Optimizer of Sobolev ineq:

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geq \mathcal{E}^2 \int_{\mathbb{R}^d} |u|^p, \qquad p = \frac{2d}{d-2}, \; d \geq 3$$

EL eq: $\quad -\Delta u = c\, u^{p-1}, \quad u \geq 0$

$$h(x) = \frac{1}{(1+x^2)^{\frac{d-2}{2}}} \quad \text{is a sol.}$$

<u>Wrong argument last time:</u>

If $\quad h(x) \sim \dfrac{1}{|x|^\alpha} \quad$ as $|x| \to \infty$

$$\left.\begin{aligned} \Delta h &\underset{\lesssim}{\sim} \frac{1}{|x|^{\alpha+2}} \\ h^{p-1} &\sim \frac{1}{|x|^{\alpha(p-1)}} \end{aligned}\right\} \quad \alpha + 2 = \alpha(p-1) \Rightarrow \alpha = \frac{2}{p-2} = \frac{d-2}{2}$$

Check again:

$$h = \frac{1}{(1+x^2)^\alpha}$$

$$\partial_i h = \frac{-2\alpha x_i}{(1+x^2)^{\alpha+1}}$$

$$\partial_i^2 h = -\frac{2\alpha}{(1+x^2)^{\alpha+1}} + \frac{2\alpha(\alpha+1) \cdot 2x_i^2}{(1+x^2)^{\alpha+2}}$$

$$\Delta h = \sum_i = \frac{-2d\alpha}{(1+x^2)^{\alpha+1}} + \frac{4\alpha(\alpha+1)x^2}{(1+x^2)^{\alpha+2}}$$

$$= \frac{-2d\alpha + 4\alpha(\alpha+1)}{(1+x^2)^{\alpha+1}} - \frac{4\alpha(\alpha+1)}{(1+x^2)^{\alpha+2}}$$

Our choice: $-2d\alpha + 4\alpha(\alpha+1) = 0$

$$\Rightarrow 2(\alpha+1) = d \Rightarrow \alpha = \frac{d}{2} - 1 = \frac{d-2}{2}.$$

Also: $\alpha(p-1) = \alpha+2$

$$\Leftrightarrow \alpha = \frac{2}{p-2} = \frac{d-2}{2}.$$

Last time:

If $(*)$ $\displaystyle\int_{\mathbb{R}^d} S_i(x)\,|u(x)|^p\,dx = 0, \quad \forall i = 1,\dots,d+1$

$\Rightarrow$ $u = h$ (up to sym.)

here
$$S_i(x) = \begin{cases} \dfrac{2x_i}{1+x^2} & \text{if } i = 1,\dots,d \\[2ex] \dfrac{1-x^2}{1+x^2} & \text{if } i = d+1 \end{cases}$$

We proved that under $(*)$, then u is the optimizer

$$\int_{\mathbb{R}^d} |\nabla u|^2 - 2d(d-2)\int_{\mathbb{R}^d} \frac{|u(x)|^2}{1+x^2}\,dx \geq 0$$

($h(x)$ is the opt.)

Why we can assume (*) (center of mass cond)?

$$\int_{\mathbb{R}^d} S_i(x)\, |u(x)|^p\, dx = 0, \quad \forall i = 1, \dots, d+1.$$

Note: $$\sum_{i=1}^{d+1} |S_i(x)|^2 = 1, \quad \forall x \in \mathbb{R}^d$$

$$\mathbb{S}^d = \{\omega \in \mathbb{R}^{d+1}, |\omega| = 1\}$$

$$S : \mathbb{R}^d \to \mathbb{S}^d \setminus \{(0, 0, \dots, -1)\}$$

If we extend

$$S : \mathbb{R}^d \cup \{\infty\} \to \mathbb{S}^d \qquad \text{bijective}$$

(related conformal $I(x) = \frac{x}{|x|^2}$)

Key point: Our Sobolev optimizers have the natural sym + "rotations on $\mathbb{S}^d$"

Lemma (Change of variables)

Define $$u(x) = \left(\frac{2}{1+x^2}\right)^{\frac{d}{p}} f(S(x))$$

$u : \mathbb{R}^d \to \mathbb{C}$, $f : \mathbb{S}^d \to \mathbb{C}$

Then:
$$\int_{\mathbb{R}^d} |u|^p = \int_{\mathbb{S}^d} |f|^p$$

Proof: Consider

$$|S(x) - S(y)|^2 = \sum_{i=1}^{d+1} |S_i(x) - S_i(y)|^2$$

$$= \sum_{i=1}^{d} \left| \frac{2x_i}{1+x^2} - \frac{2y_i}{1+y^2} \right|^2 + \left| \frac{1-x^2}{1+x^2} - \frac{1-y^2}{1+y^2} \right|^2$$

$$= \frac{1}{(1+x^2)^2(1+y^2)^2} \left[\sum_{i=1}^{d} \left| 2x_i(1+y^2) - 2y_i(1+x^2) \right|^2 + \left| (1-x^2)(1+y^2) - (1-y^2)(1+x^2) \right|^2 \right]$$

$$\sum_{i=1}^{d} \left| 2x_i(1+y^2) - 2y_i(1+x^2)\right|^2$$

$$+ \left|(1-x^2)(1+y^2) - (1-y^2)(1-x^2)\right|^2 \Big]$$

$$\left|2x_i(1+y^2) - 2y_i(1+x^2)\right|^2$$

$$= 4x_i^2(1+y^2)^2 + 4y_i^2(1+x^2)^2 - 4x_iy_i(1+x^2)(1+y^2)$$

$$\sum_i = 4x^2(1+y^2)^2 + 4y^2(1+x^2)^2 - 8xy(1+x^2)(1+y^2)$$

$$= 4(x-y)^2(1+x^2)(1+y^2)$$

$$+ 4x^2(1+y^2)^2 + 4y^2(1+x^2)^2 - 4(x^2+y^2)(1+x^2)(1+y^2)$$

$$= 4(x-y)^2(1+x^2)(1+y^2)$$

$$+ \underbrace{4x^2(1+y^2)(y^2-x^2) + 4y^2(1+x^2)(x^2-y^2)}$$

$$-4(x^2-y^2)^2$$

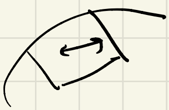
And: $\left|(1-x^2)(1+y^2) - (1-y^2)(1-x^2)\right|^2 = 4(x^2-y^2)^2.$

Conclusion:

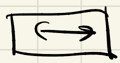
$$|S(x)-S(y)|^2 = \frac{1}{(1+x^2)(1+y^2)}\, 4|x-y|^2$$

If $|y-x| \ll 1$

$$|S(x)-S(y)|^2 \simeq \frac{4}{(1+x^2)^2}\, |x-y|$$



S^d



$\mathbb{R}^d$

$$\Rightarrow \quad |S(x)-S(y)| \simeq \left(\frac{2}{1+x^2}\right)|x-y|, \quad |x-y| \ll 1$$

$$\Rightarrow \quad dS(x) = \left(\frac{2}{1+x^2}\right) dx$$

$$\Rightarrow \int_{\mathbb{R}^d} \left(\frac{2}{1+x^2}\right)^d |f(S(x))|^p\, dx = \int_{S^d} |f(\omega)|^p\, d\omega$$

$$\Vert$$

$$\int_{\mathbb{R}^d} |u(x)|^p\, dx \qquad \square$$

Lemma: Take $u(x) = \left(\frac{2}{1+x^2}\right)^{\frac{d}{p}} f(S(x))$ (**)

$$\int_{\mathbb{R}^d} |\nabla u|^2 = \int_{\mathbb{S}^d} \left[|\nabla f|^2 + \frac{d(d-2)}{4} |f|^2 \right].$$

(exercise)

Recall that we want:

$$(*) \int_{\mathbb{R}^d} S_i(x) \, |u(x)|^p \, dx = 0 \quad , \forall i = 1, .., d+1$$

$$\Longleftrightarrow \int_{\mathbb{R}^d} S(x) \, |u(x)|^p \, dx = 0 \quad \text{on } \mathbb{R}^{d+1}$$

$$\overset{(**)}{\Longleftrightarrow} \int_{\mathbb{R}^d} S(x) \left(\frac{2}{1+x^2}\right)^d |f(S(x))|^p \, dx = 0 \text{ on } \mathbb{R}^{d+1}$$

$$\Longleftrightarrow \int_{\mathbb{S}^d} \omega \, |f(\omega)|^p \, dx = 0 \text{ on } \mathbb{R}^{d+1}$$

Here we used $u \overset{(**)}{\longleftrightarrow} f$ and we map the desired equality $(*)$ to an equality on $\mathbb{S}^d$.

We will prove that up to dilations & rotations we can achieve (*).

Equivalently, we need to show that $\forall f: \mathbb{S}^d \to \mathbb{C}$, L^p-inte. then up to dilations & rotations, we get

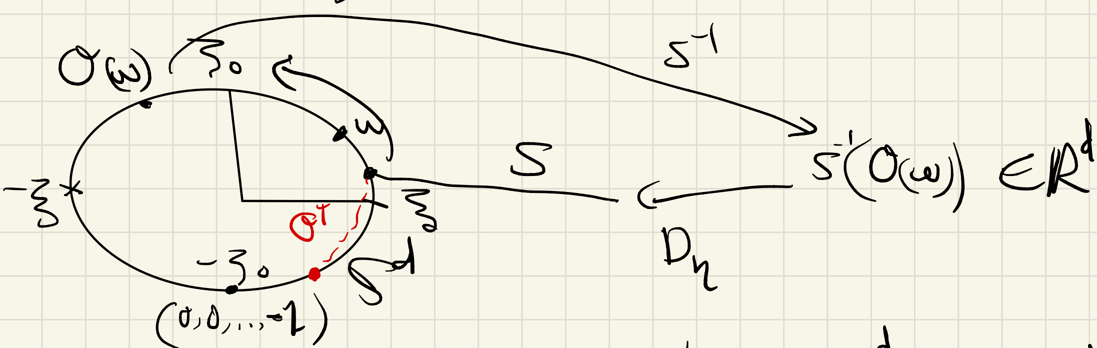
$$\int_{\mathbb{S}^d} \omega \, |f(\omega)|^p \, d\omega = 0.$$

<u>Notation:</u> Take $\eta > 0$ and $\xi \in \mathbb{S}^d$.

Define $D_\eta(x) = \eta x$, $\forall x$

$\mathcal{O}: \mathbb{S}^d \longrightarrow \mathbb{S}^d$ rotation s.t.

$\mathcal{O}\xi \longrightarrow \xi_0 = (0, 0, \dots, 1)$



Consider $\gamma = \mathcal{O}^T S D_\eta S^{-1} \mathcal{O} : \mathbb{S}^d \setminus \{-\xi\} \to \mathbb{S}^d$

$\quad \| $

$\gamma_{\eta,\xi}$. Note: $\boxed{\gamma|_{\eta=1} = 1, \ \gamma|_{\eta \to 0} = \xi}$

Thus we have a family of transformation

$$\gamma_{\eta,\xi} : S^d \longrightarrow S^d \quad \left(\gamma_{\eta,\xi}(-\xi) = -\xi \right)$$

Define $F: B \to B$, B = unit ball in $\mathbb{R}^{d+1}$ closed

$$F(r\xi) = \int_{S^d} \underbrace{\gamma_{1-r,\xi}(\omega)}_{\to\, \xi \text{ as } r \to 1} |f(\omega)|^p \, d\omega$$

$0 \le r \le 1$

$\xi \in S^d$

<u>Claim:</u> F is continuous!

And $\lim_{r \to 1} F(r\xi) = \xi \int_{S^d} |f(\omega)|^p \, d\omega$

In particular:

$$\langle x, F(x) \rangle \xrightarrow[x \sim \partial B = S^d]{} \cancel{|x|^2} \int_{\mathbb{R}^d} |f(\omega)|^p \, d\omega > 0$$

This implies that

$$\langle x, F(x) \rangle \geq \varepsilon, \quad \forall\, r_\varepsilon \le |x| \le 1$$

(Exercise + Brouwer fixed point theorem)

$\Longrightarrow$

$$x \overset{g}{\longmapsto} x - \tau F(x) \quad : \quad B \to B \text{ if } \tau > 0 \text{ small}$$

$\Rightarrow$ $\exists$ a fixed point for g

$\Rightarrow$ $\exists\, x :$ $F(x) = 0$.

$$\Rightarrow \exists\, \gamma : \quad \int_{S^d} \gamma(\omega)\, |f(\omega)|^p \, d\omega = 0$$

$\Rightarrow$ up to dilations & rotations

$$\int \omega \, |f(\omega)|^p \, d\omega = 0.$$

(i.e. $\int \omega \, |\tilde{f}(\omega)|^p \, d\omega = 0$, $\tilde{f}(\omega) = J_\gamma^{\frac{1}{p}} f(\omega)$)

$$\begin{tikzcd}
u \arrow[r, "(**)"] & f \arrow[d, "\gamma"] \\
\tilde{u} & \tilde{f} \arrow[l, "(**)"]
\end{tikzcd}
\qquad \left(u(x) = \left(\frac{1}{1+x^2}\right)^{\frac{d}{p}} f(S(x)) \right)$$

now $\displaystyle\int_{\mathbb{R}^d} S(x)\, |\tilde{u}(x)|^p \, dx = 0$ (center of mass condition) here $\int |\tilde{u}|^p = \int |u|^p$, $\int |\nabla \tilde{u}|^2 = \int |\nabla u|^2$

→ $\tilde{u}$ is also an optimiser of Sobolev ineq.

→ $\tilde{u} = h(x)$ up to dilations & translations

→ same for u.

$\square$

<u>Remark:</u> If u is radial, then it is very easy to get

$$\int_{\mathbb{R}^d} S(x) |u(x)|^p dx = 0.$$

(up to dilations)

In fact, $\forall i = 1, \dots, d$, $S_i(x) = \frac{2x_i}{1+x^2}$ odd

$$\rightarrow \int_{\mathbb{R}^d} \underbrace{S_i(x) |u(x)|^p}_{\text{odd}} dx = 0$$

When $i = d+1$, in principle

$$\int_{\mathbb{R}^d} S_{d+1}(x) |u(x)|^p dx = \int_{\mathbb{R}^d} \frac{1-x^2}{1+x^2} |u(x)|^p dx \neq 0$$

but we can consider

$$G(R) = \int_{\mathbb{R}^d} \frac{R - |x|^2}{R + |x|^2} |u(x)|^p \, dx$$

Clearly:

$$\lim_{R \to \infty} G(R) = \int_{\mathbb{R}^d} |u(x)|^p \, dx > 0$$

$$\lim_{R \to 0} G(R) = -\int |u(x)|^p \, dx < 0$$

(Dominated C.V, $\left| \frac{R - |x|^2}{R + |x|^2} \right| \leq 1$)

$\Rightarrow$ by the continuity, $\exists R > 0$ s.t.

$$0 = G(R) = \int_{\mathbb{R}^d} \frac{R - |x|^2}{R + |x|^2} |u(x)|^p \, dx$$

$\searrow$ dilation
(R = 1)

$$0 = \int_{\mathbb{R}^d} \frac{1 - |x|^2}{1 + |x|^2} |u(x)|^p \, dx$$

Chapter 4: Rearrangement method

Elementary rearr. inequality:

$a_1 \geq a_2 \geq \dots \ (\in \mathbb{R}), \quad b_1 \geq b_1 \geq \dots$

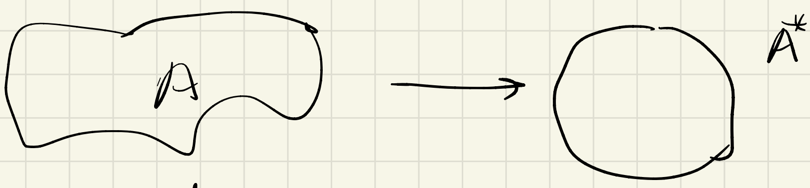
$$\sum_{i=1}^{N} a_i b_{N-i} \leq \sum_{i=1}^{N} a_i b_{\sigma(i)} \leq \sum_{i=1}^{N} a_i b_i$$

σ permutation $\{1, 2, \dots, N\}$

(proved by induction, $a_1 b_2 + b_2 a_1 \leq a_1 b_1 + a_2 b_2$
$\Leftrightarrow (a_1 - a_2)(b_1 - b_2) \geq 0$)

Radially symmetric decreasing rearrangement:

If $A \subset \mathbb{R}^d \longrightarrow A^* =$ ball centered at 0
$|A^*| = |A|$



If $f: \mathbb{R}^d \to \mathbb{R}_+$

$$f(x) = \int_0^\infty \mathbb{1}(\{f(x) > t\})\, dt$$

Def
$$\Rightarrow \quad f^*(x) = \int_0^\infty \mathbb{1}(\{f > t\}^*)(x)\,dt$$

The def of f^* depends only on the level set of f

In particular:

$$|\{f^*(x) > t\}| = |\{f(x) > t\}|$$

Thus f^*: radially symmetric decreasing function which has the same level sets of f

(If $f : \mathbb{R}^d \to \mathbb{C}$, then $f^* = (|f|)^*$)

Simplest rearr ineq

If $f, g : \mathbb{R}^d \to \mathbb{R}_+$ measurable, then:

$$\int_{\mathbb{R}^d} f(x)\, g(x)\, dx \le \int_{\mathbb{R}^d} f^*(x)\, g^*(x)\, dx$$

Moreover if $g = g^*$ strictly $\downarrow$, then "=" occur iff $f = f^*$.

Proof:

$$f(x) = \int_0^\infty \mathbb{1}(\{f(x) > t\})\, dt$$

$$g(x) = \int_0^\infty \mathbb{1}(\{g(x) > s\})\, ds$$

$$\Rightarrow \int_{\mathbb{R}^d} f(x)\, g(x)\, dx = \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \mathbb{1}(\{f(x) > t\})\, \mathbb{1}(\{g(x) > s\})\, dt\, ds\, dx$$

$$\overset{\text{Fubini}}{=} \int_0^\infty \int_0^\infty dt\, ds \left[\int_{\mathbb{R}^d} \mathbb{1}(\{f(x) > t\})\, \mathbb{1}(\{g(x) > s\})\, dx \right]$$

$$\int_{\mathbb{R}^d} \mathbb{1}_A(x)\, \mathbb{1}_B(x)\, dx = |A \cap B| \le |A^* \cap B^*| = \min(|A|, |B|)$$

$\square$

Riesz rearr. ineq.:

$f, g, h : \mathbb{R}^d \to \mathbb{R}_+$ measurable

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\, g(x-y)\, h(y)\, dx\, dy$$

$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f^*(x)\, g^*(x-y)\, h^*(y)\, dx\, dy$$

Pólya - Szegö ineq.:

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geq \int_{\mathbb{R}^d} |\nabla u^*|^2$$