Discrete spectrum asymptotics for certain finite band lattice operators

Alexander Sobolev
joint with L. Parnovski

University College London

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Plan

- Introduction
- Main result
- Proofs

Lattice operators: introduction

The underlying space: $l^2(\Gamma)$ with some lattice $\Gamma \subset \mathbb{R}^d$, $d \geq 1$.
The unperturbed operator H_0:

$$H_0 u(\mathbf{n}) = h_0(\mathbf{n}) u(\mathbf{n}), \mathbf{n} \in \Gamma.$$

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The standard Schrödinger operator: $\Gamma = \mathbb{Z}^d$ and
potential: $h_0(\mathbf{n}) = V(\mathbf{n})$,
unperturbed Hamiltonian: $b(\mathbf{n}, \pm\mathbf{e}_j) = 1$, where $\mathbf{e}_j, j = 1, 2, \dots, d$ is the canonical basis of $\mathbb{Z}^d$, and $b(\mathbf{n}, \boldsymbol{\theta}) = 0$ for other $\boldsymbol{\theta}$.

We are interested in self-adjoint operators, so

$$h_0(\mathbf{n}) = \overline{h_0(\mathbf{n})}, \quad b(\mathbf{n}, \boldsymbol{\theta}) = \overline{b(\mathbf{n} + \boldsymbol{\theta}, -\boldsymbol{\theta})}.$$

Also:
$h_0(\mathbf{n}) \to \infty$ as $|\mathbf{n}| \to \infty$ and
$b(\mathbf{n}, \boldsymbol{\theta}) h_0(\mathbf{n})^{-1} \to 0$ as $\mathbf{n} \to \infty$, so B is H_0-compact.

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More specifically:

$$h_0(\mathbf{n}) = |\mathbf{n}|^{2m}, m > 0.$$

Classes of perturbations $\mathbf{S}_\alpha(w; R)$, $\alpha \in \mathbb{R}$,
$w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta, \beta \in (0,1)$, $\langle \boldsymbol{\xi} \rangle = \sqrt{1 + |\boldsymbol{\xi}|^2}$:
$b(\mathbf{n}, \boldsymbol{\theta}) = 0$ for $|\boldsymbol{\theta}| > R$, and

$$\| b \|_s^{(\alpha)} := \max_{0 \le p \le s} \sup_{\mathbf{n}, \boldsymbol{\theta}} \ w(\mathbf{n})^{-\alpha + p} |\nabla_{\mathbf{n}}^p b(\mathbf{n}, \boldsymbol{\theta})| < \infty,$$

for all s. Here

$$\nabla_{\mathbf{n}} b(\mathbf{n}, \boldsymbol{\theta}) = \begin{bmatrix} b(\mathbf{n} + \mathbf{e}_1, \boldsymbol{\theta}) - b(\mathbf{n}, \boldsymbol{\theta}) \\ \vdots \\ b(\mathbf{n} + \mathbf{e}_d, \boldsymbol{\theta}) - b(\mathbf{n}, \boldsymbol{\theta}) \end{bmatrix}.$$

One-dimensional case

"One-sided" Jacobi matrix:

$$J = \begin{bmatrix} h_0(1) & b(1,1) & 0 & 0 & \cdots & 0 \\ b(2,-1) & h_0(2) & b(2,1) & 0 & \cdots & 0 \\ 0 & b(3,-1) & h_0(3) & b(3,1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

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$$h_0(n) - h_0(n-1) \geq cn^z, |b(n, \pm 1)| \leq Cn^{z-\epsilon}, z > -1, \epsilon > 0.$$

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Then (see [BMZ], 2010)]:

$$|\lambda_n(J) - h_0(n)| \leq Cn^{z-2\epsilon}, \ |\lambda_n(J) - h_0(n) - d(n)| \leq Cn^{z-3\epsilon},$$

with

$$d(n) = -\frac{|b(n,1)|^2}{h_0(n+1) - h_0(n)} - \frac{|b(n,-1)|^2}{h_0(n-1) - h_0(n)}.$$

A straightforward generalization:

$$d(\mathbf{n}) = -\sum_{\phi\neq\mathbf{0}} \frac{|b(\mathbf{n},\phi)|^2}{h_0(\mathbf{n}+\phi) - h_0(\mathbf{n})}.$$

Problem: **resonances**, i.e. points $\mathbf{n}$ and ϕ where $h_0(\mathbf{n}+\phi) - h_0(\mathbf{n}) = 0$!

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$$0 \le \iota \le 1, \quad \iota(t) = \begin{cases} 1, & |t| \le 1/4; \\ 0, & |t| \ge 1/2, \end{cases}$$

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$$\Upsilon(\mathbf{n},\boldsymbol{\theta}) = 1 - \iota\left(\frac{|\boldsymbol{\theta}\cdot(\mathbf{n}+\boldsymbol{\theta}/2)|}{\langle\mathbf{n}\rangle^\beta|\boldsymbol{\theta}|}\right).$$

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Let $b^o(\mathbf{n}) = b(\mathbf{n}, \mathbf{0})$, and

$$d(\mathbf{n}) = -\sum_{\phi \neq \mathbf{0}} \frac{|b(\mathbf{n}, \phi)|^2}{h_0(\mathbf{n}+\phi) - h_0(\mathbf{n})} \Upsilon(\mathbf{n}, \phi).$$

$B = \{b^o(\mathbf{n})\}, \ D = \{d(\mathbf{n})\}.$

Main result

Theorem (Main Theorem)

Let $b \in \mathbf{S}_\alpha(w; R)$, $w(\boldsymbol{\xi}) = \langle \boldsymbol{\xi} \rangle^\beta$ *with* $\alpha \geq 0$, $\beta \in (0, 1)$ *s.t.*

$$2m - 2 > \beta(\alpha - 2).$$

Let

$$\delta_3 = 3\alpha\beta - 2(2m - 2) - 4\beta,$$

and

$$\omega_0 = \begin{cases} d - 3 + \beta + \frac{2}{d}, & \beta < \frac{1}{d}, \\ d - 3 + 3\beta, & \beta \geq \frac{1}{d}, \end{cases}$$

Then for any $\omega > \omega_0$,

$$N(\rho^{2m} - C\rho^{\delta_3}; H_0 + B^o + D) - C\rho^\omega \leq N(\rho^{2m}; H) \leq N(\rho^{2m} + C\rho^{\delta_3}; H_0 + B^o + D) + C\rho^\omega.$$

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3. The counting function $N(\rho^{2m}; H_0 + B^o + D)$ equals the number of lattice points satisfying

$$h_0(\mathbf{n}) + b(\mathbf{n}, \mathbf{0}) + d(\mathbf{n}) < \rho^{2m}.$$

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4. Lattice point counting: for any lattice $\Lambda \subset \mathbb{R}^d$,

$$\#\{\mathbf{m} \in \Lambda : |\mathbf{m}| < \rho\} = \frac{\mathrm{w}_d}{|\Lambda|}\rho^d + O(\rho^{d-2+\epsilon_d}), \epsilon_d \geq 0.$$

$\epsilon_d = \frac{2}{d+1}$ (Landau, 1915), and $\epsilon_d = 0$ for $d \geq 5$ (Götze, 2004).

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For $d \geq 3$ and $\beta < 1/3$ one has $\omega_0 < d - 2$.
If $d \geq 3$, $2m = 2$, $\alpha = 0$ and $\beta < 1/3$, then $\omega_0 < d - 2$ and $\delta_3 = -4\beta$.

The "gauge" method: Take 1

Goal: to find a unitary U such that U^*HU is almost diagonal, Rozenblum '78.
Ansatz: $U = U(1), U(t) = e^{i\Psi t}, t \in \mathbb{R}$, where Ψ is a self-adjoint bounded operator.
Rewrite $A(t) = U(-t)HA(t)$:

$$A(t) = H + i\int_0^t U(-t')[H,\Psi]U(t')dt' = H_0 + B + i[H_0,\Psi] + R_2.$$

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Thus

$$i(h_0(\mathbf{n}) - h_0(\mathbf{n}+\boldsymbol{\theta}))\psi(\mathbf{n},\boldsymbol{\theta}) = -b(\mathbf{n},\boldsymbol{\theta}).$$

Need to exclude the points where $h_0(\mathbf{n}+\boldsymbol{\theta}) - h_0(\mathbf{n}) = 0$!

Partition of lattice operators

Take the function ι and define for arbitrary $L > 0$:

$$\zeta(\mathbf{n}, \boldsymbol{\theta}; L) = \iota\left(\frac{|\boldsymbol{\theta}(\mathbf{n} + \boldsymbol{\theta}/2)|}{L|\boldsymbol{\theta}|}\right), \quad \varphi(\mathbf{n}, \boldsymbol{\theta}; L) = 1 - \zeta(\mathbf{n}, \boldsymbol{\theta}; L).$$

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Set

$$b^{\natural}(\mathbf{n}, \boldsymbol{\theta}; \rho) = \begin{cases} b(\mathbf{n}, \boldsymbol{\theta})\varphi(\mathbf{n}, \boldsymbol{\theta}; \rho^{\beta}), & \boldsymbol{\theta} \neq \mathbf{0}, \\ 0, & \boldsymbol{\theta} = \mathbf{0}, \end{cases}$$

$$b^{\flat}(\mathbf{n}, \boldsymbol{\theta}; \rho) = \begin{cases} b(\mathbf{n}, \boldsymbol{\theta})\zeta(\mathbf{n}, \boldsymbol{\theta}; \rho^{\beta}), & \boldsymbol{\theta} \neq \mathbf{0}, \\ 0, & \boldsymbol{\theta} = \mathbf{0}, \end{cases}$$

$$b^{\circ}(\mathbf{n}, \boldsymbol{\theta}) = \delta_{\boldsymbol{\theta},\mathbf{0}} b(\mathbf{n}, \mathbf{0}).$$

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$$b^{o}(\mathbf{n}, \boldsymbol{\theta}) = \delta_{\boldsymbol{\theta}, \mathbf{0}} b(\mathbf{n}, \mathbf{0}).$$

$b^{\flat}(\,\cdot\,, \boldsymbol{\theta}; \rho)$'s are supported on the *resonant layers*

$$\Lambda(\boldsymbol{\theta}) = \{\boldsymbol{\xi} \in \mathbb{R}^d : |\boldsymbol{\theta} \cdot \boldsymbol{\xi}| < \rho^{\beta}|\boldsymbol{\theta}|\}.$$

Trubowitz, Skriganov, Karpeshina ...

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Lemma (Commutator equation)

Let $b \in \mathbf{S}_\alpha(w; R)$. Then the matrix Ψ with the entries

$$\begin{cases} \psi(\mathbf{n}, \boldsymbol{\theta}) = -i\dfrac{b^\natural(\mathbf{n}, \boldsymbol{\theta})}{h_0(\mathbf{n}+\boldsymbol{\theta}) - h_0(\mathbf{n})}, \ \boldsymbol{\theta} \neq 0, \\ \psi(\mathbf{n}, \mathbf{0}) = 0, \end{cases}$$

solves the equation

$$\mathrm{ad}(H_0; \Psi) + B^\natural = 0.$$

Moreover, the operator Ψ is self-adjoint, with $\psi \in \mathbf{S}_\sigma$ where

$$\sigma = \alpha - (2m-2)\beta^{-1} - 1.$$

Thus

$$A(1) = H_0 + B^o + B^\flat + R_2, \ \|R_2\| \le C\rho^{\delta_2},$$
$$\delta_2 = 2\alpha\beta - (2m-2) - 2\beta.$$

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After two iterations:

$$A(1) = H_0 + B^o + D + B^\flat + B_2^\flat + R_3, \|R_3\| \le C\rho^{\delta_3},$$
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One can do further iterations.

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One can do further iterations.
Next step is to study the operator $A(1)$.
Crucial point: structure of resonant zones.
$d = 2$: easy
$d \ge 2$: not easy, Parnovski-Sobolev 2010.

Resonant zones

Parnovski-Sobolev 2010.
RZ's are parametrized by subspaces spanned by the $\boldsymbol{\theta} \in \Theta_R := \{\boldsymbol{\eta} \in \Gamma : |\boldsymbol{\eta}| \le R\}$, denoted $\mathfrak{V} \in \mathcal{V}(R, n), 1 \le n \le d$. Let $\beta_1 = \beta, \beta_2, \dots, \beta_d$, $\sigma_1, \sigma_2, \dots, \sigma_{d-1}$ be s.t.

$$\begin{cases} 0 < \beta_1 < \cdots < \beta_{d-1} < \beta_d \le 1, \\ \sigma_1 < \cdots < \sigma_{d-1} < \beta_1. \end{cases}$$

Define

$$\Lambda_{\mathfrak{V}} = \bigcup_{\boldsymbol{\theta} \in \Theta_R \cap \mathfrak{V}} \Lambda(\boldsymbol{\theta}), \quad \Lambda := \Lambda_{\mathbb{R}^d}.$$

For any $\mathfrak{V} \in \mathcal{V}(R, n)$, $n = 1, 2, \dots, d$ define $\mathrm{M}(\mathfrak{V}) \subset \mathbb{R}^d$. If $n = 1$, then

$$\mathrm{M}(\mathfrak{V}) = \Lambda(\boldsymbol{\theta}), \quad \text{if} \quad \mathfrak{V} = \mathfrak{V}(\boldsymbol{\theta}), \ \boldsymbol{\theta} \in \Theta_R.$$

If $n \geq 2$, then for every subspace $\mathfrak{U} \in \mathcal{V}(R, p), p = 1, 2, \dots, n-1$, $\mathfrak{U} \subset \mathfrak{V}$, set

$$\mathsf{K}_{\mathfrak{U}}(\mathfrak{V}) = \left\{ \boldsymbol{\xi} \in \Lambda_{\mathfrak{V}} : |(\boldsymbol{\xi}_{\mathfrak{V}})_{\mathfrak{U}^\perp}| < \rho^{\beta_n} - \rho^{\sigma_p} \right\}, \ p = 1, 2, \dots, n-1.$$

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Now

$$\mathsf{M}(\mathfrak{V}) = \bigcap_{p=1}^{n-1} \bigcap_{\substack{\mathfrak{U} \in \mathcal{V}(R,p) \\ \mathfrak{U} \subset \mathfrak{V}}} \mathsf{K}_{\mathfrak{U}}(\mathfrak{V}), \quad \mathfrak{V} \in \mathcal{V}(R, n), n \geq 2.$$

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and

$$\Xi(\mathfrak{V}) := \begin{cases} \mathrm{M}(\mathfrak{V}) \setminus \bigcup_{m>n} \left(\bigcup_{\mathfrak{W} \in \mathcal{V}(R,m)} \mathrm{M}(\mathfrak{W}) \right), n \leq d-1, \\ \mathrm{M}(\mathfrak{V}), n = d. \end{cases}$$

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The subsets $\Xi(\mathfrak{V})$ give produce invariant subspaces of $A(1)$.