Chapter 4: Existence for Poison's equation on domains

Let Ω open in $\mathbb{R}^d$. Consider Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}, \quad (f,g) \text{ given data}$$

So far we only discussed the uniqueness and regularity of solution u. In this chapter, we study the existence issue. We focus on 2 different situations:

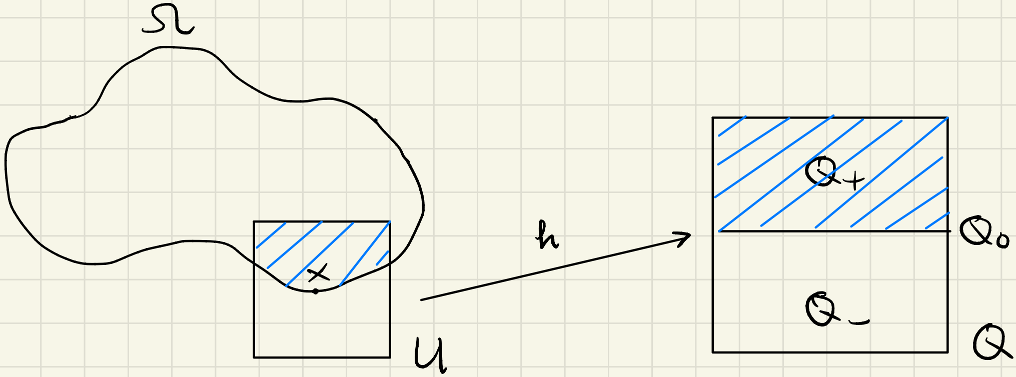
-) Classical solutions: Assuming $f \in C^2(\bar{\Omega})$ and $g \in C(\partial\Omega)$, we derive an explicit solution $u \in C^2(\bar{\Omega})$ using Green's functions.

-) Weak solutions: Assuming $f \in L^p(\Omega)$, $g \in L^p(\partial\Omega)$, we prove the existence of a weak solution using "Energy method".

In both cases, we will assume that $\partial\Omega$ is sufficiently smooth, such that the boundary constraint can be formulated properly.

Def: (C^1-domains) Let $\Omega \subset \mathbb{R}^d$ be open. We say that Ω is of class C^1 (or $\partial\Omega$ is C^1) if $\forall x \in \partial\Omega$, $\exists$ an open set $U \subset \mathbb{R}^d$ s.t. $x \in U$ and a bijective map $h : U \to Q$

$$\begin{cases} h \in C^1(\overline{U}),\ h^{-1} \in C^1(\overline{Q}) \quad (C^1\text{-diffeomorphism}) \\ h(U \cap \Omega) = Q_+,\ h(U \setminus \overline{\Omega}) = Q_-,\ h(U \cap \partial\Omega) = Q_0 \end{cases}$$



here

$$Q = \{ x = (x_1, \dots, x_{d-1}, x_d) = (x', x_d) \in \mathbb{R}^d : |x'| < 1 \text{ and } |x_d| < 1 \}$$

$$Q_0 = Q \cap (\mathbb{R}^{d-1} \times \{0\}) = \{ x = (x', 0) : |x'| < 1 \}$$

$$Q_+ = Q \cap \mathbb{R}^d_+ = \{ x = (x', x_d) : |x'| < 1,\ 0 < x_d < 1 \}$$

$$Q_- = Q \cap \mathbb{R}^d_- = \{ x = (x', x_d) : |x'| < 1,\ -1 < x_d < 0 \}$$

__More generally:__ $\partial\Omega$ is C^k if h, h^{-1} are C^k functions

Remark: We may replace Q by $B(0,1)$ as well, namely $\partial\Omega$ is C^1 if $\forall x_0 \in \partial\Omega$, $\exists$ open set $U \subset \mathbb{R}^d$, $x_0 \in U$ and a C^1 diffeomorphism $h: U \to B(0,1)$ s.t.

$h(U \cap \Omega) = B(0,1) \cap \{x_d > 0\}$, $h(U \cap \partial\Omega) = B(0,1) \cap \{x_d = 0\}$

Remark: Equivalent definition (Evans' book App. C)

$\partial\Omega$ is C^1 if $\forall x_0 \in \partial\Omega$, $\exists r > 0$ and a C^1 function $\gamma: \mathbb{R}^{d-1} \to \mathbb{R}$ s.t. (upon relabeling and reorienting the coordinates axes if necessary) we have:

$$\Omega \cap B(x_0, r) = \{x = (x', x_d) \in B(x_0, r) : x_d > \gamma(x')\}$$

More generally: $\partial\Omega$ is C^k if γ is a C^k function.

Proof of the equivalence of 2 definitions

Def 2 $\Rightarrow$ Def 1: Clearly we can take

$$h(x', x_d) = (x', x_d - \gamma(x')) \in C^1$$

$$h^{-1}(x', x_d) = (x', x_d + \gamma(x')) \in C^1$$

Def 1 $\Rightarrow$ Def 2: we need Inverse function theorem and Implicit function Theorem.

·) Writting $h = (h_1, \dots, h_d)$. Since h is C^1-diff, i.e. h is invertible near $x_0 \in \partial\Omega$, by the Inverse Function Theorem, the Jacobi matrix

$$J_h(x_0) = \left(\partial_j h_i(x_0)\right)_{1 \leq i,j \leq d}$$

is invertible. Consequently,

$$\nabla h_d(x_0) \neq \vec{0}.$$

By reordering and reorienting the axes, we may assume that

$$\partial_d h_d(x_0) > 0.$$

By the continuity, $\exists r_0 > 0$ s.t.

$$\partial_d h_d > 0 \quad \text{on} \quad B(x_0, r).$$

By the Implicit Function Theorem, $\exists C^1$ function $\gamma : \mathbb{R}^d \to \mathbb{R}$ s.t. in $B(x_0, r)$

$$(x', x_d) \in \partial\Omega \Leftrightarrow h_d(x', x_d) = 0 \Leftrightarrow x_d = \gamma(x').$$

Moreover, since $\partial_d h_d > 0$ on $B(x_0, r)$, the mapping $x_d \mapsto h_d(x', x_d)$ is strictly increasing. Therefore, in $B(0, r)$:

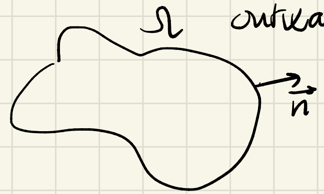
$$(x', x_d) \in \Omega \Leftrightarrow h_d(x', x_d) > 0 = h_d(x', \gamma(x')) \Leftrightarrow x_d > \gamma(x').$$

Integration by parts: If $\Omega \subset \mathbb{R}^d$ be open, bounded, $\partial\Omega \in C^1$. Then we have Gauss-Green formulas:

1) For all $u, v \in C^1(\bar{\Omega})$

$$\int_\Omega (\partial_i u) v = -\int_\Omega u(\partial_i v) + \int_{\partial\Omega} u v n_i \, dS$$

$\vec{n} = (n_1, \dots, n_d)$ unit outward normal vector



2) For all $u, v \in C^2(\bar{\Omega})$:

$$\int_\Omega u(-\Delta v) = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} u \frac{\partial v}{\partial \vec{n}} \, dS .$$

Formula of classical solutions via Green's function

Let $\Omega \subset \mathbb{R}^d$ open, bounded, $\partial\Omega$ is C^1. Assume

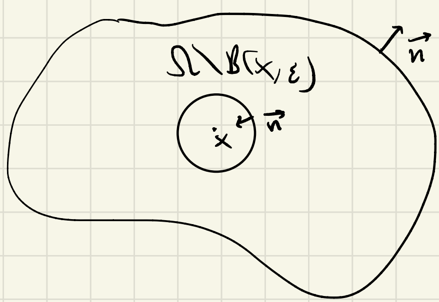
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \qquad \text{with } u \in C^2(\overline{\Omega}).$$

Let G be the fundamental solution of Laplace eq. in $\mathbb{R}^d$. For every $x \in \Omega$, let $\varepsilon > 0$ s.t. $B(x,\varepsilon) \subset \Omega$.

By integration by parts:

$$\int_{\Omega \setminus B(x,\varepsilon)} \Big(u(y) \Delta G(y-x) - G(y-x)\, \Delta u(y) \Big)\, dy$$

$$= \int_{\partial B(x,\varepsilon) \cup \partial\Omega} \Big(u(y) \frac{\partial G}{\partial \vec{n}}(y-x) - G(y-x) \frac{\partial u}{\partial \vec{n}}(y) \Big)\, dS(y)$$



We have as usual:

$$\frac{\partial G}{\partial \vec{n}} = \nabla G \cdot \vec{n} = \frac{(-y)}{d|B_1||y|} \cdot \left(-\frac{y}{|y|}\right) = \frac{1}{d|B_1|\varepsilon^{d-1}} \text{ on } \partial B(x,\varepsilon)$$

and hence

$$\int_{\partial B(x,\varepsilon)} u(y) \frac{\partial G}{\partial \vec{n}}(y-x)\, dS(y) = \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y) \xrightarrow[\varepsilon \to 0]{} u(x)$$

Moreover,

$$\left| \int_{\partial B(x,\varepsilon)} G(y-x) \frac{\partial u}{\partial \vec{n}}\, dS(y) \right| \leq C \varepsilon^{d-1} \sup_{\partial B(0,\varepsilon)} |G| \xrightarrow{\varepsilon \to 0} 0$$

Moreover, $\Delta G(y-x) = 0$ for $y \neq x$. Thus $\forall u \in C^2(\overline{\Omega})$ we have:

$$u(x) = \int_{\partial \Omega} \left(G(y-x) \frac{\partial u}{\partial \vec{n}}(y) - u(y) \frac{\partial G}{\partial \vec{n}}(y-x) \right) dS(y)$$
$$- \int_{\Omega} G(y-x)\, \Delta u(y)\, dy, \ \forall x \in \Omega.$$

Using $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$ we obtain

$$u(x) = \int_{\partial \Omega} \left(G(y-x) \frac{\partial u}{\partial \vec{n}}(y) - g(y) \frac{\partial G}{\partial \vec{n}}(y-x) \right) dS(y)$$
$$+ \int_{\Omega} G(x-y) f(y)\, dy.$$

Problem: $\frac{\partial G}{\partial \vec{n}}\Big|_{\partial\Omega}$ is unknown!

To resolve this problem, we need to introduce a "corrector" function $\phi_x = \phi_x(y)$ which solves

$$\begin{cases} \Delta \phi_x = 0 & \text{in } \Omega \\ \phi_x(y) = G(y-x) & \text{on } \partial\Omega \end{cases}, \quad x \in \Omega,\ y \in \bar{\Omega}$$

By integration by part again:

$$-\int_\Omega \phi_x(y)\,\Delta u(y)\,dy = \int_{\partial\Omega} \left(u(y)\frac{\partial \phi_x}{\partial \vec{n}}(y) - \phi_x(y)\frac{\partial u}{\partial \vec{n}}(y)\right) dS(y)$$

$$\Rightarrow \int_\Omega \phi_x(y) f(y)\,dy = \int_{\partial\Omega} \left(g(y)\frac{\partial \phi_x}{\partial \vec{n}}(y) - G(y-x)\frac{\partial u}{\partial \vec{n}}(y)\right) dS(y)$$

Theorem: Let $\Omega \subset \mathbb{R}^d$ be open, bounded, $\partial\Omega \in C^1$. Define Green function: $\tilde{G}(x,y) = G(y-x) - \phi_x(y)$, $x \in \Omega, y \in \bar{\Omega}$. If $u \in C^2(\bar{\Omega})$ solves Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Omega \end{cases}$$

then:

$$\boxed{u(x) = -\int_{\partial\Omega} g(y)\frac{\partial \tilde{G}}{\partial n}(x,y)\,dS(y) + \int_\Omega f(y)\tilde{G}(x,y)\,dy}$$

Here $\quad \dfrac{\partial \tilde{G}}{\partial \vec{n}}(x,y) = \nabla_y \tilde{G}(x,y) \cdot \vec{n}_y$ on $\partial\Omega$.

<u>Exercise:</u> The Green function satisfies the symmetry

$$\tilde{G}(x,y) = \tilde{G}(y,x) \;, \forall x,y \in \Omega, \; x \neq y.$$

In general, it is not easy to construct Green function (i.e. the corrector function) for a given domain Ω. However, it can be done explicitly in some specific cases.

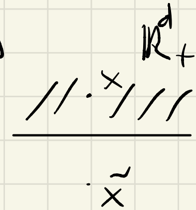
<u>Green function for a half-space:</u>

$$\Omega = \mathbb{R}^d_+ = \{ x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \; ; \; x_d > 0 \}$$

This domain is unbounded, so the previous computation does not apply directly. However, we can still try to compute the Green function, and then prove that the representation formula is valid.

<u>Def:</u> (Reflection) $x = (x', x_d) \longrightarrow \tilde{x} = (x', -x_d)$.

Then clearly $\Phi_x(y) = G(y - \tilde{x})$ satisfies



$$\begin{cases} \Delta \Phi_x = 0 & \text{in } \mathbb{R}^d_+ \\ \Phi_x = G(y - \tilde{x}) = G(y - x) & \text{on } \partial \mathbb{R}^d_+ \end{cases}$$

as $|y - \tilde{x}| = |y - x|$ if $y_d = 0$.

Define the Green function

$$\tilde{G}(x,y) = G(y-x) - \phi_x(y) = G(y-x) - G(y-\tilde{x}).$$

Then:
$$\frac{\partial \tilde{G}}{\partial y_d}(x,y) = \frac{\partial G}{\partial y_d}(y-x) - \frac{\partial G}{\partial y_d}(y-\tilde{x})$$

$$= -\frac{1}{d|B_1|}\left[\frac{y_d - x_d}{|y-x|^d} - \frac{y_d + x_d}{|y-x|^d}\right] = \frac{2x_d}{d|B_1|\,|x-y|^d}$$

Define Poisson's kernel for $y \in \partial \mathbb{R}^d_+$

$$K(x,y) = -\frac{\partial \tilde{G}}{\partial \vec{n}_y}(x,y) = +\frac{\partial \tilde{G}}{\partial y_d}(x,y) = \frac{2x_d}{d|B_1|\,|x-y|^d}.$$

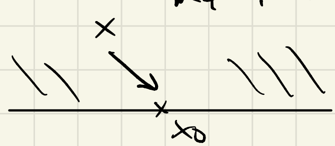
<u>Theorem</u> Assume $g \in C(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1})$ and let

$$u(x) = \int_{\partial \mathbb{R}^d_+} K(x,y)\, g(y)\, dy, \quad K(x,y) = \frac{+2x_d}{d|B_1|\,|x-y|^d}$$

Then $u \in C^\infty(\mathbb{R}^d_+) \cap L^\infty(\mathbb{R}^d_+)$. Moreover:

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^d_+ \\ \lim\limits_{\substack{x \to x_0 \\ x \in \mathbb{R}^d_+}} u(x) = g(x_0), & \forall x_0 \in \partial \mathbb{R}^d_+ \end{cases}$$

$\mathbb{R}^d_+ = \{x_d > 0\}$



Proof: Note that for each $y \in \partial \mathbb{R}^d_+$,

$$x \mapsto K(x,y) = \frac{+2x_d}{d|B_1|} \cdot \frac{1}{|x-y|^d}$$

is harmonic on $\mathbb{R}^d_+$. In fact, if $i \neq d$, then

$$\partial_{x_i} K(x,y) = -\frac{2x_d}{|B_1|} \cdot \frac{x_i - y_i}{|x-y|^{d+2}}$$

$$\partial^2_{x_i} K(x,y) = -\frac{2x_d}{|B_1|}\left(\frac{1}{|x-y|^{d+2}} - (d+2)\frac{(x_i-y_i)^2}{|x-y|^{d+4}}\right)$$

while

$$\partial_{x_d} K(x,y) = \frac{2}{d|B_1|}\frac{1}{|x-y|^d} - \frac{2x_d}{|B_1|} \cdot \frac{x_d - \cancel{y_d}}{|x-y|^{d+2}}$$

$$\partial^2_{x_d} K(x,y) = -\frac{2}{|B_1|} \cdot \frac{x_d - \cancel{y_d}}{|x-y|^{d+2}} - \frac{4x_d}{|B_1|\,|x-y|^{d+2}}$$

$$+ \frac{2x_d}{|B_1|} \cdot (d+2)\frac{(x_d - \cancel{y_d})^2}{|x-y|^{d+4}}$$

$$\Rightarrow \Delta_x K(x,y) = \sum_{i=1}^{d-1} \partial^2_{x_i} K(x,y) + \partial^2_{x_d} K(x,y)$$

$$= -\frac{2x_d}{|B_1|}\left(\frac{d-1}{|x-y|^{d+2}} - (d+2)\sum_{i=1}^{d-1}\frac{(x_i-y_i)^2}{|x-y|^{d+4}} + \frac{1+2}{|x-y|^{d+2}} - (d+2)\frac{(x_d-y_d)^2}{|x-y|^{d+4}}\right) = 0$$

(We may also argue that $\forall x \in \mathbb{R}^d_+$, $y \mapsto \tilde{G}(x,y)$ is harmonic in $\mathbb{R}^d_+ \setminus \{x\}$ $\Rightarrow$ by the symmetry, $\forall y \in \mathbb{R}^d_+$, $x \mapsto \tilde{G}(x,y)$ is harmonic in $\mathbb{R}^d_+ \setminus \{y\}$. Therefore $\forall y \in \mathbb{R}^d_+$, $x \mapsto -\frac{\partial \tilde{G}}{\partial y_d}(x,y) = K(x,y)$ is harmonic in $\mathbb{R}^d_+ \setminus \{y\}$. By a limiting argument, $\forall y \in \partial \mathbb{R}^d_+$, $x \mapsto K(x,y)$ is harmonic in $\mathbb{R}^d_+$.)

Moreover, it is straightforward to check that $\forall x \in \mathbb{R}^d_+$, (exercise) $\int_{\partial \mathbb{R}^d_+} K(x,y)\,dy = 1$, $\quad K(x,y) \geq 0$

Since $g \in L^\infty \Rightarrow u \in L^\infty$. Moreover,

$$\Delta_x u(x) = \int_{\partial \mathbb{R}^d_+} \Delta_x K(x,y)\, g(y)\,dy = 0 \,, \quad \forall x \in \mathbb{R}^d_+$$

$\Rightarrow u \in C^\infty(\mathbb{R}^d_+)$.

Finally, we check the boundary condition.

Let $x_0 \in \partial \mathbb{R}^d_+$. $\forall x \in \mathbb{R}^d_+$, we have:

$$|u(x) - g(x_0)| = \left| \int_{\partial \mathbb{R}^d_+} K(x,y)(g(y) - g(x_0))\, dy \right|$$

$$\leq \int_A + \int_B K(x,y)\, |g(y) - g(x_0)|\, dy$$

where

$$A = \{ y \in \partial \mathbb{R}^d_+ : |y - x_0| \leq L\, |x - x_0| \}$$

$$B = \{ y \in \partial \mathbb{R}^d_+ : |y - x_0| > L\, |x - x_0| \}$$

On A: Since g is continuous in $\partial \mathbb{R}^d_+ = \mathbb{R}^{d-1}$,

$$\int_A K(x,y)\, |g(y) - g(x_0)|\, dy$$

$$\leq \sup_{|y - x_0| \leq L |x - x_0|} |g(y) - g(x_0)| \underbrace{\int_{z \in \partial \mathbb{R}^d_+} K(x,z)\, dz}_{=1} \to 0 \quad \text{as } x \to x_0$$

On B: By the triangle inequality if $L \geq 2$

$$y \in B \Rightarrow |y - x| \geq |y - x_0| - |x - x_0| \geq |y - x_0| - \frac{1}{L} |y - x_0|$$

$$\geq \frac{1}{2} |y - x_0|$$

Hence $\int_B K(x,y)\, |g(y) - g(x_0)|\, dy$

$$\leq C \|g\|_{L^\infty} \int_B \frac{x_d}{|x - y|^d}\, dy \leq C \int_B \frac{x_d}{|y - x_0|^d}\, dy$$

$$= C x_d \int_{\substack{|y-x_0|^d > L|x-x_0| \\ y \in \partial \mathbb{R}^d_+ \sim \mathbb{R}^{d-1}}} \frac{1}{|y-x_0|^d} \, dy$$

$$= C x_d \int_{\substack{z \in \mathbb{R}^{d-1} \\ |z| > L|x-x_0|}} \frac{1}{|z|^d} \, dz$$

$$\leq \frac{C x_d}{L|x-x_0|} \leq \frac{C}{L} \to 0 \text{ as } L \to \infty$$

Here we use that $\forall x \in \mathbb{R}^d_+$, $\forall x_0 \in \partial \mathbb{R}^d_+$, then

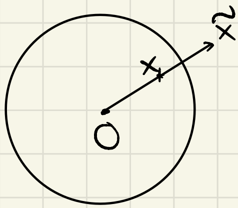
$$x_d = |x_d - (x_0)_d| \leq |x - x_0|. \qquad \square$$

Green's function for a ball $B(0,1)$

Def (Reflection / duality through unit sphere)

For every $x \in \mathbb{R}^d \setminus \{0\}$, define $\tilde{x} = \dfrac{x}{|x|^2}$

Corrector function:



Recall that for every $x \in B(0,1)$,

we want to find a function

$$\phi_x = \phi_x(y) \text{ s.t. } \begin{cases} \Delta_y \phi_x(y) = 0 & \text{in } B(0,1) \\ \phi_x(y) = G(y-x) & \text{on } \partial B(0,1). \end{cases}$$

Lemma: When $d \geq 3$, a solution for the corrector is

$$\phi_x(y) = G(|x|(y-\tilde{x})), \quad \forall x \in B(0,1), \ \forall y \in \overline{B(0,1)}.$$

Remark: At first sight ϕ_x is well-defined only when $x \neq 0$. However, when $x \to 0$, then

$$|x|(y-\tilde{x}) = |x|y - \frac{x}{|x|} \to \partial B(0,1) \quad \text{and hence}$$

$G(|x|(y-\tilde{x})) \to G(z)$ with $|z|=1$ (G is radial).

Proof: Note that $x \in B(0,1) \Rightarrow y \neq \tilde{x}, \ \forall y \in \overline{B(0,1)}$

$$y \mapsto \phi_x(y) = G(|x|(y-\tilde{x})) = \frac{1}{d(d-2)|B_1|(|x|\,|y-\tilde{x}|)^{d-2}}$$

is harmonic in $B(0,1)$. Moreover, if $y \in \partial B(0,1)$,

$$|x|\,|y-\tilde{x}| = |x|\,\left|y - \frac{x}{|x|^2}\right| = \left|\,|x|y - \frac{x}{|x|}\right|$$

$$= \sqrt{|x|^2|y|^2 - 2\,x\cdot y + 1} = \sqrt{|x|^2 - 2x\cdot y + |y|^2}$$

$$= \sqrt{|x-y|^2} = |x-y|$$

Hence $\phi_x(y) = G(y-x)$, $\forall\, y \in \partial B(0,1)$, $\forall\, x \in B(0,1)$

<u>Poisson's formula for a ball</u> Assume $u \in C^2(\overline{B(0,1)})$

solves $\begin{cases} \Delta u = 0 & \text{in } B(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$

Then recall

$$u(x) = -\int_{\partial B(0,1)} g(y)\, \frac{\partial \tilde{G}}{\partial \vec{n}}(x,y)\, dS(y)$$

where $\tilde{G}(x,y) = G(y-x) - \phi_x(y) = G(y-x) - G(|x|(y-\tilde{x}))$

Direct computation: $G(z) = \dfrac{1}{d(d-2)|B_1|\,|z|^{d-2}}$

$$\frac{\partial G}{\partial y_i}(y-x) = -\frac{(y_i - x_i)}{d|B_1|\,|y-x|^d}$$

$$\frac{\partial G}{\partial y_i}\left[G(|x|(y-\tilde{x}))\right] = -\frac{y_i - (\tilde{x})_i}{d|B_1|\,|x|^{d-2}\,|y-\tilde{x}|^d}$$

$$= -\frac{|x|^2 y_i - x_i}{d|B_1|\,|x-y|^d} \qquad (y \in \partial B(0,1))$$

$$\Rightarrow \frac{\partial \tilde{G}}{\partial y_i} = \frac{\partial G}{\partial y_i}(y-x) - \frac{\partial}{\partial y_i}\left[G\left(|x|\,(y-\tilde{x})\right)\right]$$

$$= \frac{y_i\,(|x|^2-1)}{d\,|B_1|\,|x-y|^d}$$

$$\Rightarrow \frac{\partial G}{\partial \vec{n}_y} = \sum_{i=1}^{d} \frac{\partial G}{\partial y_i} \cdot \frac{y_i}{|y|} = \frac{x^2-1}{d\,|B_1|\,|x-y|^d} \quad \text{on } \partial B(0,1).$$

The function

$$K(x,y) = -\frac{\partial G}{\partial \vec{n}}(x,y) = \frac{1-|x|^2}{d\,|B_1|\,|x-y|^d}, \quad x \in B(0,1),\ y \in \partial B(0,1)$$

is called the Poisson's kernel for ball $B(0,1)$

More generally, $\forall r > 0$, the function

$$K(x,y) = \frac{r^2-|x|^2}{r\,d\,|B_1|} \cdot \frac{1}{|x-y|^d}, \quad x \in B(0,r),\ y \in \partial B(0,r)$$

is called the Poisson's kernel for ball $B(0,r)$.

Here we derived the kernel for $d \geq 3$, but the same holds for $d=2$ as well.

<u>Theorem</u> (Poisson formula for balls)

Let $d \geq 2$. Let $r > 0$. Let $g \in C(\partial B(0,r))$ and

$$u(x) = \int_{\partial B(0,r)} K(x,y)\, g(y)\, dS(y)$$

where $K(x,y) = \dfrac{r^2 - x^2}{d|B_1|\, r} \cdot \dfrac{1}{|x-y|^d}$, $x \in B(0,r)$, $y \in \partial B(0,r)$

Then:

$$u \in C^\infty(B(0,r)) \quad \text{and} \quad \Delta u = 0 \quad \text{in } B(0,r).$$

Moreover, $\forall x_0 \in \partial B(0,r)$

$$\lim_{\substack{x \to x_0 \\ x \in B(0,r)}} u(x) = g(x_0).$$

<u>Proof:</u> Let $B = B(0,1)$, i.e. $r = 1$, for simplicity.

First, we prove that $\forall y \in \partial B$, the function

$x \mapsto K(x,y) = \dfrac{1-x^2}{d|B_1|} \cdot \dfrac{1}{|x-y|^d}$ is harmonic in B.

We have:

$$\frac{\partial}{\partial x_i} K(x,y) = \frac{-2x_i}{d|B_1|\,|x-y|^d} - \frac{(1-|x|^2)}{|B_1|} \cdot \frac{(x_i - y_i)}{|x-y|^{d+2}}$$

$$\frac{\partial}{\partial x_i} K(x,y) = \frac{-2x_i}{d|B_1|\,|x-y|^d} - \frac{(1-|x|^2)}{|B_1|} \cdot \frac{(x_i-y_i)}{|x-y|^{d+2}}$$

$$\frac{\partial^2}{\partial x_i^2} K(x,y) = -\frac{2}{d|B_1|\,|x-y|^d} + \frac{2x_i\cdot(x_i-y_i)}{|B_1|\,|x-y|^{d+2}} +$$

$$+\frac{2x_i}{|B_1|}\cdot\frac{(x_i-y_i)}{|x-y|^{d+2}} - \frac{(1-|x|^2)}{|B_1|}\cdot\frac{1}{|x-y|^{d+2}}$$

$$+\frac{1-|x|^2}{|B_1|}\cdot(d+2)\frac{(x_i-y_i)^2}{|x-y|^{d+4}}$$

$$\Rightarrow \Delta_x K(x,y) = -\frac{2}{|B_1|\,|x-y|^d} + \frac{4x\cdot(x-y)}{|B_1|\cdot|x-y|^{d+2}}$$

$$- d\,\frac{(1-|x|^2)}{|B_1|}\cdot\frac{1}{|x-y|^{d+2}} + \frac{1-|x|^2}{|B_1|}\cdot\frac{(d+2)}{|x-y|^{d+2}}$$

$$= \frac{2}{|x-y|^{d+2}}\Big[\underbrace{-|x-y|^2 + 2x\cdot(x-y) + 1-|x|^2}_{\substack{-|x|^2+2x\cdot y+|y|^2+2|x|^2-2x\cdot y\\ +1-|x|^2 = |y|^2-1=0}}\Big]$$

$$= 0$$

Moreover, $\forall x \in B$,

$$\int_{\partial B} K(x,y)\,dS(y) = 1 \qquad (\text{exercise})$$

Hence, $u \in L^\infty(B)$ since

$$|u(x)| = \left| \int_{\partial B} K(x,y) g(y) \, dS(y) \right| \leq \|g\|_{L^\infty}$$

and $\forall x \in B$ we have:

$$\Delta_x u(x) = \int_{\partial B} \Delta_x K(x,y) \, g(y) \, dS(y) = 0.$$

It remains to check the boundary condition.

Let $x_0 \in \partial B$ and $x \to x_0$, $x \in B$. Then:

$$|u(x) - g(x_0)| = \left| \int_{\partial B} K(x,y) \, (g(y) - g(x_0)) \, dS(y) \right|$$

$$\leq \int_{A_1} + \int_{A_2} K(x,y) \, |g(y) - g(x_0)| \, dS(y)$$

where $A_1 = \{ y \in \partial B : |y - x_0| \leq |x - x_0|^\alpha \}$ $\quad 0 < \alpha < \frac{1}{d}$

$A_2 = \{ y \in \partial B : |y - x_0| > |x - x_0|^\alpha \}$,

<u>On A_1:</u> $\displaystyle \int_{A_1} K(x,y) \, |g(y) - g(x_0)| \, dS(y)$

$$\leq \sup_{\substack{|z - x_0| \leq |x - x_0|^\alpha \\ z \in \partial B}} |g(z) - g(x_0)| \underbrace{\int_{\partial B} K(x,y) \, dS(y)}_{=1}$$

$\longrightarrow 0$ when $x \to x_0$, as $g \in C(\partial B)$

On A_2: We use $|g(y)-g(x_0)| \leq 2\|g\|_{L^\infty}$ and

on A_2: $|y-x_0| > |x-x_0|^\alpha$, $|x-x_0|$ small, $\alpha<1$

$$\Rightarrow |y-x| \geq |y-x_0| - |x-x_0| \geq \frac{1}{2}|x-x_0|^\alpha$$

$$\Rightarrow K(x,y) = \frac{1-|x|^2}{d|B_1|\,|x-y|^d} \leq C\,\frac{|x-x_0|}{|y-x_0|^d} \leq C|x-x_0|^{1-d\alpha}$$

Hence:

$$\int_{A_2} K(x,y)\,|g(y)-g(x_0)|\,dS(y)$$

$$\leq C \int_{A_2} |x-x_0|^{1-d\alpha}\,dS(y) \leq C|x-x_0|^{1-d\alpha} \to 0$$

as $x \to x_0$

if $1-d\alpha>0 \iff \alpha < 1/d$.

Thus $u(x)-g(x_0) \to 0$ as $x \to x_0$. $\square$

Energy method:

Let $\Omega \subset \mathbb{R}^d$ be open, bounded, C^1-boundary.

Consider:
$$\begin{cases} -\Delta u = \rho & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Integrating against a test function $\varphi \in C_c^\infty(\Omega)$ we obtain:

$$0 = \int_\Omega (-\Delta u - \rho)\varphi = \int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega \rho \varphi$$

Key observation: This is the derivative of the functional

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \rho u$$

Hence, if u is a minimizer, then it solves eq.

Interestingly, the reverse is also true, i.e.

$$\boxed{\text{solving equation} \Leftrightarrow \text{minimizes energy functional}}$$

The boundary condition does not appear in the energy functional, but it can be encoded into the set of admissible functions (candidates of solutions).

If we stay in classical solutions, then we have:

<u>Theorem</u> (Dirichlet principle) Let $\Omega \subset \mathbb{R}^d$ be open. Let $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$. Then TFAE:

(1) $u \in C^2(\overline{\Omega})$ solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Omega \end{cases}$

(2) u is a minimizer of the variational problem

$$E = \inf_{v \in A} \mathcal{E}(v) = \inf_{v \in A} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v \right)$$

where $A = \{ v \in C^2(\overline{\Omega}) \; ; \; v = g \text{ on } \partial\Omega \}$.

<u>Proof:</u> $\boxed{(1) \Rightarrow (2)}$ Consider real-valued functions.

$\forall v \in A$, $u - v = 0$ on $\partial\Omega$. Hence from $-\Delta u = f$ in Ω and integration by parts

$$0 = \int_\Omega (-\Delta u - f)(u - v) = \int_\Omega \nabla u \, \nabla(u - v) - \int_\Omega f (u - v)$$

$$= \mathcal{E}(u) - \mathcal{E}(v) + \underbrace{\frac{1}{2} \int_\Omega |\nabla u - \nabla v|^2}_{\geq 0}$$

$$\Rightarrow \quad \mathcal{E}(u) \leq \mathcal{E}(v), \quad \forall v \in A.$$

The complex-valued case: $0 = \int (-\Delta u - f)(\overline{u - v}) = \ldots$

$\boxed{(2) \Rightarrow (1)}$ Again consider the real-valued case.

For all $\varphi \in C_c^\infty(\Omega)$, we have $u + t\varphi \in A$. Hence:

$$\mathcal{E}(u) \le \mathcal{E}(u + t\varphi), \ \forall t \in \mathbb{R}$$

$$\Rightarrow 0 = \frac{d}{dt} \mathcal{E}(u+t\varphi)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{1}{2}\int_\Omega |\nabla u + t\nabla\varphi|^2 - \int_\Omega f(u+t\varphi)\right]\Big|_{t=0}$$

$$= \int_\Omega \nabla u \cdot \nabla\varphi - \int_\Omega f\varphi = \int_\Omega (-\Delta u - f)\varphi, \ \forall \varphi \in C_c^\infty$$

$$\Rightarrow \quad -\Delta u - f = 0 \text{ in } \Omega. \qquad \square$$

<u>Uniqueness</u>: the above proof also implies that there is at most one solution / one minimizer

<u>Central question</u>: Existence of a minimizer?

<u>Direct method of calculus of variations</u>:

- Prove that $E = \inf_{v \in A} \mathcal{E}(v)$ is finite.
- Take a minimizing sequence $\{v_n\} \subset A$ s.t. $\mathcal{E}(v_n) \to E$.
- Extract a subsequence $v_n \to u$ in A and $\liminf_{n\to\infty} \mathcal{E}(v_n) \ge \mathcal{E}(u)$.

Here the first two steps are easy. However, the last step is difficult since it requires a "compactness" which is not available for A. To resolve this, we need to enlarge $A \rightsquigarrow$ weak solutions.

Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be open. Recall that

$$W^{1,p}(\Omega) = \{ f \in L^p(\Omega) : \partial_i f \in L^p(\Omega), \forall i = 1, \dots, d \}$$

This is a Banach space with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{i=1}^{d} \|\partial_i f\|_{L^p(\Omega)}.$$

Note that $C_c^\infty(\Omega)$ is dense in $W^{1,p}_{loc}(\Omega)$, but in general $C_c^\infty(\Omega)$ may be not dense in $W^{1,p}(\Omega)$.

<u>Def</u>: $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1,p}(\Omega)}$.

This is a closed subspace of $W^{1,p}(\Omega)$, and hence it is always a Banach space.

Heuristically, functions in $W_0^{1,p}(\Omega)$ vanish on $\partial\Omega$, i.e. $u=0$ on $\partial\Omega$. This will be made precise via "Trace operator" later, but for now we have:

<u>Theorem:</u> Let $\Omega \subset \mathbb{R}^d$ be open, bounded, with C^1-boundary. Let $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$, $1 \leq p < \infty$. Then TFAE:

(i) $u=0$ on $\partial\Omega$.

(ii) $u \in W_0^{1,p}(\Omega)$.

<u>Proof:</u> $\boxed{(i) \Rightarrow (ii)}$ We need

<u>Lemma:</u> If $u \in W^{1,p}(\Omega)$ and u is compactly supported inside Ω, then $u \in W_0^{1,p}(\Omega)$.

<u>Proof:</u> Assume $\operatorname{supp} u = K \subset \Omega$. Take $\chi \in C_c^\infty(\Omega)$ s.t. $\chi = 1$ on K. Take $\{u_n\} \subset C^\infty(\Omega)$ s.t.

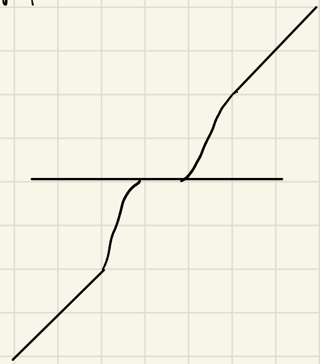
$$u_n \to u \quad \text{in} \quad W^{1,p}_{loc}(\Omega)$$

Then $\begin{cases} \chi u_n \to \chi u \quad \text{in} \quad W^{1,p}(\Omega) \quad (\text{check!}) \\ \chi u_n \in C_c^\infty(\Omega), \ \forall n \end{cases}$

Thus $u \in W_0^{1,p}(\Omega)$.

Now assume that $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Take $G \in C^1(\mathbb{R})$ s.t.

$$\begin{cases} |G(t)| \leq t \\ G(t) = 0 & \text{if } |t| \leq 1 \\ G(t) = t & \text{if } |t| \geq 2 \end{cases}$$



Then $|\nabla G| \leq C \Rightarrow$ by Chain-rule

$$\begin{cases} u_n = \frac{1}{n} G(nu) \in W^{1,p}(\Omega) \text{ and} \\ \nabla u_n = G'(nu) \nabla u \end{cases}$$

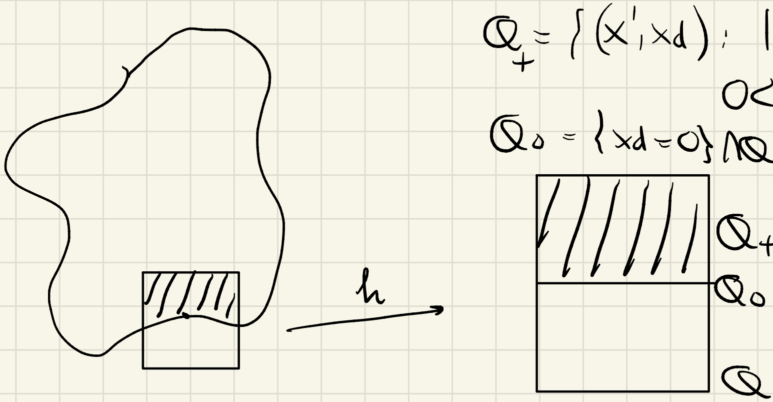
We can verify that $u_n \to u$ in $W^{1,p}(\Omega)$.

Moreover, $u_n \in W_0^{1,p}(\Omega)$ by the Lemma $\Rightarrow u \in W_0^{1,p}$.

$\boxed{(ii) \Rightarrow (i)}$

$Q_+ = \{(x', x_d) : |x'| < 1, 0 < x_d < 1\}$

$Q_0 = \{x_d = 0\} \cap Q$



Assume $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. We prove that $u = 0$ on $\partial\Omega$.

Special case $\Omega = Q_+$:

we prove that if $u \in W_0^{1,p}(Q_+) \cap C(\overline{Q_+})$, then $u = 0$ on Q_0. Let $\{u_n\} \subset C_c^\infty(Q_+)$, $u_n \to u$ in $W^{1,p}(Q_+)$. We have, for $(x', x_d) \in Q_+$:

$$u_n(x', x_d) - \underbrace{u_n(x', 0)}_{=0} = \int_0^{x_d} \partial_d u_n(x', t)\, dt$$

$$\Rightarrow \quad |u_n(x', x_d)| \le \int_0^{x_d} |\partial_d u_n(x', t)|\, dt$$

$$\Rightarrow \quad \frac{1}{\varepsilon} \int_0^\varepsilon \int_{|x'|<1} |u_n(x', x_d)|\, dx'\, dx_d \le \int_0^\varepsilon \int_{|x'|<1} |\partial_d u_n(x', x_d)|\, dx'\, dx_d$$

for all $n \in \mathbb{N}$ and $0 < \varepsilon < 1$.

Take $n \to \infty$ and use $u_n \to u$ in $W^{1,p}(\Omega)$

$$\frac{1}{\varepsilon} \int_0^\varepsilon \int_{|x'|<1} |u(x', x_d)|\, dx'\, dx_d \le \int_0^\varepsilon \int_{|x'|<1} |\partial_d u(x', x_d)|\, dx'\, dx_d$$

Take $\varepsilon \to 0$ and use $u \in C(\overline{\Omega})$

$$\int_{|x'|<1} |u(x', 0)|\, dx' \le 0$$

$\Rightarrow \quad u(x', 0) = 0$ for all $|x'| < 1$

i.e., $u = 0$ on Q_0.

General case: Ω open, bounded, C^1-boundary.

$\forall x \in \partial\Omega$, $\exists U_x$ open & h bijective $U_x \to Q$ s.t. $h(U_x \cap \Omega) = Q_+$ and $h, h^{-1} \in C^1$.

Then
$$\partial\Omega \subset \bigcup_{x\in\partial\Omega} U_x \quad \text{since } x \in U_x$$

Since $\partial\Omega$ is a compact set, $\exists$ finite family of open set $\{U_i\}_{i=1}^N$ among all $\{U_x\}_{x\in\partial\Omega}$ s.t.
$$\partial\Omega \subset \bigcup_{i=1}^N U_i$$

We can also find an open set $U_0 \subset \overline{U_0} \subset \Omega$ s.t.
$$\Omega \subset \bigcup_{i=0}^N U_i .$$

Lemma: (Partition of unity) $\exists \{\chi_i\}_{i=0}^{\infty} \subset C^\infty(\mathbb{R}^d)$ s.t.

1) $\chi_i \geq 0, \forall i$ and $\sum_{i=0}^N \chi_i = 1$ in $\mathbb{R}^d$
2) $\forall i = 1, 2, \dots, N$, $\text{supp}\, \chi_i \subset U_i$, i.e. $\chi_i \in C_c^\infty(U_i)$
3) $\text{Supp}\, \chi_0 \subset \mathbb{R}^d \setminus \partial\Omega$. In particular, $\chi_0|_\Omega \in C_c^\infty(\Omega)$. (exercise)

Now given $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, we prove $u = 0$ on $\partial\Omega$. We decompose

$$u = \sum_{i=0}^{N} \chi_i u.$$

Clearly $\chi_0 u = 0$ on $\partial\Omega$ since $\operatorname{supp} \chi_0 \subset \mathbb{R}^d \setminus \partial\Omega$.

For $1 \leq i \leq N$, since $\operatorname{supp} \chi_i \subset U_i$, it suffices to show that $\chi_i u = 0$ on $U_i \cap \Omega$.

Note that for every $1 \leq i \leq N$ (exercise)

$$\chi_i u \in W_0^{1,p}(U_i \cap \Omega) \cap C(\overline{U_i \cap \Omega}).$$

Then up to the C^1-diffeomorphism $h: U_i \to Q$ ($h: U_i \cap \Omega \to Q_+$) we get (exercise)

$$(\chi_i u) \circ h^{-1} \in W_0^{1,p}(Q_+) \cap C(\overline{Q_+}).$$

The special case $\rightarrow$ $(\chi_i u) \circ h^{-1} = 0$ on Q_0

$\Rightarrow$ $\chi_i u = 0$ on $h(Q_0) = U_i \cap \partial\Omega$. $\square$

Variational problem on $H_0^1(\Omega)$:

Let $\Omega \subset \mathbb{R}^d$ be open, bounded, C^1 boundary. Consider:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow \quad \int_\Omega \nabla u \nabla \varphi = \int f \varphi, \quad \forall \varphi \in C_c^\infty(\Omega)$$

If $f \in L^2$ and $\nabla u \in L^2$, then by a density argument this is equivalent to

$$\boxed{\int_\Omega \nabla u \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)}$$

<u>Theorem</u> (Dirichlet, Riemann, Poincaré, Hilbert)

Let $\Omega \subset \mathbb{R}^d$ be open, bounded, C^1-boundary. Let $f \in L^2(\Omega)$. Then there exist a unique solution $u \in H_0^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H_0^1(\Omega).$$

Moreover, u is the unique minimizer for

$$E = \inf_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega f v \right).$$

Nontrivial: why is $E > -\infty$?

Theorem (Poincare inequality) Let $\Omega \subset \mathbb{R}^d$ open, bounded. Then $\exists\, C = C(\Omega) > 0$ such that

$$C \int_\Omega |\nabla v|^2 \geq \int |v|^2, \quad \forall v \in H_0^1(\Omega).$$

Remark: $H^1(\Omega)$ is a Hilbert space with norm

$$\|v\|_{H^1(\Omega)} = \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)^{1/2}$$

Then $H_0^1(\Omega)$ is a closed subspace (by its definition) $\Rightarrow$ $H_0^1(\Omega)$ is a Hilbert space with H^1-norm. The Poincare inequality says that

$$\|v\|_{H^1} \geq \|\nabla v\|_{L^2} \geq \frac{1}{2C}\|v\|_{L^2} + \frac{1}{2}\|\nabla v\|_{L^2} \geq \frac{1}{C'}\|v\|_{H^1}$$

$\Rightarrow$ we can think of $H_0^1(\Omega)$ as a Hilbert space with the equivalent norm

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}.$$

Proof of Poincare inequality

By a density argument we need to prove $\exists C>0$:

$$C\int_\Omega |\nabla v|^2 \geq \int_\Omega |v|^2, \quad \forall v \in C_c^\infty(\Omega).$$

Assume by contradiction that it does not hold.

Then $\exists \{v_n\}_{n=1}^\infty \subset C_c^\infty(\Omega)$ s.t.

$$\int_\Omega |v_n|^2 = 1, \qquad \int_\Omega |\nabla v_n|^2 \to 0 \text{ as } n\to\infty.$$

We may extend v_n by 0 outside of Ω, and hence $\{v_n\} \subset C_c^\infty(\mathbb{R}^d)$. By the Fourier transform:

$$\int_\Omega |v_n|^2 = \int_{\mathbb{R}^d} |v_n|^2 = \int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = 1$$

$$\int_\Omega |\nabla v_n|^2 = \int_{\mathbb{R}^d} |\nabla v_n|^2 = \int_{\mathbb{R}^d} |2\pi k|^2 |\hat{v}_n(k)|^2 dk \to 0.$$

For $\varepsilon>0$ small we can split:

$$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk = \left(\int_{|k|\leq\varepsilon} + \int_{|k|>\varepsilon}\right) |\hat{v}_n(k)|^2 dk.$$

Clearly:
$$\int_{|k|>\varepsilon} |\hat{v}_n(k)|^2 dk \leq \int_{\mathbb{R}^d} \frac{|k|^2}{\varepsilon^2} |\hat{v}_n(k)|^2 dk \xrightarrow[n\to\infty]{} 0$$

On the other hand:

For $p, q, s \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{2p}+\frac{1}{s}=1$

by Hölder and Young inequalities:

$$\int_{|k|\leq \varepsilon} |\hat{v}_n(k)|^2 dk \leq \left(\int_{|k|\leq\varepsilon} 1\right)^{\frac{1}{q}} \left(\int_{|k|\leq\varepsilon} |\hat{v}_n(k)|^{2p}\right)^{\frac{1}{p}}$$

$$\leq C\,\varepsilon^{\frac{d}{q}} \|\hat{v}_n\|^2_{L^{2p}}$$

$$\leq C\,\varepsilon^{\frac{d}{q}} \|v_n\|^2_{L^s} \leq C\,\varepsilon^{\frac{d}{q}} \xrightarrow{\varepsilon\to 0} 0$$

Here in the last estimate we used $\|v_n\|_{L^s} \leq C$ since $s\in[1,2]$ and $\|v_n\|_{L^2}=1$ and $\operatorname{supp} v_n \subset \Omega$ bounded. Thus in summary:

$\int_{\mathbb{R}^d} |\hat{v}_n(k)|^2 dk \xrightarrow{n\to\infty} 0$ which contradicts to $\|v_n\|_{L^2}=1$. $\square$

<u>Exercise:</u> Let $\Omega\subset\mathbb{R}^d$ be open, bounded, C^1-boundary and $u\in W^{1,p}(\Omega)$. Then TFAE:

(a) $u\in W_0^{1,p}(\Omega)$

(b) $\bar{u}(x) = \begin{cases} u(x) & \text{if } x\in\Omega \\ 0 & \text{if } x\in\mathbb{R}^d\setminus\Omega \end{cases} \in W^{1,p}(\mathbb{R}^d)$.

Proof of the existence and uniqueness of

$$E = \inf_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega \rho v \right)$$

Step 1: Why is $E > -\infty$?

By Hölder and Poincare inequality, $\forall v \in H_0^1(\Omega)$:

$$\mathcal{E}(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega \rho v$$

$$\geq \frac{1}{4} \int_\Omega |\nabla v|^2 + \frac{1}{4C} \|v\|_{L^2}^2 - \|\rho\|_{L^2} \|v\|_{L^2}$$

$$\geq \frac{1}{4} \int_\Omega |\nabla v|^2 - C \|\rho\|_{L^2}^2 \geq -C \|\rho\|_{L^2}^2$$

Thus $E \geq -C \|\rho\|_{L^2}^2 > -\infty$.

Step 2: Let $\{v_n\} \subset H_0^1(\Omega)$ be a minimizing sequence. Then $\mathcal{E}(v_n) \to E \Rightarrow \|\nabla v_n\|_{L^2}^2$ is bounded. Thus $\{v_n\}$ is bounded in the Hilbert space $H_0^1(\Omega)$ (with norm $\|v\|_{H_0^1} = \|\nabla v\|_{L^2}$).

By the Banach-Alaoglu theorem, up to a subsequence, $v_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$.

Recall:

<u>Theorem</u> (Banach - Alaoglu) If H is a Hilbert space, and $\{v_n\}_{n=1}^{\infty}$ is bounded in H, then there exists a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ s.t. $v_{n_k} \to v_0$ weakly in H, namely $\langle v_{n_k}, \varphi \rangle \to \langle v_0, \varphi \rangle$, $\forall \varphi \in H$.

<u>Remark:</u> 1) By Riesz's representation theorem, the weak convergence is equivalent to

$$F(v_{n_k}) \to F(v_0), \ \forall F \in H^* = \underbrace{\mathcal{L}(H, \mathbb{R})}_{\text{space of linear continuous functions } H \to \mathbb{R} \text{ (or } \mathbb{C})}.$$

2) From $v_{n_k} \to v_0$ we have Fatou's lemma:

$$\liminf_{k \to \infty} \|v_{n_k}\| \geq \|v_0\|$$

Since $\|v_0\|^2 = \langle v_0, v_0 \rangle = \liminf_{k \to \infty} \underbrace{\langle v_0, v_{n_k} \rangle}_{\leq \|v_0\| \|v_{n_k}\|} \leq \|v_0\| \liminf_{k \to \infty} \|v_{n_k}\|$.

<u>Exercise:</u> If $v_n \rightharpoonup v_0$ in a Hilbert space, then:

$$v_n \to v_0 \text{ strongly} \iff \|v_n\| \to \|v_0\|.$$

Now we go back to Step 2 in the existence of minimizer for E. From $v_n \rightharpoonup u$ weakly in H_0^1, we obtain by Fatou's lemma:

$$\liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 = \liminf_{n\to\infty} \|v_n\|^2_{H_0^1(\Omega)} \geq \|u\|_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2$$

Moreover, consider $F: H_0^1(\Omega) \to \mathbb{R}$

$$F(\varphi) = \int_\Omega \rho\varphi$$

Then F is linear and continuous since

$$|F(\varphi)| \leq \left|\int_\Omega \rho\varphi\right| \leq \|\rho\|_{L^2}\|\varphi\|_{L^2} \leq C\|\rho\|_{L^2}\|\varphi\|_{H_0^1(\Omega)}$$

by Poincare inequality. Hence

$$v_n \rightharpoonup u \Rightarrow F(v_n) \to F(u) \text{ , i.e. } \int_\Omega \rho v_n \to \int_\Omega \rho u .$$

In summary:

$$\liminf_{n\to\infty} \mathcal{E}(v_n) \geq \mathcal{E}(u) \Rightarrow E \geq \mathcal{E}(u)$$

Thus u is a minimizer.

<u>Uniqueness:</u> If u_1, u_2 are two minimizer, then

$$0 \geq 4\left(\mathcal{E}(u_1) + \mathcal{E}(u_2) - 2\,\mathcal{E}\left(\frac{u_1+u_2}{2}\right)\right)$$

$$= \int_\Omega \left(2|\nabla u_1|^2 + 2|\nabla u_2|^2 - |\nabla(u_1+u_2)|^2\right) = \int_\Omega |\nabla(u_1-u_2)|^2 .$$

Moreover, $u_1, u_2 \in H_0^1(\Omega) \Rightarrow u_1 - u_2 \in H_0^1(\Omega) \Rightarrow$

$$0 \geq \int_\Omega |\nabla(u_1-u_2)|^2 \geq \frac{1}{C}\int_\Omega |u_1-u_2|^2 \Longrightarrow u_1 = u_2 . \ \square$$

<u>Exercise</u>: Prove that $u \in H_0^1(\Omega)$ is a minimizer for E if and only if $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi$, $\forall \varphi \in H_0^1(\Omega)$.

<u>Remark</u>: We can also argue by using Riesz's representation theorem as follows: from

$$\varphi \mapsto F(\varphi) = \int_\Omega f\varphi \quad \text{is linear, continuous from } H_0^1(\Omega) \to \mathbb{R}$$

we know that $\exists!\, u \in H_0^1(\Omega)$ s.t.

$$\langle u, \varphi \rangle_{H_0^1} = F(\varphi), \quad \forall \varphi \in H_0^1(\Omega)$$

$$\Longleftrightarrow \quad \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi, \quad \forall \varphi \in H_0^1(\Omega).$$

Theory of trace:

Theorem: Let $\Omega \subset \mathbb{R}^d$ be open, bounded, C^1-boundary.
Then $\exists!$ a linear, bounded operator

$$T : H^1(\Omega) \to L^2(\partial\Omega)$$

such that $\forall u \in H^1(\Omega) \cap C(\bar{\Omega})$ we have:

$$Tu = u_{|\partial\Omega} \text{ in the usual restriction sense.}$$

Remark. The boundedness of T means

$$\|Tu\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}, \ \forall u \in H^1(\Omega).$$

Thus the key point is the inequality

$$\|u_{|\partial\Omega}\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}, \ \forall u \in H^1(\Omega) \cap C(\bar{\Omega}).$$

Let us first consider the simple case.

Lemma (Trace inequality for $\mathbb{R}^d_+$) We have:

$$\|u_{|\partial\mathbb{R}^d_+}\|_{L^2(\partial\mathbb{R}^d_+)} \leq \|u\|_{H^1(\mathbb{R}^d_+)}, \ \forall u \in C^1_c(\mathbb{R}^d).$$

Recall:
$$\mathbb{R}^d_+ = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : \ x_d > 0\}$$
$$\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \{0\} \simeq \mathbb{R}^{d-1}.$$

Proof of the lemma: (for real-valued functions)

For all $u \in C_c^1(\mathbb{R}^d)$ and $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$

$$|u(x',0)|^2 = -\int_0^\infty \partial_d |u(x',x_d)| \, dx_d$$

$$= -\int_0^\infty 2\, u(x',x_d)\, \partial_d u(x',x_d)\, dx_d$$

$$\leq \int_0^\infty \left(|u(x',x_d)|^2 + |\partial_d u(x',x_d)|^2 \right) dx_d$$

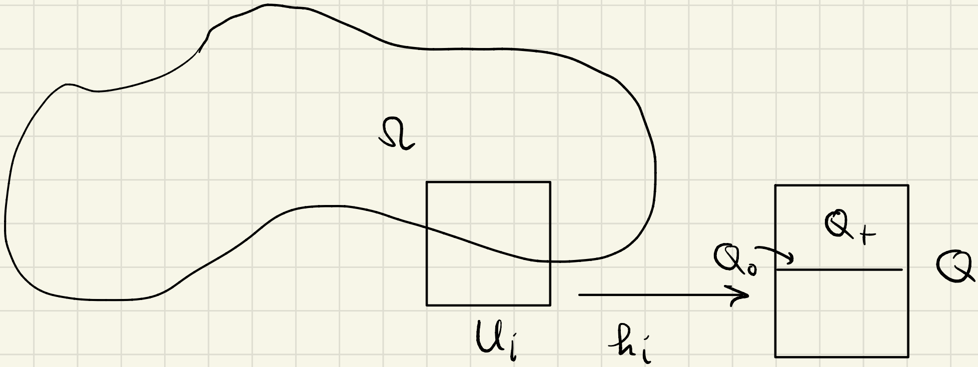
$$\Rightarrow \int_{\mathbb{R}^{d-1}} |u(x',0)|^2 dx' \leq \int_{\mathbb{R}^{d-1}} \left(\int_0^\infty (\cdots) dx_d \right) dx'$$

$$= \int_{\mathbb{R}^d_+} \left(|u|^2 + |\partial_d u|^2 \right) dx \leq \|u\|_{H^1(\mathbb{R}^d_+)} \quad \square$$

Lemma: (Trace inequality for Ω) let $\Omega \subset \mathbb{R}^d$ be open, bounded, C^1-boundary. Then: $\exists\, C = C_\Omega > 0$

$$\| u_{|\partial\Omega} \|_{L^2(\partial\Omega)} \leq C \, \|u\|_{H^1(\Omega)}, \quad \forall u \in C_c^1(\mathbb{R}^d).$$

Proof: We use a partition of unity to go back to the previous case.



Since $\partial\Omega$ is C^1 and compact, $\exists$ finite open sets $\{U_i\}_{i=1}^N \subset \mathbb{R}^d$ s.t.

$$\partial\Omega \subset \bigcup_{i=1}^N U_i$$

and $\forall i = 1, 2, \dots, N$, $\exists$ a C^1-diffeomorphism $h_i : U_i \to Q$ s.t. $h_i(U_i) = Q$, $h_i(U_i \cap \Omega) = Q_+$, $h_i(U_i \cap \partial\Omega) = Q_0$.

Recall: $Q = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^d : |x'| < 1, |x_d| < 1\}$

$$Q_+ = Q \cap \{x_d > 0\}, \quad Q_0 = Q \cap \{x_d = 0\}.$$

Partition of unity:

$\exists \{\theta_i\}_{i=0}^N \subset C^\infty(\mathbb{R}^d)$ s.t.

1) $\sum_{i=0}^N \theta_i = 1$ and $1 \geq \theta_i \geq 0 \quad \forall i = 0, \dots, N.$

2) $\forall i = 1, \dots, N$, $\theta_i \in C_c^\infty(U_i)$.

3) $\operatorname{supp} \theta_0 \subset \mathbb{R}^d \setminus \partial\Omega$.

For $u \in C_c^1(\mathbb{R}^d)$ we decompose:

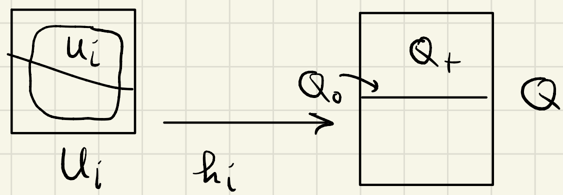
$$u = \sum_{i=0}^{N} \theta_i u = \sum_{i=0}^{N} u_i, \quad u_i = \theta_i u \in C_c^1(\mathbb{R}^d).$$

In particular:

$$u|_{\partial\Omega} = \sum_{i=1}^{N} u_i|_{\partial\Omega}.$$

For each $i = 1, 2, \ldots, N$, define $v_i : Q \to \mathbb{R}$ by

$$v_i(y) = u_i(h_i^{-1}(y))$$



Since $u_i \in C_c^1(U_i)$ and h_i is C^1-diff

$\Rightarrow v_i \in C_c^1(Q)$

We may extend v_i by 0 outside Q

$\Rightarrow v_i \in C_c^1(\mathbb{R}^d)$.

Then by the trace inequality for $\mathbb{R}^d_+$, we have

$$\|v_i\|_{L^2(\partial\mathbb{R}^d_+)} \leq \|v_i\|_{H^1(\mathbb{R}^d_+)}$$

$$\|v_i\|_{L^2(\partial\mathbb{R}^d_+)} = \|v_i\|_{L^2(Q_0)} \geq \|u_i\|_{L^2(U_i \cap \partial\Omega)} = \|u_i\|_{L^2(\partial\Omega)}$$

$$\|v_i\|_{H^1(\mathbb{R}^d_+)} = \|v_i\|_{H^1(Q_+)} \leq C\|u_i\|_{H^1(U_i \cap \Omega)} \leq C\|u\|_{H^1(\Omega)}$$

In summary:

$$\|u\|_{L^2(\partial\Omega)} = \Big\| \sum_{i=1}^{N} u_i \Big\|_{L^2(\partial\Omega)} \leq \sum_{i=1}^{N} \|u_i\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad \square$$

To conclude, we need a density result.

<u>Theorem</u> (Density) Let $\Omega \subset \mathbb{R}^d$ be open, bounded, with C^1-boundary. Then $C_c^\infty(\mathbb{R}^d)_{|\Omega}$ is dense in $H^1(\Omega)$, i.e. $\forall u \in H^1(\Omega)$, $\exists \{u_n\} \subset C_c^\infty(\mathbb{R}^d)$ s.t.

$$u_{n|\Omega} \to u \quad \text{in} \quad H^1(\Omega).$$

Moreover, if $u \in C(\overline{\Omega})$, then $u_n \to u$ in $C(\overline{\Omega})$.

This density theorem is a consequence of:

<u>Theorem</u> (Extension) If $\Omega \subset \mathbb{R}^d$ is open, bounded, with C^1 boundary, then $\exists$ an extension

$B: H^1(\Omega) \to H^1(\mathbb{R}^d)$ linear bounded, i.e.

$$Bu_{|\Omega} = u, \quad \|Bu\|_{H^1(\mathbb{R}^d)} \leq C \|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega)$$

Moreover, we can construct B such that:

1) Bu is supported in any given $\tilde{\Omega} \supset\supset \Omega$.
2) $\|Bu\|_{L^2} \leq C \|u\|_{L^2}$.
3) If $u \in C(\overline{\Omega})$, then $Bu \in C_c(\mathbb{R}^d)$.

Proof of Density theorem from Extension theorem

Let $u \in H^1(\Omega)$. Then $\exists Bu \in H^1(\mathbb{R}^d)$, compactly supported, such that $Bu|_\Omega = u$.

Take $g \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} g = 1$, $g_n(x) = n^d g(nx)$.

Then $u_n := g_n * Bu \in C_c^\infty(\mathbb{R}^d) \longrightarrow Bu$ in $H^1(\mathbb{R}^d)$

$\Rightarrow$ $u_n|_\Omega \to u$ in $H^1(\Omega)$.

Moreover, if $u \in C(\overline{\Omega})$, then $Bu \in C_c(\mathbb{R}^d)$, and hence $u_n = g_n * Bu \to Bu$ in $C(\mathbb{R}^d)$

$\Rightarrow$ $u_n|_{\overline{\Omega}} \to (Bu)|_{\overline{\Omega}} = u$ in $C(\overline{\Omega})$. $\square$

Concluding the Trace theorem:

Define $T: C^1(\overline{\Omega}) \to L^2(\partial\Omega)$ by $Tu = u|_{\partial\Omega}$.

By the trace inequality and the fact that $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, $\exists!$ extension

$$T: H^1(\Omega) \to L^2(\partial\Omega) \quad \text{linear bounded.}$$

Moreover, if $u \in H^1(\Omega) \cap C(\overline{\Omega})$, then $\exists \{u_n\} \subset C_c^\infty(\mathbb{R}^d)$ s.t. $u_n \to u$ in $H^1(\Omega) \cap C(\overline{\Omega})$, and hence

$$\left.\begin{array}{l} T(u_n) \longrightarrow T(u) \\ \| \\ u_n|_{\partial\Omega} \longrightarrow u|_{\partial\Omega} \end{array}\right\} \Rightarrow T(u) = u|_{\partial\Omega}. \quad \square$$

It remains to prove the extension theorem.

<u>lemma</u> (Extension for Q)

Let $u \in H^1(Q_+)$ and define $Bu : Q \to \mathbb{R}$ by

$$Bu(x) = \begin{cases} u(x) & \text{if } x \in Q_+ \\ u(x', -x_d) & \text{if } x = (x', x_d) \in Q_- \end{cases}$$

Then $Bu \in H^1(Q)$ and

$$\|Bu\|^2_{L^2(Q)} = 2\|u\|^2_{L^2(Q_+)}, \quad \|\nabla(Bu)\|^2_{L^2(Q)} = 2\|\nabla u\|^2_{L^2(Q_+)}$$

<u>Proof:</u> It is clear that $(x = (x', x_d))$

$$\int_Q |Bu|^2 = \int_{Q_+} |Bu|^2 + \int_{Q_-} |Bu|^2$$

$$= \int_{Q_+} |u|^2 + \int_{Q_-} |u(x', -x_d)|^2 \, dx$$

$$= \int_{Q_+} |u|^2 + \int_{Q_+} |u(x', x_d)|^2 \, dx = 2\int_{Q_+} |u|^2.$$

For the derivative, for all $i = 1, 2, \dots, d-1$ we have

$$\partial_i (Bu)(x) = \begin{cases} \partial_i u(x) & \text{if } x \in Q_+ \\ \partial_i u(x', -x_d) & \text{if } x = (x', x_d) \in Q_- \end{cases}$$

The difficult part is in $\partial_d (Bu)$. We claim:

$$\partial_d (Bu)(x) = g(x) := \begin{cases} \partial_d u(x) & \text{if } x \in Q_+ \\ -(\partial_d u)(x', -x_d) & \text{if } x = (x', x_d) \in Q_- \end{cases}$$

To see that, take $\varphi \in C_c^\infty(Q)$ and compute:

$$\int_Q (Bu)(\partial_d \varphi) = \int_{Q_+} u \, \partial_d \varphi + \int_{Q_-} u(x', -x_d)(\partial_d \varphi)(x', x_d) \, dx$$

$$= \int_{Q_+} u \, \partial_d u + \int_{Q_+} u(x', x_d)(\partial_d \varphi)(x', -x_d) \, dx$$

$$= \int_{Q_+} u \, \partial_d \varphi - \int_{Q_+} u(x', x_d) \, \partial_d [\varphi(x', -x_d)] \, dx$$

$$= \int_{Q_+} u \, \partial_d \tilde{\varphi}$$

where $\tilde{\varphi}(x', x_d) = \varphi(x', x_d) - \varphi(x', -x_d)$

Claim:
$$\int_{Q_+} u \, \partial_d \tilde{\varphi} = -\int_{Q_+} (\partial_d u) \, \tilde{\varphi}.$$

Using this claim, we conclude that

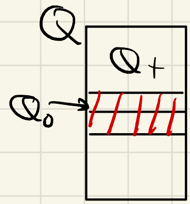
$$\int_Q (Bu)(\partial_d \varphi) = \int_Q u \, \partial_d \tilde{\varphi} = -\int_{Q_+} (\partial_d u) \, \tilde{\varphi}$$

$$= -\int_Q g \varphi, \quad \forall \varphi \in C_c^\infty(Q) \Rightarrow \partial_d (Bu) = g.$$

Finally we prove the claim: $\forall \varphi \in C_c^\infty(Q)$:

$$\int_{Q_+} u\, \partial \tilde{\varphi} = -\int_{Q_+} (\partial_d u)\, \tilde{\varphi}, \quad \tilde{\varphi}(x) = \varphi(x) - \varphi(x', -x_d)$$

If $\tilde{\varphi}$ were in $C_c^\infty(Q_+)$, then it would be obvious.
However, $\tilde{\varphi}$ is not necessarily in $C_c^\infty(Q_+)$, so we need to be careful.



$\eta_\varepsilon(x_d) = 0$ if $|x_d| \le \varepsilon$

Take $\eta_\varepsilon(x_d) = \eta_0\left(\frac{x_d}{\varepsilon}\right)$

where $\eta_0 \in C^\infty$, $\eta_0(x_d) = 1$ if $|x_d| \geq 2$

$\eta_0(x_d) = 0$ if $|x_d| \le 1$

Since $\eta_\varepsilon(x_d) = 0$ if $|x_d| \le \varepsilon$ and $\tilde{\varphi} \in C_c^\infty(Q)$ we find that $\eta_\varepsilon \tilde{\varphi} \in C_c^\infty(Q_+)$. Hence:

$$\int_{Q_+} u\, \partial(\eta_\varepsilon \tilde{\varphi}) = -\int_{Q_+} (\partial_d u)(\eta_\varepsilon \tilde{\varphi}).$$

Since $\eta_\varepsilon(x_d) \to 1$ as $\varepsilon \to 0$, $\forall x_d > 0$

$$\Rightarrow \int_{Q_+} (\partial_d u)(\eta_\varepsilon \tilde{\varphi}) \to \int_{Q_+} (\partial_d u)\, \tilde{\varphi}$$

by Dominated Convergence (here $(\partial_d u)\, \tilde{\varphi} \in L^1(Q)$)

Next, consider:

$$\int_{Q_+} u\, \partial_d (\eta_\varepsilon \tilde{\varphi}) = \int_{Q_+} u (\partial_d \eta_\varepsilon) \tilde{\varphi} + \int_{Q_+} u\, \eta_\varepsilon (\partial_d \tilde{\varphi})$$

By Dominated convergence again:

$$\int_{Q_+} u \ \eta_\varepsilon \ (\partial_d \tilde{\varphi}) \underset{\varepsilon \to 0}{\longrightarrow} \int_{Q_+} u\, (\partial_d \tilde{\varphi}).$$

Finally we need to prove that:

$$\int_Q u (\partial_d \eta_\varepsilon) \tilde{\varphi} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Since $|\partial_d \eta_\varepsilon(x_d)| \leq \frac{C}{\varepsilon} \mathbb{1}(|x_d| \leq 2\varepsilon)$

and $|\tilde{\varphi}(x)| = |\varphi(x', x_d) - \varphi(x', -x_d)|$

$$\leq C \ |x_d| \leq C\varepsilon \quad \text{if } |x_d| \leq 2\varepsilon$$

we find that:

$$\left| \int_Q u (\partial_d \eta_\varepsilon) \tilde{\varphi} \right| \leq C \int_{Q \cap \{ |x_d| \leq 2\varepsilon \}} |u| \underset{\varepsilon \to 0}{\longrightarrow} 0$$

by Dominated convergence again. Thus:

$$\begin{array}{ccc} \int_{Q_+} u\, \partial_d (\eta_\varepsilon \tilde{\varphi}) & = & - \int_{Q_+} (\partial_d u)\, \eta_\varepsilon \tilde{\varphi} \\ \downarrow {\scriptstyle \varepsilon \to 0} & & \downarrow \\ \int_{Q_+} u\, \partial_d \tilde{\varphi} & & - \int_{Q_+} (\partial_d u)\, \tilde{\varphi} \end{array} \quad \square$$

By using this lemma we can prove the extension theorem by a partition of unity. This technique is similar to the proof of trace theorem, so we omit the details. $\square$

Now we are ready to discuss the Poisson equation with inhomogeneous boundary condition.

<u>Theorem</u>: Let Ω be open, bounded, C^1-boundary. Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. Then $\exists!$ $u \in H^1(\Omega)$ s.t.

$$\begin{cases} -\Delta u = f & \text{in } D'(\Omega) \\ u|_{\partial\Omega} = g & \text{on } \partial\Omega \end{cases}$$

Here $u|_{\partial\Omega} = T(u) \in L^2(\partial\Omega)$ defined by trace oper.

Moreover, if Ω is connected and $g \neq$ constant, then u is the unique minimizer for the variational problem:

$$E = \inf_{v \in M} \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v \right\}$$

where $M = \{ v \in H^1(\Omega), \ v|_{\partial\Omega} = g \text{ on } \partial\Omega \}$

We will need:

<u>Lemma</u> (Poincaré inequality) Let Ω be open, bounded, connected, with C^1-boundary. Then $\forall g \in L^2(\partial\Omega)$ s.t. $g \neq$ constant, $\exists C > 0$ s.t.

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \forall u \in M$$

where $M = \{v \in H^1(\Omega) : v_{|\partial\Omega} = g\}$

<u>Proof:</u> We assume by contradiction that it fails. Then $\exists \{u_n\} \subset H^1(\Omega)$, $u_n|_{\partial\Omega} = g$ s.t.

$$\|\nabla u_n\|_{L^2(\Omega)} \to 0, \quad \|u_n\|_{L^2(\Omega)} = 1.$$

Since $\{u_n\}$ is bounded in $H^1(\Omega)$, by the Banach-Alaoglu theorem, up to a subsequence

$$u_n \rightharpoonup u_0 \text{ weakly in } H^1(\Omega).$$

Since $\nabla u_n \to 0$ strongly in L^2 and $\nabla u_n \rightharpoonup \nabla u_0$ weakly in L^2, we have $\nabla u_0 = 0 \Rightarrow u_0 = \text{const}$ (here we need Ω to be connected) $\Rightarrow u_{0|\partial\Omega} = \text{const}$.

On the other hand, note that M is convex and closed in $H^1(\Omega)$ since the trace operator $T: H^1(\Omega) \to L^2(\partial\Omega)$ is continuous. Therefore, M is also weakly closed in $H^1(\Omega)$ by the Hahn-Banach theorem. Thus from $\{u_n\} \subset M$, $u_n \rightharpoonup u_0$ weakly in $H^1(\Omega)$

$\Rightarrow u_0 \in M \Rightarrow u_0|_{\partial\Omega} = g$.

We get a contradiction since $g \neq$ const. $\square$

Now we are ready to consider Poisson's equation.

<u>Proof of Theorem</u>: First let us assume that Ω is connected and $g \neq$ const.

<u>Step 1</u>: We prove that $E = \inf_{v \in M} \mathcal{E}(v)$ has a minimizer.

By Poincare's inequality, $\forall v \in M$:

$$\begin{aligned}
\mathcal{E}(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f v \\
&\geq \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\geq \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} - C\|f\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} \\
&\geq \frac{1}{4}\|\nabla v\|^2_{L^2(\Omega)} - C\|f\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus $E = \inf_{v \in M} \mathcal{E}(v) > -\infty$. Moreover, taking a minimizing sequence $\{v_n\} \subset M$, $\mathcal{E}(v_n) \to E$, we find that $\|\nabla v_n\|_{L^2(\Omega)}$ is bounded, and hence $\|v_n\|_{H^1(\Omega)}$ is bounded (by Poincare inequality again). By Banach–Alaoglu theorem, up to a subsequence, we have $v_n \rightharpoonup u$ weakly in $H^1(\Omega)$. Hence:

$$\begin{cases} \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^2 \geq \int_\Omega |\nabla u|^2, \text{ as } \nabla v_n \rightharpoonup \nabla u \text{ in } L^2 \\ \int_\Omega v_n f \to \int_\Omega u f \quad \text{as } v_n \to u \text{ in } L^2 \end{cases}$$

$$\Rightarrow \quad E = \lim_{n\to\infty} \mathcal{E}(v_n) \geq \mathcal{E}(u).$$

Note that $\{v_n\} \subset M$, $v_n \rightharpoonup u$ in $H^1(\Omega)$ and M is weakly closed in $H^1(\Omega)$ (as argued in the proof of Poincare inequality), therefore $u \in M$. This means that u is a minimizer for $E = \inf_{v \in M} \mathcal{E}(v)$.

Step 2: Now we prove that if u is a minimizer for E, then $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

In fact, $\forall \varphi \in C_c^\infty(\Omega)$ we have:

$$\mathcal{E}(u) \leq \mathcal{E}(\underbrace{u+t\varphi}_{\in M}), \forall t \in \mathbb{R}$$

$$\Rightarrow 0 = \frac{d}{dt} \mathcal{E}(u+t\varphi)\Big|_{t=0} = \int_\Omega \nabla u \cdot \nabla \varphi - \int_\Omega f\varphi$$

Thus:

$$\int_\Omega u(-\Delta\varphi) = \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi, \forall \varphi \in C_c^\infty(\Omega)$$

$$\Rightarrow \quad -\Delta u = f \text{ in } \mathcal{D}'(\Omega).$$

Step 3: We prove that Poisson equation has at most 1 solution. Assume that u_1, u_2 are 2 solutions. Then $u = u_1 - u_2$ solves

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = 0 & \text{on } \Omega \end{cases} \Rightarrow u = 0.$$

Step 4: If $g = c_0$ a constant, then Poisson equation can be rewritten, with $\tilde{u} = u - c_0$,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = c_0 & \text{on } \Omega \end{cases} \Leftarrow \begin{cases} -\Delta \tilde{u} = f & \text{in } \Omega \\ \tilde{u} = 0 & \text{on } \Omega \end{cases}$$

has a unique solution.

If Ω is not connected, then by considering connected components of Ω, we can prove that Poisson's equation always has a unique solution (for all $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$).

Two final remarks:

① We can describe $H^1_0(\Omega)$ as the kernel of the trace operator $T: H^1(\Omega) \to L^2(\partial\Omega)$.

Theorem: Let $\Omega \subset \mathbb{R}^d$ be open, bounded, with C^1-boundary. Then:

$$H^1_0(\Omega) = \{ u \in H^1(\Omega), \ T(u) = 0 \text{ on } \partial\Omega \}.$$

Recall that if $u \in H^1(\Omega) \cap C(\overline{\Omega})$, then

$$T(u) = u_{|\partial\Omega} \quad \text{the usual restriction.}$$

In this case we recover a result proved before

Proof: Easy direction

$$H^1_0(\Omega) \subset \{ u \in H^1(\Omega) : T(u) = 0 \}$$

follows from the facts that $C^\infty_c(\Omega)$ is dense in $H^1_0(\Omega)$ and the trace operator is continuous.

Now let us consider the difficult direction: given $u \in H^1(\Omega)$ and $T(u) = 0$, we need to show that $u \in H^1_0(\Omega)$.

Let us think of the case Ω as $\mathbb{R}^d_+$ for simplicity.
By a density argument, take $\{u_n\} \subset C_c^\infty(\mathbb{R}^d)$ s.t. $u_n|_\Omega \to u$ in $H^1(\Omega)$. Then $T(u_n) \to T(u) = 0$ in $L^2(\partial\Omega)$. We have, for $x = (x', x_d) \in \mathbb{R}^{d-1} \times [0, \infty)$

$$u_n(x', x_d) = u_n(x', 0) + \int_0^{x_d} \partial_d u_n(x', t)\, dt$$

$$\Rightarrow \; |u_n(x', x_d)|^2 \leq 2|u_n(x', 0)|^2 + 2x_d \int_0^{x_d} |\partial_d u_n(x', t)|^2 dt$$

$$\Rightarrow \int_{\mathbb{R}^{d-1}} |u_n(x', x_d)|^2 dx' \leq 2\|T(u_n)\|^2_{L^2(\mathbb{R}^{d-1})} + 2x_d \int_0^{x_d} |\partial_d u_n(x', t)|^2 dt$$

$$\overset{(n \to \infty)}{\Longrightarrow} \int_{\mathbb{R}^{d-1}} |u(x', x_d)|^2 dx' \leq 2x_d \int_0^{x_d} |\partial_d u(x', t)|^2 dt \quad (*)$$

for a.e. $x_d \in (0, \infty)$.

Now let us use $(*)$ to approximate u by a function which is 0 close to $\partial\Omega$.

Take $\eta \in C^\infty(\mathbb{R}_+)$ s.t. $\eta(t) = \begin{cases} 1 & \text{if } t \geq 2 \\ 0 & \text{if } t \leq 1 \end{cases}$

$0 \leq \eta \leq 1$

Then

$$v_n(x) = u(x)\,\eta(nx_d) \text{ satisfies } \begin{cases} v_n \in H^1(\Omega) \\ v_n = 0 \text{ if } x_d \le \frac{1}{n} \end{cases}$$

Hence, by a result proved before, $v_n \in H_0^1(\Omega)$.

Thus it remains to show that $v_n \to u$ in $H^1(\Omega)$.

We have:

$$\int_{\mathbb{R}^d_+} |v_n - u|^2 = \int_{\mathbb{R}^d_+} |u(x)|^2 \, |1 - \eta(nx_d)|^2 \, dx$$

$$\le \int_{\mathbb{R}^d_+} |u(x)|^2 \, \mathbb{1}\left(x_d \le \frac{2}{n}\right) dx \to 0$$

by Dominated C.V.

Similarly, $\forall i = 1, 2, \dots, d-1$,

$$\int_{\mathbb{R}^d_+} |\partial_i (v_n - u)|^2 = \int_{\mathbb{R}^d_+} |\partial_i u(x)|^2 \left(1 - \eta(nx_d)\right)^2 dx \to 0$$

Let us consider:

$$\partial_d v_n = (\partial_d u)\,\eta(nx_d) + u\, n\, \eta'(nx_d)$$

Again:

$$\int_{\mathbb{R}^d_+} \left|\partial_d u(x)\,\eta(nx_d) - \partial_d u(x)\right|^2 dx \to 0$$

by Dominated C.V.

For the most difficult term we use $(*)$ and

$$|\eta'(nx_d)| \leq C \,\mathbb{1}\left(x_d \leq \frac{2}{n}\right)$$

to bound:

$$\int_{\mathbb{R}^d_+} |u(x) n\, \eta'(nx_d)|^2 \, dx = \int_0^\infty \left(\int_{\mathbb{R}^{d-1}} |u(x', x_d)|^2 dx' \right) n^2 |\eta'(nx_d)|^2 dx_d$$

$$\leq C \int_0^\infty \left(x_d \int_0^{x_d} |\partial_d u(x', t)|^2 dx' \, dt \right) n^2 \,\mathbb{1}\left(x_d \leq \frac{2}{n}\right) dx_d$$

$$\leq C \int_0^{2/n} \left[\left(\frac{2}{n}\right) \int_0^{\frac{2}{n}} |\partial_d u(x', t)|^2 dx \, dt \right] n^2 \, dx_d$$

$$\leq C \int_0^{2/n} |\partial_d u(x', t)|^2 dx \, dt \longrightarrow 0$$

by Dominated C.V.

This ends the proof when $\Omega \rightsquigarrow \mathbb{R}^d_+$. The general case of Ω can be treated using a partition of unity. $\square$

② Recall that the variational characterization of Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \Omega \end{cases}$$

is

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in M$$

where $\quad M = \{ v \in H^1(\Omega) : v = g \text{ on } \partial\Omega \}$.

Here we cannot replace M by $H^1(\Omega)$.
In fact, if $u \in H^2(\Omega)$ and

$$(**) \quad \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f \varphi, \quad \forall \varphi \in H^1(\Omega)$$

then u satisfies the Neumann condition:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n} = 0 \quad \text{on } \partial\Omega.$$

To see that, note that using $(**)$ for $\varphi \in C_c^\infty(\Omega)$ we get $-\Delta u = f$ in $L^2(\Omega)$, and hence by integration by part:

for all $\varphi \in H^1(\Omega)$ we have:

$$\int_\Omega f\varphi = \int_\Omega (-\Delta u)\varphi = \underbrace{\int_\Omega \nabla u \cdot \nabla\varphi}_{= \int_\Omega f\varphi \text{ by } (**)} - \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \varphi \, dS$$

$$\Rightarrow \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \cdot \varphi \, dS = 0, \quad \forall \varphi \in H^1(\Omega)$$

$$\Rightarrow \frac{\partial u}{\partial \vec{n}} = 0 \quad \text{on } \partial\Omega.$$

Here $u \in H^2(\Omega) \Rightarrow \nabla u \in H^1(\Omega)$

$\Rightarrow \dfrac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}$ is well-defined on $\partial\Omega$

by trace theorem. $\square$