Statics and Dynamics of Magnetic Vortices

I.M.Sigal
based on the joint with S. Gustafson and T. Tzaneteas

Münich, April 2012

Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations:

$$-\Delta_A \Psi = \kappa^2 (1 - |\Psi|^2) \Psi$$
$$\operatorname{curl}^2 A = \operatorname{Im}(\bar{\Psi} \nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

Origin of Ginzburg-Landau Equations

Superconductivity. $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*; $|\Psi|^2$ gives the density of (Cooper pairs of) superconducting electrons. $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential. $\mathrm{Im}(\bar{\Psi}\nabla_A\Psi)$ is the superconducting current.
(Homogenous solutions:
(a) $(\Psi \equiv 1,\ A \equiv 0)$, the perfect superconductor,
(b) $(\Psi \equiv 0,\ A$ with $\mathrm{curl}\, A$ constant), the normal metal.)

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively.
(One can think of A as a connection on the principal $U(1)$- bundle $\mathbb{R}^2 \times U(1)$, and Ψ, as the section of this bundle.)

Similar equations appear in the theory of superfluidity and of fractional quantum Hall effect.

Ginzburg-Landau Energy

Ginzburg-Landau equations are the Euler-Lagrange equations for the *Ginzburg-Landau energy functional*

$$\mathcal{E}_\Omega(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}.$$

Superconductors: $\mathcal{E}(\Psi, A)$ is the difference in (Helmhotz) free energy between the superconducting and normal states.

In the $U(1)$ Higgs model case, $\mathcal{E}_\Omega(\Psi, A)$ is the energy of a static configuration in the $U(1)$ Yang-Mills-Higgs classical gauge theory.

Quantization of Flux

From now on we let $d = 2$. Finite energy states (Ψ, A) are classified by the topological degree

$$\deg(\Psi) := \deg\left(\frac{\Psi}{|\Psi|}\bigg|_{|x|=R}\right),$$

where $R \gg 1$. For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi \deg(\Psi) \in 2\pi\mathbb{Z},$$

where $B := \operatorname{curl} A$ is the magnetic field associated with the vector potential A.

Type I and II Superconductors

Two types of superconductors:

$\kappa < 1/\sqrt{2}$: Type I superconductors, exhibit first-order phase transitions from the non-superconducting state to the superconducting state (essentially, all pure metals);

$\kappa > 1/\sqrt{2}$: Type II superconductors, exhibit second-order phase transitions and the formation of vortex lattices (dirty metals and alloys).

For $\kappa = 1/\sqrt{2}$, Bogomolnyi has shown that the Ginzburg-Landau equations are equivalent to a pair of first-order equations. Using this Taubes described completely solutions of a given degree.

Symmetries

Gauge symmetry: for any sufficiently regular function $\eta : \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma_\gamma : \ (\Psi(x), \ A(x)) \mapsto (e^{i\eta(x)}\Psi(x), \ A(x) + \nabla\eta(x));$$

Translation symmetry: for any $h \in \mathbb{R}^2$,

$$T_h : \ (\Psi(x), \ A(x)) \mapsto (\Psi(x+h), \ A(x+h));$$

Rotation and reflection symmetry: for any $R \in O(2)$

$$T_R : \ (\Psi(x), \ A(x)) \mapsto (\Psi(Rx), \ R^{-1}A(Rx)).$$

Equivariant Pairs

A key class of solutions is provided by equivariant pairs.

Given a subgroup, G, of the group of rigid motions (a semi-direct product of the groups of translations and rotations) , an equivariant pair is a state (Ψ, A) s.t. $\forall g \in G$, $\exists \gamma = \gamma(g)$ s.t.

$$T_g(\Psi, A) = \Gamma_\gamma(\Psi, A),$$

where T_g is the action of G and Γ_γ is the action of for the gauge group.

This leads to two classes of solutions:

$G = O(2) \Longrightarrow$ magnetic vortices (labeled by the equivalence classes of the homomorphisms of S^1 into $U(1)$, i.e. by $n \in \mathbb{Z}$).

$G =$ the group of lattice translations for a lattice $\mathcal{L} \Longrightarrow$ Abrikosov lattices.

Vortices

"Radially symmetric" (more precisely, *equivariant*) solutions:

$$\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta) \ ,$$

where (r, θ) are the polar coordinates of $x \in \mathbb{R}^2$.

$$\deg(\Psi^{(n)}) = n \in \mathbb{Z}.$$

The profiles are exponentially localized:

$$|1 - f^{(n)}(r)| \leq ce^{-m_f r}, \quad |1 - a^{(n)}(r)| \leq ce^{-m_a r},$$

Here m_f = *coherence length* and m_a = *penetration depth*.

$$\kappa = m_f / m_a.$$

$(\Psi^{(n)}, A^{(n)})$ = the *magnetic n-vortex* (superconductors) or *Nielsen-Olesen* or *Nambu string* (the particle physics).

The exponential decay is due to the Higgs mechanism of mass generation.

Stability/Instability of Vortices

Theorem

1. *For Type I superconductors all vortices are stable.*
2. *For Type II superconductors, the* ± 1*-vortices are stable, while the* n*-vortices with* $|n| \geq 2$*, are not.*

The statement of Theorem I was conjectured by Jaffe and Taubes on the basis of numerical observations (Jacobs and Rebbi, . . .).

Abrikosov Vortex Lattice States

A pair (Ψ, A) for which all the physical characteristics $|\Psi|^2$, $B(x) := \operatorname{curl} A(x)$ and $J(x) := \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$ are doubly periodic with respect to a lattice $\mathcal{L}$ is called the *Abrikosov (vortex) lattice state*.

One can show that (Ψ, A) is an Abrikosov lattice state if and only if it is an equivariant pair for the group of lattice translations for a lattice $\mathcal{L}$. Explicitly,

$$T_s(\Psi, A) = \Gamma_{g_s}(\Psi, A), \ \forall s \in \mathcal{L},$$

where $g_s : \mathbb{R}^2 \to \mathbb{R}$ is, in general, a multi-valued differentiable function, with differences of values at the same point $\in 2\pi\mathbb{Z}$. The last equation implies that satisfies

$$g_{s+t}(x) - g_s(x+t) - g_t(x) \in 2\pi\mathbb{Z}.$$

Existence of Abrikosov Lattices

Let H_{c1} and H_{c2} be the magnetic fields at which the first vortex enters the superconducting sample and the normal material becomes superconducting.

Theorem (High magnetic fields). For for every $\mathcal{L}$ and $\bar{B} < H_{c2}$, but close to H_{c2}, there exists a non-trivial $\mathcal{L}$-lattice solution, with one quantum of flux per cell and with average magnetic flux per cell equal to $\bar{B}$.

If $\kappa > 1/\sqrt{2}$, then the minimum of the average energy per cell is achieved for the triangular lattice.

Theorem (Low magnetic fields). For every $\mathcal{L}$, n and $\bar{B} > H_{c1}$ (but close to H_{c1}), there exist non-trivial $\mathcal{L}$-*lattice* solution, with n quanta of flux per cell and with average magnetic flux per cell $= \bar{B}$.

References

- Existence for $H \approx H_{c2}$: The Abrikosov lattice solutions were discovered in 1957 by A. Abrikosov.

Rigorous results: Odeh, Barany - Golubitsky - Tursky, Dutour, Tzaneteas - IMS (also for $b > H_{c2},\ \kappa < \kappa_{\rm c}(\tau) := \frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{\beta(\tau)}}$, where $\beta(\tau)$ is the Abrikosov 'constant').

Energy minimization by triangular lattices: Dutour, Tzaneteas - IMS, using results of Aftalion - Blanc - Nier, on minimization of the Abrikosov function.

Finite domains: Almog, Aftalion - Serfaty.

- Existence for $H \approx H_{c2}$: Aydi - Sandier and others ($\kappa \to \infty$) and Tzaneteas - IMS (all κ's).

More references can be found in the books by E. Sandier and S. Serfaty, 2007, and S. Fournais, B. Helffer, 2010, and the review S. J. Gustafson, I. M. Sigal and T. Tzaneteas, JMP, 2010.

Time-Dependent Eqns. Superconductivity

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A\Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \mathrm{Im}(\overline{\Psi}\nabla_A\Psi)$, and $J_n = \sigma E$.

Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations ($\Phi = 0$)

$$\partial_t^2 \Psi = \Delta_A \Psi + \kappa^2 (1 - |\Psi|^2) \Psi$$
$$\partial_t^2 A = - \operatorname{curl}^2 A + \operatorname{Im}(\bar{\Psi} \nabla_A \Psi),$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics (written here in the *temporal gauge*).

Consider two types of perturbations:
(a) Gauge -periodic perturbations of the same periodicity as the Abrikosov lattice;
(b) Finite energy perturbations, those satisfying,

$$\lim_{Q \to \mathbb{R}^2} (\mathcal{E}_Q(\Psi, A) - \mathcal{E}_Q(\Psi_0, A_0)) < \infty.$$

Lattice shapes can be parameterized by $\tau \in \mathbb{C}$, satisfying $|\tau| \geq 1$, $\operatorname{Im} \tau > 0$, $-\frac{1}{2} < \operatorname{Re} \tau \leq \frac{1}{2}$, and $\operatorname{Re} \tau \geq 0$, if $|\tau| = 1$.

Theorem (Tzaneteas - IMS)

There is $0 < \kappa_*(\tau) < \frac{1}{\sqrt{2}}$ s.t. the Abrikosov vortex lattice solutions for high magnetic fields are

(i) *asymptotically stable for $\kappa > \kappa_*(\tau)$;*
(ii) *unstable for $\kappa < \kappa_*(\tau)$.*

Abrikosov's 'Constant'

Moreover, 1) $\kappa_*^{\rm per}(\tau) := \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\beta(\tau)}}$, where $\beta(\tau)$ is the Abrikosov 'constant':

$$\beta(\tau) = \frac{\langle|\xi|^4\rangle_\Omega}{\langle|\xi|^2\rangle_\Omega^2}.$$

Here Ω is an elementary cell of the lattice and $\xi \neq 0$ is the unique solution of the equation

$$-\Delta_{A_0}\xi = \kappa^2\xi, \qquad A_0 := \frac{\kappa^2}{2}(-x_2, x_1),$$

satisfying, for any $\nu \in \mathcal{L}$,

$$\xi(x+\nu) = e^{\frac{i\kappa^2}{2}\nu\wedge x}\xi(x).$$

2) $\kappa_*^{\rm loc}(\tau)(=\kappa_*(\tau)$ for energy finite perturbations) satisfies

$$\kappa_*^{\rm per}(\tau) \leq \kappa_*^{\rm lot}(\tau) < \frac{1}{\sqrt{2}}.$$

Key Steps in Proof

Collar lemma for the infinite-dimensional manifold $\mathcal{M} = \{T_g^{sym} u_* : g \in G\}$ of $\mathcal{L}-$ lattice solutions;

Estimates of the hessian, $\mathcal{E}''(u_*)$ at a Abrikosov lattice solution $u_* = (\Psi_*, A_*)$;

Differential inequalities for Lyapunov - type functionals.

Hessian

Theorem. The Hessian $\mathcal{E}''(u_*)$ at a Abrikosov lattice solution $u_* = (\Psi_*, A_*)$ has the following properties:

- $\mathcal{E}''(u_*)$ is real-linear and symmetric in the inner product $\langle w_1, w_2 \rangle = \int (\mathrm{Re}\,\bar{\xi}_1 \xi_2 + \alpha_1 \cdot \alpha_2)$, where $w_i = (\xi_i, \alpha_i)$;
- $\mathrm{null}\,\mathcal{E}''(u_*) = \mathcal{Z} := \{G_\gamma : \ \gamma \in H_2(\mathbb{R}^2, \mathbb{R})\}$, where $G_\gamma := (i\gamma\Psi_*, \nabla\gamma)$;
- If $\theta := \inf_{v \perp \mathcal{Z}, \|v\|=1} \langle v, \mathcal{E}''(u_*) v \rangle$, then

  $\theta > 0$ for $\kappa > \kappa_*(\tau)$ (u_* is linearly stable)

  $\theta < 0$ for $\kappa < \kappa_*(\tau)$ (u_* is linearly unstable).

Idea of Proof of Stability

The key point: $u_* = (\Psi_*, A_*)$ is equivariant $\Longrightarrow$ the Hessian $\mathcal{E}''(u_*)$ commutes with magnetic translations,

$$T_s = e^{-ig_s} S_s,$$

where S_s is the translation operator $S_s f(x) = f(x+s)$ and, recall, $g_s : \mathbb{R}^2 \to \mathbb{R}$ is a multi-valued differentiable function, satisfying

$$g_{s+t}(x) - g_s(x+t) - g_t(x) \in 2\pi\mathbb{Z}.$$

This co-cycle condition implies

$$T_{t+s} = T_t T_s.$$

Hence T_s defines a unitary representation of $\mathcal{L}$ on $L^2(\mathbb{R}^2; \mathbb{C}) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$.

Direct Fibre Integral (Bloch Decomposition)

The Hessian operator $\mathcal{E}''(u_*)$ commutes with the representation T_s of $\mathcal{L}$ $\Rightarrow$ it can be decomposed into the fiber direct integral

$$U\mathcal{E}''(u_*)U^{-1} = \int_{\Omega^*}^{\oplus} L_k d\mu_k$$

on the space

$$\mathscr{H} = \int_{\Omega^*}^{\oplus} \mathscr{H}_k d\mu_k,$$

where Ω^* is the fundamental cell of the reciprocal lattice (the dual group to $\mathcal{L}$ under addition mod the reciprocal lattice), and $d\mu_k = \frac{dk}{|\hat{\Omega}|}$ is the Haar measure on Ω^*.

Fibre Spaces and Operators

Above, $U : L^2(\mathbb{R}^2; \mathbb{C}) \times L^2(\mathbb{R}^2; \mathbb{R}^2) \to \mathscr{H}$ is a unitary operator given by

$$(Uv)_k(x) = \sum_{s \in \mathcal{L}} \chi_s^{-1} T_s v(x),$$

where $\chi : \mathcal{L} \to U(1)$ is a character of the representation T_s of $\mathcal{L}$, explicitly given by

$$\chi_s = e^{ik \cdot s}, \quad k \in \Omega^*,$$

L_k is the restriction of the operator $\mathcal{E}''(u_*)$ to $H_2(\Omega)$, satisfying

$$T_s v(x) = \chi_s v(x), \qquad s \in \text{basis}.$$

Θ-function

In the leading order in $\epsilon := \sqrt{\frac{\kappa^2 - b}{\kappa^2[(2\kappa^2-1)\beta(\tau)+1]}}$, the ground states of the fiber operators, L_k, are expressed through the entire functions, $\Theta_k(z),\ z = x^1 + ix^2 \in \mathbb{C},\ k \in \Omega^*$, satisfying the periodicity conditions

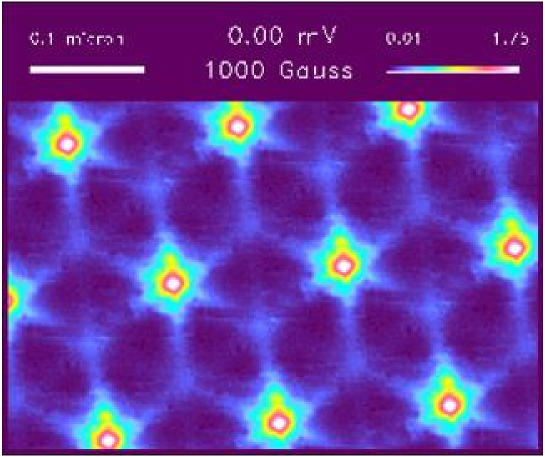
$$\Theta_k(z+1) = \Theta_k(z),$$

$$\Theta_k(z+\tau) = e^{i(k_2 - k_1\tau)} e^{-2inz} e^{-in\tau z} \Theta_k(z),$$

where τ is the shape parameter of the lattice $\mathcal{L}$.

Using this and perturbation theory, one finds inf L_k (the lowest band function) which give inf $\mathcal{E}''(u_*)$.

Abrikosov Lattice. Experiment



Extensions

Microscopic corrections to macroscopic solutions

Dynamics of Several Vortices

Consider a dynamical problem with initial conditions, describing several vortices, with the centers at points z_1, $z_2, \ldots$ and with the degrees n_1, n_2, $\ldots$, glued together, e.g.

$$\psi_{\underline{z},\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j),$$

$$A_{\underline{z},\chi}(x) = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x) \ ,$$

where $\underline{z} = (z_1, z_2, \ldots)$ and χ is an arbitrary real function.
We will assume that $R(\underline{z}) := \min_{j \neq k} |z_j - z_k| \gg 1$.

Vortex Dynamics: Superconductors

The *superconductor model* (Gustafson - IMS):
For initial data (Ψ_0, A_0) close to some $(\Psi_{\underline{z}_0,\chi_0}, A_{\underline{z}_0,\chi_0})$ with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} \leq \epsilon \ll 1$ we have

$$(\Psi(t), A(t)) = (\Psi_{\underline{z}(t),\chi(t)}, A_{\underline{z}(t),\chi(t)}) + O(\epsilon \log^{1/4}(1/\epsilon))$$

and that the vortex dynamics is governed by the system

$$\gamma_{n_j}\dot{z}_j = -\nabla_{z_j} W(\underline{z}) + O(\epsilon^2 \log^{3/4}(1/\epsilon)).$$

Here $W(\underline{z}) \sim \sum_{j\neq k}(const)n_j n_k \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}}$ is the *effective energy* and $\gamma_n > 0$.

Vortex Dynamics: $U(1)$-Higgs Model

The *Higgs model* (Gustafson - IMS):
For times up to $O\left(\frac{1}{\sqrt{\epsilon}}\log\left(\frac{1}{\epsilon}\right)\right)$, the effective dynamics is given by

$$\gamma_{n_j}\ddot{z}_j = -\nabla_{z_j} W(\underline{z}(t)) + o(\epsilon).$$

with the same effective energy/Hamiltonian

$W(\underline{z}) \sim \sum_{j\neq k}(const)n_j n_k \frac{e^{-|z_j-z_k|}}{\sqrt{|z_j-z_k|}}$ and with $\gamma_n > 0$.

Previous Results

Extensive literature on the *Gross-Pitaevski equation* (superfluids).

The *Gorkov-Eliashberg-Schmidt equations*:
Non-rigorous results: Manton ($\kappa \approx \frac{1}{\sqrt{2}}$), Atiyah - Hitchin ($\kappa \approx \frac{1}{\sqrt{2}}$), Perez - Rubinstein, Chapman-Rubinstein-Schatzman, W.E.

Rigorous results: Stuart, Demoulini - Stuart (both, $\kappa \approx \frac{1}{\sqrt{2}}$), Spirn (independently), Sandier - Serfaty (bounded domains, large κ and h_{ex} below $C \log \kappa$, Tice, Serfaty - Tice (the dynamics with applied field and external current).

No results on the $U(1)$ - Higgs model.

Thank-you for your attention.