{aligned} u &= \frac{1}{2}\int_{x-t}^{x+t} \left(h(y) - g'(y)\right) dy + g(x+t) \\ &= \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy \end{aligned}}$$

Theorem ($d=1$) Let $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define u by d'Alembert formula as above.

Then .) $u \in C^2(\mathbb{R} \times (0,\infty))$

.) $\partial_t^2 u - \partial_x^2 u = 0$

.) $u = g$, $\partial_t u = h$ when $t \to 0$.

Proof: Exercise

Remark: If $g \in C^k$ and $h \in C^{k-1}$, then $u \in C^k$ (but not better).

Reflection method: Replace $\mathbb{R}$ by $\mathbb{R}_+ = (0,\infty)$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{on } \mathbb{R}_+ \times (0,\infty) \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R}_+ \times \{t=0\}, \quad g(0) = h(0) = 0 \\ u = 0 & \text{on } \{x=0\} \times \{t>0\} \end{cases}$$

Define:
$$\tilde{u}(x,t) = \begin{cases} u(x,t) & , x \geq 0, \ t \geq 0 \\ -u(-x,t) & , x \leq 0, \ t \geq 0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ -g(-x) & x \leq 0 \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x) & x \geq 0 \\ -h(-x) & h \leq 0 \end{cases}$$

$$\rightarrow \begin{cases} \partial_t^2 \tilde{u} - \partial_x^2 \tilde{u} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ \tilde{u} = \tilde{g}, \ \partial_t \tilde{u} = \tilde{h} & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

By d'Alembert formula
$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+t) + \tilde{g}(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(y)\, dy$$

$$\Rightarrow u(x,t) = \begin{cases} \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy, & x \geq t \geq 0 \\ \frac{1}{2}\left[g(x+t) - g(t-x)\right] + \frac{1}{2}\int_{t-x}^{x+t} h(y)\, dy, & t \geq x \geq 0 \end{cases}$$

$\downarrow$

(solution of the heat eq in $\mathbb{R}_+ \times (0,\infty)$)

$\boxed{d \geq 2}$

$$(*) \quad \begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u = g,\ \partial_t u = h & \mathbb{R}^d \times \{t=0\} \end{cases}$$

<u>Idea:</u> Averaging of u over spheres $\rightsquigarrow$ 1D problem

<u>Def:</u> For $x \in \mathbb{R}^d$, $t > 0$, $r > 0$,

$$U_r(x,t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y)$$

Similarly $G_r(x)$, $H_r(x)$ average over $\partial B(x,r)$

<u>Lemma:</u> (Euler - Poisson - Darboux equation) :

If $u \in C^2(\mathbb{R}^d \times [0,\infty))$ solves $(*)$, then $\forall x \in \mathbb{R}^d$:

.) $(r,t) \mapsto U \in C^2([0,\infty) \times [0,\infty))$,

.) $$\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{d-1}{r} \partial_r U = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}_+ \\ U = G,\ \partial_t U = H & \text{on } \mathbb{R}_+ \times \{t=0\} \end{cases}$$

<u>Note:</u> $\partial_r^2 + \frac{d-1}{r} \partial_r$ is the radial part of Δ.

Proof: We compute for $r>0$:

$$\partial_r U_r(x,t) \overset{?}{=} \frac{r}{d} \fint_{B(x,r)} \Delta_x u(y,t)\,dy$$

In fact, LHS is:

$$\partial_r \fint_{\partial B(x,r)} u(y,t)\,dS(y) = \partial_r \fint_{\partial B(0,1)} u(x+rz)\,dS(z)$$

$$= \fint_{\partial B(0,1)} \nabla u(x+rz)\cdot \underbrace{z}_{\frac{z}{|z|}}\,dS(z)$$

$$= \fint_{\partial B(0,r)} \nabla u(y)\cdot \frac{y-x}{r}\,dS(y)$$

$$= \fint_{\partial B(0,r)} \frac{\partial u}{\partial \vec{n}}\,dS(y)$$

$$= \frac{1}{|\partial B(0,r)|}\int_{B(x,r)} \Delta_x u\,dy \quad \text{(Green formula)}$$

$$= \frac{r}{d} \fint_{B(x,r)} \Delta_x u(y,t)\,dt$$

(The computation is similar to the proof of the mean-value theorem for Poisson eq.)

Thus we conclude that:

(1) $$\partial_r U_r(x,t) = \frac{r}{d} \fint_{B(x,r)} \Delta_x u(y,t)\,dy$$

(2) Next:

$$\partial_r^2 U_r(x,t) = \partial_r \left[\frac{r}{d} \fint_{B(x,r)} \Delta_x u(y,t)\,dy \right]$$

$$= \partial_r \left[\frac{1}{d|B_1|r^{d-1}} \int_{B(x,r)} \Delta_x u\,dy \right)$$

$$= -\left(\frac{d-1}{d}\right) \fint_{B(x,r)} \Delta_x u\,dy + \underbrace{\frac{1}{d|B_1|r^{d-1}} \int_{\partial B(x,r)} \Delta_x u\,dS}_{\fint_{\partial B(x,r)} \Delta_x u\,dS}$$

(3) Obviously

$$\partial_t^2 U = \partial_t^2 \fint_{\partial B(x,r)} u\,dS = \fint_{\partial B(x,r)} \overset{\Delta_x u}{\overset{\shortparallel}{(\partial_t^2 u)}}\,dS.$$

Conclusion:

$$\partial_t^2 U - \partial_r^2 U - \frac{d-1}{r}\partial_r U = 0 \quad \text{(from (1),(2),(3))}$$

The above computation also show that

$$U \in C^2(\mathbb{R}_+ \times [0,\infty))$$

Moreover, $\partial_r U_r(x,t) \to 0$ as $r \to 0^+$.

$$\partial_r^2 U(x,t) \xrightarrow{r\to 0} \left(\frac{1}{d} - 1\right)\Delta_x u + \Delta_x u = \frac{1}{d}\Delta_x u$$

$$\Rightarrow \quad U \in C^2([0,\infty) \times [0,\infty)).$$

Finally, when $t=0$ $\begin{cases} u = g \\ \partial_t = h \end{cases} \Rightarrow \begin{cases} U = G \\ \partial_t U = H \end{cases}$

$\square$.

<u>Q</u>: How to solve the Euler-Poisson-Darboux eq?

In general, odd d is easier than even d.

We will consider $d=3$ and $d=2$.

$\boxed{d=3}$ Def: $\tilde{U} = rU$, $\tilde{G} = rG$, $\tilde{H} = rH$.

Then:
$$\begin{cases} \partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \tilde{U} = \tilde{G},\ \partial_t \tilde{U} = \tilde{H} & \text{when } t=0 \\ \tilde{U} = 0 & \text{when } r=0 \end{cases}$$

Thus, by d'Alembert's formula, for $0 \le r \le t$

$$\tilde{U}_r(x,t) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(y)\,dy$$

$$\Rightarrow U_r(x,t) = \frac{1}{2}\left[\frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{r}\right] + \frac{1}{2r}\int_{t-r}^{t+r} \tilde{H}$$

Taking $r \to 0$

$$\boxed{\begin{aligned} u(x,t) &= \tilde{G}'(t) + \tilde{H}(t). \\ &= \partial_t\left(t \int g\,dS\right) + t ⨍ h\,dS \end{aligned}}$$

Using $\displaystyle\fint_{\partial B(x,t)} g(y)\, dS(y) = \fint_{\partial B(0,1)} g(x+tz)\, dS(z)$

$$\Rightarrow \partial_t \fint_{\partial B(x,t)} g\, dS = \fint_{\partial B(0,1)} \nabla g(x+tz) \cdot z\, dz$$

$$= \fint_{\partial B(x,t)} \nabla g(y) \cdot \left(\frac{y-x}{t}\right) dS(y)$$

$$\Rightarrow \partial_t \left(t \fint_{\partial B(x,t)} g(y)\, dS(y) \right)$$

$$= \fint_{\partial B(x,t)} \Big(g + \nabla g \cdot (y-x) \Big)\, dS(y)$$

<u>Conclusion:</u> (Kirchhoff's formula in 3D)

$$\boxed{u(x,t) = \fint_{\partial B(x,t)} \Big(g(y) + \nabla g \cdot (y-x) + t h(y) \Big)\, dS(y)}$$

for all $x \in \mathbb{R}^3$, $t > 0$.

$\boxed{d=2}$ The transformation $\tilde{u} = rU$ does not work!

Idea: Think of 2D problem as 3D with "x_3" hidden.

Write $\bar{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$

$$\Rightarrow \begin{cases} \partial_t^2 \bar{u} - \Delta_x \bar{u} = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u} = \bar{g},\ \partial_t \bar{u} = \bar{h} & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

We use Kirchhoff's formula

$$u(x,t) = \bar{u}(\bar{x},t)$$
$$= \frac{\partial}{\partial t}\left(t \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} \right) + t \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}$$

Note:
$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, dS$$

$$= \frac{1}{4\pi t^2} \int_{B(x,t)} g(y)\, 2\left(1+|\nabla \gamma|^2\right)^{1/2} dy = \dots$$

where $\gamma(y) = \left(t^2 - |y-x|^2\right)^{1/2}$, $y \in B(x,t)$.

$$\ldots = \frac{1}{4\pi t^2} \int_{B(x,t)} g(y) \frac{2t}{\sqrt{t^2 - |y-x|^2}} \, dy$$

$$= \frac{t}{2} \fint_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |y-x|^2}} \, dy$$

Similary:

$$\fint_{\partial \bar{B}(\bar{x},t)} \bar{h} \, d\bar{S} = \frac{t}{2} \fint_{B(x,t)} \frac{h(y)}{\sqrt{t^2 - |y-x|^2}} \, dy$$

$$\Rightarrow u(x,t) = \partial_t \left(\frac{t^2}{2} \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}} dy \right)$$

$$+ \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{1/2}} dy$$

$$= (\mathrm{I}) + (\mathrm{II})$$

$$(\mathrm{I}) = \partial_t \left(\frac{1}{2} t \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}} dz \right)$$

$$=$$

$$= \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz + t \fint_{B(0,1)} \frac{\nabla g(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz$$

$$= t \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|y-x|^2}}\,dy + t \fint_{B(x,t)} \frac{\nabla g(y)\cdot(y-x)}{\sqrt{t^2-|y-x|^2}}\,dy$$

<u>Conclusion:</u> (Poisson formula for 2D)

$$\boxed{u(x,t) = \frac{t}{2} \fint_{B(x,t)} \frac{g(y) + \nabla g(y)\cdot(y-x) + t h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy}$$

for $x \in \mathbb{R}^2$, $t > 0$

<u>General dim:</u> $\rightarrow$ d odd first,

d even by d+1 odd.

(discussed in tutorial)

Wave equation in bounded set ($\Omega \subset \mathbb{R}^d$)

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0,\infty) \\ u = g,\ \partial_t u = h & \text{when } t = 0 \\ u = 0 & \text{when } x \in \partial\Omega \end{cases}$$

Spectral method: $\Omega \subset \mathbb{R}^d$ open, bounded

$\Rightarrow$ $-\Delta$ has an e.f. $(e_i)_{i=1}^\infty$ with e.v. $(\lambda_i)_{i=1}^\infty$

i.e. $\begin{cases} -\Delta e_i = \lambda_i e_i \\ e_i|_{\partial\Omega} = 0 \end{cases}$ s.t. $\cdot)$ $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_i \to \infty$

$\cdot)$ (e_i) ONB for $L^2(\Omega)$.

We write: $u(x,t) = \sum_i a_i(t) e_i(x)$

$$\Rightarrow \quad a_i''(t) + \lambda_i a_i(t) = 0$$

$$\Rightarrow \quad a_i(t) = a_i(0) \cos\left(\sqrt{\lambda_i}\, t\right) + \frac{a_i'(0)}{\sqrt{\lambda_i}} \sin\left(\sqrt{\lambda_i}\, t\right)$$

Here $a_i(0)$ & $a_i'(0)$ is determined by

$$\begin{cases} g = u(t=0) = \sum_i a_i(0)\, e_i(x) \\ h = \partial_t u(t=0) = \sum_i a_i'(0)\, e_i(x) \end{cases} \Rightarrow \begin{cases} a_i(0) = (e_i, g) \\ a_i'(0) = (e_i, h) \end{cases}$$

<u>Uniqueness:</u> let $\Omega \subset \mathbb{R}^d$ open, bounded, C^1-boundary.

Then the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in } \Omega \times (0,T) \\ u = 0,\ \partial_t u = 0 & \text{in } \Omega \times \{t=0\} \\ u = 0 & \text{in } \partial\Omega \times [0,T] \end{cases}$$

has only trivial solution $u = 0$ (if $u \in C^2(\bar{\Omega} \times [0,T])$)

<u>Proof:</u> Consider the energy functional

$$e(t) = \int_\Omega |\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2$$

$$\Rightarrow e'(t) = 2\left(\int \partial_t u \cdot \partial_t^2 u + \nabla_x u \cdot \partial_t \nabla_x u\right)$$

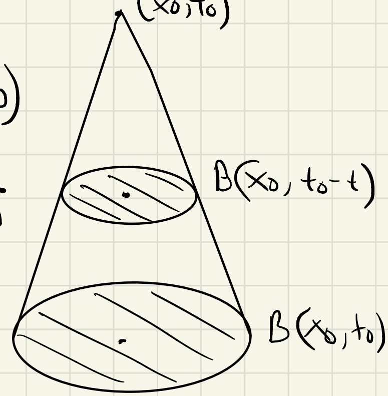
$$= 2\left(\int \partial_t u \cdot \partial_t^2 u - \Delta_x u \cdot \partial_t u\right) = 0$$

Thus $e(t) = e(0) = 0 \Rightarrow \partial_t u = 0 \Rightarrow u = 0$.

(& $u(t=0) = 0$)

<u>Remark:</u> The same uniqueness result hold in $\mathbb{R}^d$ if we assume that $u \in C^2(H^2(\mathbb{R}^d), [0,T])$.

Theorem (Propagation of speed)



Assume $\partial_t^2 u - \Delta_x = 0$ in $\mathbb{R}^d \times (0,\infty)$

$u = 0, \ \partial_t u = 0$ in $B(x_0, t_0) \times \{t = 0\}$

Then: $u = 0$ in

$C = \{(x,t) : t \in [0, t_0], \ |x - x_0| \leq t_0 - t\}$

Proof: Consider the energy functional

$$e(t) = \int_{B(x_0, t_0 - t)} \left((\partial_t u)^2 + (\nabla_x u)^2 \right) dx$$

$$\Rightarrow e'(t) = \int_{B(x_0, t_0-t)} 2 \left(\partial_t u \cdot \partial_{tt} u + \nabla_x u \cdot \nabla_x \partial_t u \right) - \int_{\partial B(x_0, t_0 - t)} (\partial_t u)^2 + (\nabla_x u)^2$$

$$= \int_{B(x_0, t_0-t)} 2 \partial_t u \underbrace{(\partial_t^2 u - \Delta_x u)}_{=0} + \int_{\partial B(x_0, t_0-t)} \underbrace{2 \frac{\partial u}{\partial \vec{n}} \cdot \partial_t u - \left((\partial_t u)^2 + (\nabla_x u)^2 \right)}_{\leq 0 \text{ by Cauchy-Schwarz}}$$

$$\Rightarrow e(t) \leq e(0) = 0 \Rightarrow e(t) = 0, \ \forall t \in [0, t_0].$$