Introduction:

- Diff equation: equation of a function & derivatives
- ODE (Ordinary differential equation)

Example: Population equation

$$\begin{cases} \frac{d}{dt} f(t) = \alpha f(t), & t>0, \alpha = 1\% \\ f(0) = a_0 > 0 \end{cases}$$

$$\Rightarrow \quad f(t) = a_0 e^{\alpha t}$$

It is a typical example of linear ODE.

There is also non linear PDE, e.g

$$\begin{cases} f'(t) = F(t, f(t)) & , t>0 \\ f(0) = a_0 \end{cases}$$

Ex:

$$\begin{cases} f'(t) = 1 + f^2(t) & , t>0 \\ f(0) = 0 \end{cases}$$

$$\leadsto f(t) = \tan(t) = \frac{\sin(t)}{\cos(t)}$$

$\leadsto$ domain of variables & functions!

• PDE (partial differential equation) : equation of a function of 2 or more variables & its partial derivatives.

<u>Recall</u>: $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ (or $\mathbb{C}$)

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}$$

$$e_i = (\underbrace{0, 0, 1}_{i\text{-th position}}, 0, \dots) \in \mathbb{R}^d$$

$$D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f(x), \quad \alpha = (\alpha_i)_{i=1}^d, \quad |\alpha| = \sum |\alpha_i|$$

$Df = \nabla f$: gradient term

$D^2 f = (D^\alpha f)_{|\alpha|=2}$ Hessian matrix

$D^k f = (D^\alpha f)_{|\alpha|=k}$: all k-order derivative

<u>Def</u>: Let $\Omega \subset \mathbb{R}^d$, F a given function

$$F\left(D^k f(x), D^{k-1} f(x), \dots, Df(x), f(x), x\right) = 0 \quad \text{on } x \in \Omega$$

is called a k-th order PDE.

<u>Linear</u>: $\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha f(x) = 0$ ⇝ easier

<u>Semi linear</u>: ⇝ more difficult

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha f(x) + F(D^{k-1} f(x), \dots, f(x), x) = 0$$

There are also more complicated nonlinear PDE.

<u>Goal</u>: For "solving" an equation, we want

- Explicit solutions, or
- Well-posed theory

($\exists$, $!$, depending continuously on data)

⇝ important to define the right set/space of data & solutions.

<u>Classical solutions</u>: "Smooth", i.e.

relevant derivatives exist & continuous

<u>Weak solutions & regularity</u>

- Schwartz (Fields medal 1950) → distribution theory
- Figalli (Fields medal 2018) ⇝ Monge-Ampère equation

$$\det(D^2 u) = f(u, \nabla u, D^2 u)$$

$$D^2 u = (D^\alpha u)_{|\alpha|=2}$$ Hessian matrix

(optimal transport & change of variables)

- Navier - Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div}(u) = \sum_i \frac{\partial}{\partial x_i} u_i = 0 \end{cases}$$

$$u = (u_i)_{i=1}^d, \quad u_i : \mathbb{R}^d \to \mathbb{R}$$

(incompressible, viscous flow)

One of 7 Millennium Prize Problems

(<u>Question</u>: in 3D × time, given an initial vector field p, $\exists$ a solution u that is smooth.)

In the course:

- Laplace / Poisson equation

$$\Delta u = \sum_{i=1}^{d} \partial_{x_i}^2 u = f$$

- Heat equation

$$\partial_t u - \Delta u = f$$

- Wave equation

$$\partial_t^2 u - \Delta u = f$$

- Schrödinger equation:

$$i\partial_t u - \Delta u = f$$

If $f = f(x)$ given $\to$ linear equation.
($f = 0 \rightsquigarrow$ fundamental solution)
If $f = f(x, u)$ or $f(x, u, \nabla u) \to$ nonlinear equation.

Goal: $\to$ Representation of solutions
Functional Analysis: $\hookrightarrow$ Abstract well-posedness
Sobolev spaces (focus on linear theory)

Some basic functional spaces:

$$C(\Omega) = \{ f : \Omega \to \mathbb{R} \text{ or } \mathbb{C} \text{ , continuous} \}$$

$$\|f\| = \sup_{x \in \Omega} |f(x)|$$

$\rightsquigarrow$ a norm space

$$C^k(\Omega) = \{ f : D^\alpha f \in C(\Omega), \ \forall |\alpha| \le k \}$$

$L^p(\Omega) \rightsquigarrow$ Lebesgue space

$\rightsquigarrow$ Sobolev space

$$W^{k,p}(\Omega) = \{ f : D^\alpha f \in L^p, \ \forall |\alpha| \le k \}$$

$$H^k(\Omega) = W^{k,2}(\Omega) = \text{Hilbert spaces}$$

$$H^k(\mathbb{R}^d) = \{ f \in L^2, \ \hat{f}(p)\, |p|^k \in L^2(\mathbb{R}^d) \}$$

Classical solutions $\rightsquigarrow C^k(\Omega)$

Weak solutions $\to$ Sobolev spaces.

Fourier transform & convolution.