Chapter 3: Weak solutions and regularity

Def: u is a weak solution of the linear equation

$$\sum_{\alpha} c_\alpha D^\alpha u = F(x) \quad \text{on } \mathcal{D}'(\Omega)$$

if

$$\int_\Omega \sum (-1)^{|\alpha|} c_\alpha \, u (D^\alpha \varphi) = \int F\varphi, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Fundamental sol of $-\Delta$ on $\mathbb{R}^d$:

$$G(x) = \begin{cases} -\frac{1}{2}|x| & d=1 \\ -\frac{1}{2\pi} \ln|x| & d=2 \\ \frac{1}{4\pi|x|} & d=3 \\ \frac{1}{|B_1|\, d(d-2)\, |x|^{d-2}} & d \geq 3 \end{cases}$$

Theorem: $-\Delta G_y = \delta_y$ in $\mathcal{D}'(\mathbb{R}^d)$

where $G_y(x) = G(x-y)$.

Proof:

$$(-\Delta G_y)(\varphi) = \int_{\mathbb{R}^d} G_y(x)\, (-\Delta \varphi(x))\, dx$$

$$= [G * (-\Delta\varphi)](y) = (-\Delta)(G * \varphi)(y)$$

$$= \varphi(y), \quad \forall \varphi \in C_c^\infty$$

$$\Rightarrow \quad -\Delta G_y = \delta_y$$

$\rightsquigarrow$ Poisson eq

Theorem: (Poisson's equation with L^1_{loc} data)

Let $f \in L^1_{loc}(\mathbb{R}^d)$ s.t. $\omega_d f \in L^1$, where

$$\omega_d(x) = \begin{cases} (1+|x|) & d=1 \\ \ln(1+|x|) & d=2 \\ \frac{1}{(1+|x|)^{d-2}} & d \geq 3 \end{cases}$$

Then:

$$u(x) = (G * f)(x) \in L^1_{loc}(\mathbb{R}^d)$$

and

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Moreover, $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and

$$\partial_i u(x) = \int_{\mathbb{R}^d} \frac{\partial G_y}{\partial x_i}(x) f(y)\, dy = (\partial_{x_i} G) * f(x).$$

Remark: We can replace $\mathbb{R}^d$ by Ω open $\subset \mathbb{R}^d$.

Then $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

Proof: For every ball $B = B(0,R)$

$$\int_B |u| \leq \int_B \left(\int_{\mathbb{R}^d} |G(x-y)||f(y)|\, dy \right) dx$$

$$\leq \int_{\mathbb{R}^d} \left(\int_B |G(x-y)|\, dx \right) |f(y)|\, dy$$

If $|y| > R$, then $y \notin B(0,R) \Rightarrow$ by the mean-value theorem

$$\int_{B(0,R)} |G(x-y)|\,dx = |G(y)| \leq C\,\omega_d(y).$$

If $|y| \leq R$, then $|x-y| \leq R + |x|$

$$\Rightarrow \int_{B(0,R)} |G(x-y)|\,dx \leq \int_{|z|\leq 2R} |G(z)|\,dz \leq C_R \quad \text{as } G \in L^1_{loc}$$

Thus

$$\int_B |u| \leq C_R \int_{|y|>R} \omega_d(y)\,|f(y)|\,dy + C_R \int_{|y|\leq R} |f(y)|\,dy < \infty$$

$$\Rightarrow \quad u \in L^1_{loc}.$$

Now we prove $-\Delta u = f$ in $\mathcal{D}'(\mathbb{R}^d)$: $\forall \varphi \in C_c^\infty$

$$(-\Delta u)(\varphi) = \int_{\mathbb{R}^d} u\,(-\Delta\varphi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)\,f(y)\,(-\Delta\varphi(x))\,dx\,dy$$

$$= \int_{\mathbb{R}^d} \Big[\underbrace{G * (-\Delta\varphi)(y)}_{-\Delta(G*\varphi) = \varphi}\Big] f(y)\,dy$$

$$= \int_{\mathbb{R}^d} \varphi(y)\,f(y)\,dy$$

$$\Rightarrow \quad -\Delta u = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Finally, we check:

$$\partial_i u(x) = (\partial_i G * f)(x) = \int_{\mathbb{R}^d} \partial_i G(x-y) f(y)\, dy.$$

Note that $|\partial_i G(x)| \leq \dfrac{C}{|x|^{d-1}} \in L^1_{loc}(\mathbb{R}^d)$.

For all ball $B = B(0,R) \subset \mathbb{R}^d$,

$$(*) \int_B \int_{\mathbb{R}^d} |\partial_i G(x-y)|\, |f(y)|\, dy\, dx$$

$$\leq \int_{\mathbb{R}^d} \left(\int_{B(0,R)} \frac{C}{|x-y|^{d-1}} dx \right) |f(y)|\, dy$$

If $|y| > 2R \Rightarrow |x-y| \geq |y| - |x| \geq \frac{1}{2}|y|$ if $|x| \leq R$

$$\Rightarrow \int_{B(0,R)} \frac{1}{|x-y|^{d-1}} dx \leq \int_{B(0,R)} \frac{C}{|y|^{d-1}} \leq C_R\, \omega_d(y).$$

If $|y| \leq 2R \Rightarrow |x-y| \leq 3R$ if $|x| \leq R$

$$\Rightarrow \int_{B(0,R)} \frac{1}{|x-y|^{d-1}} dx \leq \int_{|z| \leq 3R} \frac{1}{|z|^{d-1}} dz \leq C_R$$

Thus:

$$(*) \leq C_R \int_{|y|>2R} \omega_d(y)\, |f(y)|\, dy + C_R \int_{|y| \leq 2R} |f(y)|\, dy < \infty$$

$$\Rightarrow \partial_i G * f \in L^1_{loc}(\mathbb{R}^d).$$

Next, for all $\varphi \in C_c^\infty$,

$$(\partial_i u)(\varphi) = -\int u\,(\partial_i \varphi) = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)\, f(y)\, (\partial_i \varphi)(x)\, dx\, dy$$

$$= -\int (G * \partial_i \varphi)\, f(y)\, dy$$

$$= -\int (\partial_i G * \varphi)\, f(y)\, dy \quad \leftarrow \text{one of exercise}$$

$$= + \int (\partial_i G * f)\, \varphi$$

$$\Rightarrow \partial_i u = \underbrace{\partial_i G}_{\in L^1_{loc}} * f \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

Here we used for $\varphi \in C_c^\infty$

$$\partial_i (G * \varphi) = (\partial_i G * \varphi).$$

<u>Regularity:</u> Given $-\Delta u = f$ in $\mathcal{D}'(\mathbb{R}^d)$, How much can we say about the smoothness of u from the information on f?

First consider the Laplace equation

$$\Delta u = 0 \quad \text{on } \mathcal{D}'(\Omega).$$

<u>Lemma:</u> (Weyl) Let Ω open in $\mathbb{R}^d$. Let $T \in \mathcal{D}'(\Omega)$ s.t. $\Delta T = 0$ in $\mathcal{D}'(\Omega)$. Then $T = \varphi \in C^\infty(\Omega)$ and it is harmonic.

<u>Proof.</u> $(\Omega = \mathbb{R}^d)$ Let $f \in C_c^\infty$. Then:

$$y \mapsto T(f_y) \in C^\infty \text{ and}$$

$$\Delta_y(T(f_y)) = T((\Delta f)_y) = (\Delta T)(f_y) = 0$$

$\Rightarrow$ $T(f_y)$ is harmonic

Let $g \in C_c^\infty$ be radial s.t. $\int g = 1$. By the mean-value theorem:

$$\int_{\mathbb{R}^d} T(f_y)\, g(y)\, dy = \int_{\mathbb{R}^d} T(f_0)\, g(y)\, dy = T(f)$$

By the convolution theorem

$$\int_{\mathbb{R}^d} T(f_y)\, g(y)\, dy = T(f * g) = T(g * f)$$
$$= \int_{\mathbb{R}^d} T(g_y)\, f(y)\, dy$$

Thus: $T(f) = \int T(g_y)\, f(y)\, dy$, $\forall f \in C_c^\infty(\mathbb{R}^d)$

$\Rightarrow$ $T = T(g_y) \in C^\infty$.

Exercise: If $f \in C(\mathbb{R}^d)$ is harmonic in $\mathbb{R}^d$ and $g \in C_c(\mathbb{R})$ is radial, then:
$$\int_{\mathbb{R}^d} fg = f(0) \int_{\mathbb{R}^d} g.$$

Now consider Poisson's equation
$$-\Delta u = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d)$$
where f satisfies $w_d f \in L^1(\mathbb{R}^d)$.

Remark: Any solution has the form
$$u = G * f + h$$
where $\Delta h = 0$. By Weyl's lemma $h \in C^\infty$
$\Rightarrow$ it suffices to consider $G*f$.

Remark: Moreover, the regularity is a "local question"

Eg. $f = f_1 + f_2 = \varphi f + (1-\varphi) f$, $\varphi \in C_c^\infty$
$$\Rightarrow G*f = G*f_1 + G*f_2.$$
If $\varphi = 1$ on a ball $B \Rightarrow f_2 = 0$ on $B \Rightarrow \Delta(G*f_2) = 0$ on $B \Rightarrow G*f_2 \in C^\infty$ on B $\Rightarrow$ the regularity of $G*f$ on B is determined by $G*f_1$ only.
Thus WLOG we may assume that f has compact supp.

<u>Theorem:</u> (Low regularity for Poisson's equation)

Let $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, be compactly supported. Then

(a) $d=1 \Rightarrow G * f \in C^1$

$d=2 \Rightarrow G * f \in L^q_{loc}$, $\forall q < \infty$

$d \geq 3 \Rightarrow G * f \in L^q_{loc}$, $\forall q < \dfrac{d}{d-2}$

(b) If $d/2 < p \leq d$ then $G * f \in C^{0,\alpha}$, $\forall \alpha < 2 - \frac{d}{p}$.

i.e. $|(G*f)(x) - (G*f)(y)| \leq C|x-y|^\alpha$, $\forall x, y \in \mathbb{R}^d$.

(c) If $p > d$, then $G * f \in C^{1,\alpha}$, $\forall \alpha < 1 - \frac{d}{p}$.

<u>Example:</u> Let $B = B(0, 1/2) \subset \mathbb{R}^3$ and

$$u(x) = \omega(r) = \ln(|\ln r|), \quad r = |x|.$$

Then:

$$f(x) := -\Delta u(x) = -\omega''(r) - \frac{2\omega'(r)}{r}, \quad \forall x \in B.$$

In this case $f \in L^{3/2}(B)$ but u is <u>not</u> continuous.

In comparison, theorem (b) says that if $f \in L^{3/2+\varepsilon}$ then u is Hölder continuous!

Proof: $(d \geq 3)$ (a) $\boxed{p = 1}$ Recall the proof of Young's inequality

$$|(G * f)(x)| \leq \int_{\mathbb{R}^d} |G(x-y)||f(y)|\, dy$$

$$\leq \left(\int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy \right)^{1/q} \left(\int_{\mathbb{R}^d} |f(y)|\, dy \right)^{1/q'}$$

$$\Rightarrow |(G * f)(x)|^q \leq C \int_{\mathbb{R}^d} |G(x-y)|^q |f(y)|\, dy \quad \left(\frac{1}{q} + \frac{1}{q'} = 1 \right)$$

$\Rightarrow \forall$ Ball $B = B(0,R) \subset \mathbb{R}^d$

$$\int_B |(G * f)(x)|^q dx \leq C \int_{\mathbb{R}^d} \left(\int_{B(0,R)} |G(x-y)|^q dx \right) |f(y)|\, dy \quad (*)$$

Note that

$$|G(x)|^q \sim \frac{1}{|x|^{(d-2)q}} \in L^1_{loc} \Leftrightarrow (d-2)q < d \Leftrightarrow q < \frac{d}{d-2}.$$

If $y \in \operatorname{supp} f \Rightarrow |y| \leq R_1 \Rightarrow$

$$\int_{B(0,R)} |G(x-y)|^q dx \leq \int_{|z| \leq R+R_1} |G(z)|^q \leq C_R$$

Thus $(*) \leq C_R \int_{\mathbb{R}^d} |f(y)|\, dy < \infty \Rightarrow G * f \in L^q_{loc}$

$\forall R \qquad \forall q < \frac{d}{d-2}.$

(b) $\boxed{d/2 < p \le d}$ By the triangle inequality

$$|(G*\rho)(x) - (G*\rho)(y)|$$

$$\le C \int_{\mathbb{R}^d} \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| |\rho(z)|\, dz$$

Note: $$\left| \frac{1}{|x|^{d-2}} - \frac{1}{|y|^{d-2}} \right| = \left|\left(\frac{1}{|x|} - \frac{1}{|y|}\right)\right| \left(\frac{1}{|x|^{d-3}} + \dots + \frac{1}{|y|^{d-3}} \right)$$

$$\le C \frac{\big||x|-|y|\big|}{|x|\cdot|y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right)$$

(for all $0<\alpha<1$) $$\le C \frac{|x-y|^{\alpha} \max(|x|,|y|)^{1-\alpha}}{|x|\,|y|} \max\left(\frac{1}{|x|^{d-3}}, \frac{1}{|y|^{d-3}} \right)$$

$$\le C \frac{|x-y|^{\alpha}}{\min(|x|,|y|)^{d-2+a}}$$

$$\le C |x-y|^{\alpha} \cdot \left(\frac{1}{|x|^{d-2+\alpha}} + \frac{1}{|y|^{d-2+\alpha}} \right)$$

$$\Rightarrow \left| \frac{1}{|x-z|^{d-2}} - \frac{1}{|y-z|^{d-2}} \right| \le C |x-y|^{\alpha} \left(\frac{1}{|x-z|^{d-2+\alpha}} + \frac{1}{|y-z|^{d-2+\alpha}} \right)$$

$$\Rightarrow |G*\rho(x) - G*\rho(y)| \le C\, |x-y|^{\alpha} \left(\sup_x \int_{\mathbb{R}^d} \frac{|\rho(z)|}{|x-z|^{d-2+\alpha}}\, dz \right)$$

It remains to prove that

$$\sup_x \left| \int_{\mathbb{R}^d} \frac{|f(z)|}{|x-z|^{d-2+\alpha}} \, dz \right| < \infty$$

when $f \in L^p(B)$ with $d/2 < p \leq d$ and $0 < \alpha < 2 - \frac{d}{p}$.

Here $\quad \operatorname{supp} f \subset B = B(0, R_1) \subset \mathbb{R}^d$.

.) If $|x| > 2R_1$, then $|x-z| > R_1$ if $z \in \operatorname{supp} f$ and

$$\int_{\mathbb{R}^d} \frac{|f(z)|}{|x-z|^{d-2+\alpha}} \, dz \leq \frac{1}{R_1^{d-2+\alpha}} \|f\|_{L^1}$$

.) If $|x| \leq 2R_1$, then $|x-z| \leq 3R_1$ if $z \in \operatorname{supp} f$ and by Hölder inequality ($1/p + 1/p' = 1$)

$$\int_{\mathbb{R}^d \cap B} \frac{|f(z)|}{|x-z|^{d-2+\alpha}} \, dz \leq \left(\int_B |f(z)|^p dz \right)^{1/p} \left(\int_B \frac{1}{|x-z|^{(d-2+\alpha)p'}} dz \right)^{1/p'}$$

$$\leq \|f\|_{L^p} \left(\int_{|z| \leq 3R_1} \frac{dz}{|z|^{(d-2+\alpha)p'}} \right)^{1/p'}$$

$$< \infty$$

if $\quad (d-2+\alpha)p' < d \iff d-2+\alpha < \frac{d}{p'} = d\left(1 - \frac{1}{p}\right)$

$$\iff \alpha < 2 - \frac{d}{p}.$$

c) $\boxed{p>d}$ Recall that

$$\partial_i (G * \rho) = \underbrace{(\partial_i G)}_{\in L^1_{loc}(\mathbb{R}^d)} * \rho \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

Here we need to prove that $\partial_i G * \rho \in C^{0,\alpha}(\mathbb{R}^d)$. Then $\rho \in C^{1,\alpha}$ due to the equivalence of classical and distributional derivatives.

Similarly to (b), by the triangle inequality

$$|(\partial_i G * \rho)(x) - (\partial_i G * \rho)(y)|$$

$$\leq \int_{\mathbb{R}^d} |\partial_i G(x-z) - \partial_i G(y-z)| \, |\rho(z)| \, dz$$

Note: $G(x) = \dfrac{1}{(d-2)d|B_1||x|^{d-2}} \Rightarrow \partial_i G(x) = \dfrac{-x_i}{d|B_1| \, |x|^d}$

The rest is left as an exercise. $\square$

Theorem (High regularity for Poisson's equation)

Let $f \in C^{k,\alpha}(\mathbb{R}^d)$ be compactly supported with $k \in \{0,1,2,\dots\}$ and $0<\alpha<1$.

Then: $G*f \in C^{k+2,\alpha}(\mathbb{R}^d)$.

Remark: $f \in C$ does not imply that $G*f \in C^2$.

We will discuss an example in Exercise section.

Proof: It suffices to consider $k=0$ and use induction in k as $D^\beta(G*f) = G*(D^\beta f)$.

Since $f \in C^{0,\alpha}$ & compactly supported $\Rightarrow f \in L^p$ for all $p \le \infty$. Hence by the previous theorem:

$$G*f \in C^1, \quad \partial_i(G*f) = (\partial_i G)*f \in C.$$

Now let us compute the second derivative

$$\partial_j \partial_i (G*f) = \partial_j (\partial_i G * f).$$

Take $\varphi \in C_c^\infty$, we have:

$$-\partial_j\partial_i(G*f)(\varphi) = \int_{\mathbb{R}^d} (\partial_i G*f)(x)\partial_j\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_i G(x-y)\, f(y)\, \partial_j\varphi(x)\,dx\,dy$$

$$= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} \partial_i G(x-y)\partial_j\varphi(x)\,dx\right)dy$$

To use the integration by part for dx, we need to isolate the singularity of

$$\partial_i \partial_j G(x) = \frac{1}{|B_1||x|^d}\left(\omega_i \omega_j - \frac{1}{d}\delta_{ij}\right), \quad \omega = \frac{x}{|x|}.$$

By Dominated convergence, we write:

$$\int_{\mathbb{R}^d} \partial_i G(x-y)\, \partial_j \varphi(x)\, dx = \lim_{\varepsilon \to 0^+} \int_{|x-y| \geq \varepsilon} \partial_i G(x-y)\, \partial_j \varphi(x)\, dx$$

By Gauss-Green theorem, $\forall \varepsilon > 0$:

$$\int_{|x-y| \geq \varepsilon} \partial_i G(x-y)\, \partial_j \varphi(x)\, dx$$

$$= -\int_{\partial B(y,\varepsilon)} \partial_i G(x-y)\, \varphi(x)\, \frac{(x-y)_j}{|x-y|}\, dS(x) - \int_{|x-y| \geq \varepsilon} \partial_i \partial_j G(x-y)\, \varphi(x)\, dx$$

The first term can be computed explicitly as $\varepsilon \to 0$:

$$-\int_{\partial B(y,\varepsilon)} \partial_i G(x-y)\, \varphi(x)\, \frac{(x-y)_j}{|x-y|}\, dS(x), \quad \partial_i G(x) = \frac{-x_i}{d|B_1||x|^d}$$

$$= +\varepsilon^{d-1} \int_{\partial B(0,1)} \partial_i G(\varepsilon\omega)\, \varphi(y+\varepsilon\omega)\, \omega_j\, d\omega, \quad \omega = \frac{x}{|x|}$$

$$= +\int_{\partial B(0,1)} \frac{1}{d|B_1|}\, \omega_i \omega_j\, \varphi(y+\varepsilon\omega)\, d\omega \xrightarrow[\varepsilon \to 0]{} \frac{\delta_{ij}}{d}\, \varphi(y)$$

For the second term, we split:

$$-\int_{|x-y|\geq \varepsilon} \partial_i \partial_j G(x-y)\, \varphi(x)\, dx = -\int_{|x-y|\geq 1} - \int_{1\geq |x-y|\geq \varepsilon}$$

$$= (\mathrm{I}) + (\mathrm{II})$$

Part (I) is nice as there is no regularity ($\partial_i \partial_j G(x)$ is smooth on $|x|>1$). For (II), we use the symmetry

$$\int_{\partial B(0,r)} \partial_i \partial_j G(z)\, d\sigma(z) = 0, \quad \forall r>0$$

$$\Rightarrow \int_{1\geq |x-y|\geq \varepsilon} \partial_i \partial_j G(x-y)\, dx = 0$$

$$\Rightarrow (\mathrm{II}) = -\int_{1\geq |x-y|\geq \varepsilon} \partial_i \partial_j G(x-y)\, \varphi(x)\, dx$$

$$= -\int_{1\geq |x-y|\geq \varepsilon} \partial_i \partial_j G(x-y) \big(\varphi(x)-\varphi(y)\big)\, dx$$

$$\to -\int_{1\geq |x-y|} \partial_i \partial_j G(x-y) \big(\varphi(x)-\varphi(y)\big)\, dx$$

as $\left|\partial_i \partial_j G(x-y)\big(\varphi(x)-\varphi(y)\big)\right| \leq \dfrac{C}{|x-y|^d} \cdot |x-y| \in L^1_{loc}(dx)$

In summary:

$$-\partial_i\partial_j(G*f)(\varphi) = \int_{\mathbb{R}^d} f(y)\left(\int_{\mathbb{R}^d} \partial_i G(x-y)\,\varphi(x)\,dx\right)dy$$

$$= \frac{1}{d}\delta_{ij}\int_{\mathbb{R}^d} f(y)\varphi(y)\,dy - \int_{\mathbb{R}^d} f(y)\left(\int_{|x-y|\geq 1} \partial_i\partial_j G(x-y)\varphi(x)\,dx\right)dy$$

$$- \int_{\mathbb{R}^d} f(y)\left(\int_{1\geq |x-y|} \partial_i\partial_j G(x-y)\,(\varphi(x)-\varphi(y))\,dx\right)dy$$

$$= \frac{1}{d}\delta_{ij}\int_{\mathbb{R}^d} f(x)\varphi(x)\,dx - \int_{\mathbb{R}^d}\varphi(x)\left(\int_{|x-y|\geq 1} f(y)\,\partial_i\partial_j G(x-y)\,dy\right)dx$$

$$- \int_{\mathbb{R}^d}\varphi(x)\left(\int_{1\geq |x-y|} \partial_i\partial_j G(x-y)\,(f(y)-f(x))\,dy\right)dx$$

$\forall \varphi \in C_c^\infty$

$$\Rightarrow \partial_i\partial_j G(x) = -\frac{1}{d}\delta_{ij} f(x) + \int_{|x-y|\geq 1} \partial_i\partial_j G(x-y)\,f(y)\,dy$$

$$+ \int_{1\geq |x-y|} \partial_i\partial_j G(x-y)\,(f(y)-f(x))\,dy$$

The first term is good as $f \in C^{0,\alpha}$. The second term is also good since $\partial_i\partial_j G(x)$ is smooth on $|x|>1$ and $f\in C^{0,\alpha}$. It remains to prove that

$$W_{ij}(x) = \int_{|x-y|\leq 1} \partial_i \partial_j G(x-y)\,(\rho(y)-\rho(x))\,dy$$

is Hölder continuous. Note that

$$\left| \partial_i \partial_j G(x-y)\left(\rho(y)-\rho(x)\right)\right| \leq \frac{C}{|x-y|^d}\,|x-y|^\alpha \in L^1_{loc}(dy)$$

and W_{ij} is well-defined. Let us rewrite:

$$W_{ij}(x) = \int_{|z|\leq 1} \partial_i \partial_j G(z)\left(\rho(x+z)-\rho(x)\right)dz$$

$$\Rightarrow W_{ij}(x) - W_{ij}(y) = \int_{|z|\leq 1} \partial_i \partial_j G(z)\left(\rho(x+z)-\rho(y+z)-\rho(x)+\rho(y)\right) dz$$

Combining

$$\left|\rho(x+z)-\rho(y+z)-\rho(x)+\rho(y)\right| \leq C \min\left(|x-z|^\alpha, |y|^\alpha\right).$$

and $|\partial_i \partial_j G(z)| \leq C|z|^{-d}$ we easily obtain

$$|W_{ij}(x) - W_{ij}(y)| \leq C\,|x-y|^{\alpha'} \text{ for any } \alpha' < \alpha.$$

However, to get $\alpha' = \alpha$, we need a precise computation. Let us split, for every $x \neq y$:

$$W_{ij}(x) - W_{ij}(y) = \int_{|z|\leq 1} \cdots \leq \int_{|z| < |x-y|} + \int_{|x-y|<|z|<1}$$

For the first domain:

$$\left| \int_{|z| \le |x-y|} \partial_{ij} G(x-y) \left(f(x+z) - f(y+z) - f(x) + f(y) \right) dz \right|$$

$$\le C \int_{|z| \le |x-y|} \frac{1}{|z|^d} |z|^\alpha \, dz \le C \, |x-y|^\alpha.$$

For the second domain, we use again the fact

$$\int_{\partial B(0,r)} \partial_{ij} G(z) \, dS(z) = 0, \ \forall r > 0 \Rightarrow \int_{|x-y| \le |z| \le 1} \partial_{ij} G(z) dz = 0.$$

Hence,

$$(*) \quad \int_{|x-y| < z < 1} \partial_i \partial_j G(z) \left(f(x+z) - f(y+z) - f(x) + f(y) \right) dz$$

$$= \int_{|x-y| < |z| < 1} \partial_i \partial_j G(z) \Big(f(x+z) - f(x) - \left(f(y+z) - f(x) \right) \Big) dz$$

$$= \int_A \partial_i \partial_j G(z-x) \left(f(z) - f(x) \right) dz$$

$$- \int_B \partial_i \partial_j G(z-y) \left(f(z) - f(x) \right) dz$$

where $A = \{ z : \ |x-y| < |z-x| < 1 \}$

$B = \{ z : \ |x-y| < |z-y| < 1 \}$.

We split $\int_A = \int_{A\cap B} + \int_{A\setminus B}$, $\int_B = \int_{A\cap B} + \int_{B\setminus A}$.

<u>On the common domain $A\cap B$:</u>

$$\left|\int_{A\cap B} \left(\partial_i\partial_j G(z-x) - \partial_i\partial_j G(z-y)\right)\left(f(z)-f(x)\right) dz\right|$$

$$\leq C \int_{A\cap B} |x-y| \left(\frac{1}{|z-x|^{d+1}} + \frac{1}{|z-y|^{d+1}}\right) |z-x|^{\alpha}\, dz$$

$$\leq C\,|x-y| \int_A \frac{1}{|z-x|^{d+1-\alpha}} dz + C\,|x-y| \int_B \frac{|z-x|^{\alpha}}{|z-y|^{d+1}} dz$$

We have:

$$|x-y| \int_A \frac{1}{|z-x|^{d+1-\alpha}} dz \leq |x-y| \int_{|x-y|<|z-x|} \frac{1}{|z-x|^{d+1-\alpha}} dz$$

$$\leq |x-y| \int_{|x-y|<|z|} \frac{1}{|z|^{d+1-\alpha}} \leq C|x-y|^{\alpha}$$

Moreover, using $|z-x|^{\alpha} \leq |z-y|^{\alpha} + |x-y|^{\alpha}$ we obtain

$$|x-y| \int_B \frac{|z-x|^{\alpha}}{|z-y|^{d+1}} dz \leq |x-y| \int_B \frac{1}{|z-y|^{d+1-\alpha}} dz + \underbrace{|x-y|^{1+\alpha} \int_B \frac{1}{|z-y|^{d+1}}}_{\leq C|x-y|^{\alpha}}$$

On $A \setminus B$: We split

$$A \setminus B = E_1 \cup E_2 \quad \text{where}$$

$$E_1 = \{ z \in A : \ |x-y| \geq |z-y| \}$$

$$E_2 = \{ z \in A : \ |z-y| \geq 1 \}$$

By the triangle inequality,

$$z \in E_1 \Rightarrow |z-x| \leq |z-y| + |x-y| \leq 2|x-y|$$

$$\Rightarrow \left| \int_{E_1} \cdots \right| \leq \int_{E_1} |\partial_{ij} G(z-x)| \, |f(z) - f(x)| \, dz$$

$$\leq \int_{|z-x| \leq 2|x-y|} \frac{C}{|z-x|^d} \cdot |z-x|^\alpha \, dz$$

$$\leq C \int_{|z| \leq 2|x-y|} \frac{1}{|z|^{d-\alpha}} \, dz \leq C |x-y|^\alpha.$$

Moreover, $z \in E_2 \Rightarrow |z-x| \geq |z-y| - |x-y| \geq 1 - |x-y|$

$$\Rightarrow \left| \int_{E_2} \cdots \right| \leq \int_{1-|x-y| \leq |z| \leq 1} \frac{1}{|z|^{d-\alpha}} \, dz \leq C |x-y|^\alpha.$$

Case $B \setminus A$ is similar! $\square$

Theorem: (Regularity on general domains)

Let Ω be open in $\mathbb{R}^d$. Let $u, f \in \mathcal{D}'(\Omega)$ s.t.

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega).$$

a) If $f \in L^1_{loc}(\Omega)$, then

$d=1 \Rightarrow u \in C^1(\Omega)$

$d=2 \Rightarrow u \in L^q_{loc}(\Omega)$ for all $q < \infty$

$d=3 \Rightarrow u \in L^q_{loc}(\Omega)$ for all $q < \frac{d}{d-2}$

b) If $f \in L^p_{loc}(\Omega)$ with $d \geq p > \frac{d}{2}$, then

$u \in C^{0,\alpha}_{loc}(\Omega)$, $\forall\, 0 < \alpha < 2 - \frac{d}{p}$, i.e.

$|u(x) - u(y)| \leq C_K |x-y|^\alpha$, $\forall\, x, y \in K$

($\forall K$ compact set $\subset \Omega$)

c) If $f \in L^p_{loc}(\Omega)$ with $p > d$, then

$u \in C^{1,\alpha}_{loc}(\Omega)$, $\forall\, 0 < \alpha < 1 - \frac{d}{p}$.

d) If $f \in C^{0,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$, then

$u \in C^{2,\alpha}_{loc}(\Omega)$.

e) If $f \in C^{m,\alpha}_{loc}(\Omega)$, then $u \in C^{m,\alpha}_{loc}(\Omega)$, $m \geq 0$.

<u>Proof:</u> Take a ball B s.t. $\overline{B} \subset \Omega$. Define

$$f_B(x) = \mathbb{1}_B(x) f(x) = \begin{cases} f(x) & \text{if } x \in B \\ 0 & \text{if } x \notin B. \end{cases}$$

Then $f \in L^1_{loc}(\Omega) \Rightarrow f_B \in L^1(\mathbb{R}^d)$ & compactly supp.

Hence, we know that $G * f_B \in L^1_{loc}(\mathbb{R}^d)$ and

$$-\Delta(G * f_B) = f_B \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

On the other hand, $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

Hence: $-\Delta(G * f_B) = -\Delta u$ in $\mathcal{D}'(B)$ since $\forall \varphi \in C_c^\infty(B) \subset C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^d)$ we have:

$$(-\Delta u)(\varphi) = \int_\Omega f \varphi = \int_{\mathbb{R}^d} f_B \varphi = (-\Delta(G * f_B))(\varphi).$$

Put differently, $-\Delta(u - G * f_B) = 0$ in $\mathcal{D}'(B)$, namely $u - G * f_B$ is harmonic in B. By Weyl lemma, $u - G * f_B \in C^\infty(B) \Rightarrow$ the smoothness of u in B is the same to the smoothness of $G * f_B$ (that was studied before).

<u>Note:</u> If $f \in C^{0,\alpha}$, we need to define $f_B \in C^{0,\alpha}$ (exercise)