A central limit theorem for mean field quantum dynamics

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N-boson system: described by wave function $\psi_N \in L^2(\mathbb{R}^{3N})$.

Evolution governed by Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}$$

with Hamiltonian

$$H_N = \sum_{j=1}^N (-\Delta_{x_j} + V_{\text{ext}}(x_j)) + \lambda \sum_{i<j}^N V(x_i - x_j)$$

Mean field regime: $N \gg 1$, $\lambda \ll 1$, with $N\lambda$ fixed. Study dynamics generated by

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^N V(x_i - x_j)$$

We assume the potential to have at most Coulomb type singularities, in the sense that

$$V^2(x) \le C(1 - \Delta)$$

Self-consistent evolution: consider a factorized initial state

$$\psi_{N,0}(\mathbf{x}) = \prod_{j=1}^{N} \varphi(x_j) \qquad (\mathbf{x} = (x_1, \dots, x_N)).$$

If factorization is approximately preserved in time,

$$\psi_{N,t}(\mathbf{x}) \simeq \prod_{j=1}^{N} \varphi_t(x_j)$$

we may replace the many-body interaction by an effective one-particle potential

$$\frac{1}{N} \sum_{i \neq j}^{N} V(x_i - x_j) \simeq \frac{1}{N} \sum_{i \neq j}^{N} \int \mathrm{d}x_i \, V(x_i - x_j) |\varphi_t(x_i)|^2 \simeq \, (V * |\varphi_t|^2)(x_j)$$

The one-particle wave function φ_t must solve the self-consistent Hartree equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + \, (V * |\varphi_t|^2) \varphi_t \, .$$

Reduced Densities: For $k = 1, .., N$, the reduced k-particle density matrix is given by

$$\gamma_{N,t}^{(k)} = \text{Tr}_{k+1,\ldots,N} \; |\psi_{N,t}\rangle\langle\psi_{N,t}| \qquad \text{acting on } L^2(\mathbb{R}^{3k})$$

$\gamma_{N,t}^{(k)}$ is an operator on $L^2(\mathbb{R}^{3k})$ with kernel

$$\gamma_{N,t}^{(k)}(\mathbf{x}_k; \mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k} \, \psi_{N,t}(\mathbf{x}_k, \mathbf{x}_{N-k}) \overline{\psi}_{N,t}(\mathbf{x}'_k, \mathbf{x}_{N-k}) \, ,$$

with $\mathbf{x}_k = (x_1, \ldots, x_k)$, $\mathbf{x}_{N-k} = (x_{k+1}, \ldots, x_N)$, $\text{Tr}\, \gamma_{N,t}^{(k)} = 1$.

Convergence towards Hartree dynamics: for every fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}$, one finds

$$\gamma_{N,t}^{(k)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$$

as $N \to \infty$.

First proof by Erdős-Yau (2000), using techniques of Spohn (1980), other methods and proofs by Rodnianski-S. (2007), Fröhlich-Knowles-Schwarz (2008), Knowles-Pickl (2009).

Fock space representation: let

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2_s(\mathbb{R}^{3n}, \mathrm{d}x_1 \dots \mathrm{d}x_n)$$

Vectors in $\mathcal{F}$ are sequences $\psi = \{\psi^{(n)}\}_{n \geq 1}$ with $\psi^{(n)} \in L^2_s(\mathbb{R}^{3n})$.

Creation and annihilation operators: for $f \in L^2(\mathbb{R}^3)$, define

$$\left(a^*(f)\psi\right)^{(n)}(x_1, \dots, x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j) \psi^{(n-1)}(x_1, \dots, \widehat{x}_j, \dots, x_n)$$

$$\left(a(f)\psi\right)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1} \int \mathrm{d}x \, \overline{f(x)} \psi^{(n+1)}(x, x_1, \dots, x_n)$$

They satisfy canonical commutation realtions:

$$[a(f), a^*(g)] = (f, g)_{L^2} \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

For example,

$$\left\{0, \dots, 0, \varphi^{\otimes N}, 0, \dots\right\} = \frac{(a^*(\varphi))^N}{\sqrt{N!}} \Omega$$

where $\Omega = \{1, 0, \dots\}$ is the vacuum.

We also introduce the operator-valued distributions a_x^*, a_x s.t.

$$a^*(f) = \int \mathrm{d}x\, f(x) a_x^* \qquad \text{and} \qquad a(f) = \int \mathrm{d}x\, \overline{f(x)}\, a_x$$

We define the number of particle operator

$$\mathcal{N} = \int \mathrm{d}x\ a_x^* a_x$$

and the Hamiltonian

$$\mathcal{H}_N = \int \mathrm{d}x\ \nabla_x a_x^* \nabla_x a_x + \frac{1}{N} \int \mathrm{d}x \mathrm{d}y V(x-y) a_x^* a_y^* a_y a_x$$

Observe that

$$e^{-i\mathcal{H}_N t} \{0, \dots, 0, \varphi^{\otimes N}, 0, \dots\} = \{0, \dots, 0, e^{-iH_N t} \varphi^{\otimes N}, 0, \dots\}$$

What did we gain by formulating the problem on Fock space?

Coherent states: for $\varphi \in L^2(\mathbb{R}^3)$ define the Weyl operator

$$W(\varphi) = \exp(a^*(\varphi) - a(\varphi))$$

The coherent state with wave function φ is then given by

$$W(\varphi)\Omega = e^{-\|\varphi\|^2/2} \sum_{j=0}^{\infty} \frac{a^*(\varphi)^j}{j!}\Omega = e^{-\|\varphi\|^2/2} \left\{ 1, \varphi, \frac{\varphi^{\otimes 2}}{\sqrt{2}}, \dots \right\}$$

where $\Omega = \{1, 0, \dots\}$ is the vacuum.

- $W(\varphi)^* = W(\varphi)^{-1} = W(-\varphi)$
- $\langle W(\varphi)\Omega, \mathcal{N} W(\varphi)\Omega \rangle = \|\varphi\|^2$
- We have

$$W^*(\varphi)\, a_x\, W(\varphi) = a_x + \varphi(x),$$
$$W^*(\varphi)\, a_x^*\, W(\varphi) = a_x^* + \overline{\varphi}(x)$$

Evolution of coherent states: we consider the initial state

$$W(\sqrt{N}\varphi)\Omega = e^{-N/2}\left\{1, \sqrt{N}\varphi, \dots, \frac{N^{j/2}}{\sqrt{j!}}\varphi^{\otimes j}, \dots\right\}$$

and the one-particle density associated with its time-evolution

$$\Gamma^{(1)}_{N,t}(x;y) = \frac{1}{N}\left\langle e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega,\; a^*_y\, a_x\, e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\right\rangle$$

Expanding around $a_x \simeq \sqrt{N}\,\varphi_t(x)$, $a^*_y \simeq \sqrt{N}\,\overline{\varphi}_t(y)$, we conclude

$$\begin{aligned}
&\Gamma^{(1)}_{N,t}(x;y) - \varphi_t(x)\overline{\varphi}_t(y) \\
&\quad = \frac{1}{N}\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}\,\left(a^*_y - \sqrt{N}\overline{\varphi}_t(y)\right) \\
&\qquad\qquad\qquad\qquad \times \left(a_x - \sqrt{N}\varphi_t(x)\right)\, e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\rangle \\
&\qquad + \frac{\varphi_t(x)}{\sqrt{N}}\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}\,\left(a^*_y - \sqrt{N}\overline{\varphi}_t(y)\right)\, e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\rangle \\
&\qquad + \frac{\overline{\varphi}_t(y)}{\sqrt{N}}\langle \Omega, W^*(\sqrt{N}\varphi)e^{i\mathcal{H}_N t}\,\left(a_x - \sqrt{N}\varphi_t(x)\right)\, e^{-i\mathcal{H}_N t}W(\sqrt{N}\varphi)\Omega\rangle
\end{aligned}$$

Fluctuation dynamics: since

$$(a_y^* - \sqrt{N}\,\overline{\varphi}_t(y)) = W(\sqrt{N}\varphi_t) a_y^* W^*(\sqrt{N}\varphi_t)$$
$$(a_x - \sqrt{N}\,\varphi_t(y)) = W(\sqrt{N}\varphi_t) a_x W^*(\sqrt{N}\varphi_t)$$

we write, following ideas of Hepp (1973),

$$\Gamma^{(1)}_{N,t}(x;y) - \varphi_t(x)\overline{\varphi}_t(y) = \frac{1}{N}\left\langle \Omega, \mathcal{U}^*(t)\, a_y^* a_x \,\mathcal{U}(t)\Omega \right\rangle + \frac{\varphi_t(x)}{\sqrt{N}}\left\langle \Omega, \mathcal{U}^*(t)\, a_y^* \mathcal{U}(t)\Omega \right\rangle + \frac{\overline{\varphi}_t(y)}{\sqrt{N}}\left\langle \Omega, \mathcal{U}^*(t)\, a_x \,\mathcal{U}(t)\Omega \right\rangle$$

with

$$\mathcal{U}(t) = W(\sqrt{N}\varphi_t) e^{-i\mathcal{H}_N t} W^*(\sqrt{N}\varphi)$$

The problem reduces essentially to estimating

$$\langle \Omega, \mathcal{U}^*(t)\, \mathcal{N}\, \mathcal{U}(t)\Omega \rangle$$

uniformly in N.

Observe that fluctuation dynamics satisfies

$$i\partial_t \mathcal{U}(t) = \mathcal{L}_N(t)\mathcal{U}(t) \quad \text{with} \quad \mathcal{U}_N(0) = 1$$

with time-dependent generator

$$\begin{aligned}\mathcal{L}_N(t) = & \int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x + \int dx\, (V * |\varphi_t|^2)(x) a_x^* a_x \\ & + \int \mathrm{d}x \mathrm{d}y\, V(x-y)\varphi_t(x)\overline{\varphi}_t(y)\, a_x^* a_y \\ & + \int \mathrm{d}x \mathrm{d}y\, V(x-y)\left(\varphi_t(x)\varphi_t(y)\, a_x^* a_y^* + \overline{\varphi}_t(x)\overline{\varphi}_t(y) a_x a_y\right) \\ & + \frac{1}{\sqrt{N}} \int \mathrm{d}x \mathrm{d}y\, V(x-y)\, a_x^* \left(\overline{\varphi}_t(y) a_y + \varphi_t(y) a_y^*\right) a_x \\ & + \frac{1}{N} \int \mathrm{d}x \mathrm{d}y\, V(x-y)\, a_x^* a_y^* a_y a_x\end{aligned}$$

Growth of $\mathcal{N}$: $\mathcal{U}_N(t)$ does not preserves number of particles. Still, one can show [Rodnianski-S. (2008)]:

$$\left\langle \psi, \mathcal{U}^*(t)\, (\mathcal{N}+1)^k \mathcal{U}(t)\psi \right\rangle \leq C e^{K|t|} \langle \psi, (\mathcal{N}+1)^{2k+2}\, \psi \rangle$$

Consequence [Rodnianski-S. (2008)]: For every fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}$, there exists constants $C = C(k), K = K(k) > 0$ with

$$\mathrm{Tr}\, \left| \Gamma^{(k)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \right| \leq \frac{C e^{K|t|}}{N}$$

Limiting fluctuation dynamics [Ginibre-Velo (1979)]: as $N \to \infty$, $\mathcal{U}(t)$ approaches $\mathcal{U}_\infty(t)$ where

$$i\partial_t \mathcal{U}_\infty(t) = \mathcal{L}_\infty(t)\mathcal{U}_\infty(t)$$

with time-dependent generator

$$\begin{aligned} \mathcal{L}_\infty(t) = & \int \mathrm{d}x\, \nabla_x a_x^* \nabla_x a_x + \int dx\, (V * |\varphi_t|^2)(x) a_x^* a_x \\ & + \int \mathrm{d}x \mathrm{d}y\, V(x-y)\varphi_t(x)\overline{\varphi}_t(y)\, a_x^* a_y \\ & + \int \mathrm{d}x \mathrm{d}y\, V(x-y) \left(\varphi_t(x)\varphi_t(y)\, a_x^* a_y^* + \overline{\varphi}_t(x)\overline{\varphi}_t(y) a_x a_y \right) \end{aligned}$$

Since the generator is quadratic, $\mathcal{U}_\infty(t)$ can be described as a Bogoliubov transformation.

For $f, g \in L^2(\mathbb{R}^3)$, let $A(f,g) = a^*(f) + a(\overline{g})$.

A Bogoliubov transformation is a linear map

$$\Theta : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

which preserves canonical commutation relation, i.e.

$$\Big[A(\Theta(f_1,g_1)), A(\Theta(f_2,g_2))\Big] = \Big[A(f_1,g_1), A(f_2,g_2)\Big]$$

for all $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^3)$.

Easy to check:

$$\Theta \text{ Bogoliubov transf.} \quad \Leftrightarrow \quad \Theta^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Leftrightarrow \quad \Theta = \begin{pmatrix} U & V \\ \overline{V} & \overline{U} \end{pmatrix}$$

where $U, V : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ are s.t. $U^*U - V^*V = 1$ and $U^*\overline{V} - V^*\overline{U} = 0$.

The limiting fluctuation dynamics $\mathcal{U}_\infty(t)$ is so that

$$\mathcal{U}_\infty(t)\, A(f,g)\, \mathcal{U}^*_\infty(t) = A(\Theta_t(f,g))$$

for a time-dependent Bogoliubov transformation

$$\Theta_t = \begin{pmatrix} U_t & \overline{V}_t \\ V_t & \overline{U}_t \end{pmatrix}$$

A simple computation shows that $\Theta_{t=0} = 1$ and

$$i\partial_t \Theta_t = \begin{pmatrix} D_t & -\overline{B}_t \\ B_t & -\overline{D}_t \end{pmatrix} \Theta_t$$

with $D_t, B_t : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ given by

$$\begin{aligned} D_t f &= -\Delta f + (V * |\varphi_t|^2) f + (V * \overline{\varphi}_t f)\varphi_t \\ B_t f &= (V * \overline{\varphi}_t f)\overline{\varphi}_t \end{aligned}$$

Back to factorized data: we compute

$$\begin{aligned}\gamma^{(1)}_{N,t}(x,y) &= \frac{1}{N}\left\langle e^{-i\mathcal{H}_N t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, a^*_x a_y e^{-i\mathcal{H}_N t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega\right\rangle \\ &= \frac{d_N}{N}\left\langle e^{-i\mathcal{H}_N t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, a^*_x a_y e^{-i\mathcal{H}_N t} P_N W(\sqrt{N}\varphi)\Omega\right\rangle \\ &= \frac{d_N}{N}\left\langle e^{-i\mathcal{H}_N t}\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega, a^*_x a_y e^{-i\mathcal{H}_N t} W(\sqrt{N}\varphi)\Omega\right\rangle\end{aligned}$$

with $d_N \simeq N^{1/4}$. We introduce fluctuation dynamics:

$$\begin{aligned}\gamma^{(1)}_{N,t}(x,y) &= \frac{1}{N}\left\langle \xi, \mathcal{U}(t)(a^*_x + \sqrt{N}\overline{\varphi}_t(x))(a_y + \sqrt{N}\varphi_t(y))\mathcal{U}^*(t)\Omega\right\rangle \\ &= \overline{\varphi}_t(x)\varphi_t(y) + \frac{1}{N}\langle \xi, \mathcal{U}^*(t) a^*_x a_y \mathcal{U}(t)\Omega\rangle \\ &\quad + \frac{\overline{\varphi}_t(x)}{\sqrt{N}}\langle \xi, \mathcal{U}^*(t) a_y \mathcal{U}(t)\Omega\rangle + \frac{\varphi_t(y)}{\sqrt{N}}\langle \xi, \mathcal{U}^*(t) a^*_x \mathcal{U}(t)\Omega\rangle\end{aligned}$$

with

$$\xi = d_N W^*(\sqrt{N}\varphi)\frac{(a^*(\varphi))^N}{\sqrt{N!}}\Omega$$

As before, the problem reduces to controlling the growth of

$$\langle \xi, \mathcal{U}^*(t)\mathcal{N}\mathcal{U}(t)\Omega\rangle$$

uniformly in N.

Using the estimate

$$\|(\mathcal{N}+1)^{-1}\xi\| \lesssim 1,$$

and the bounds

$$\left\langle \psi, \mathcal{U}^*(t)\,(\mathcal{N}+1)^k\,\mathcal{U}(t)\psi \right\rangle \leq C e^{K|t|}\langle \psi, (\mathcal{N}+1)^{2k+2}\,\psi\rangle$$

one obtains:

Theorem [Chen, Lee, S. (2011)]: For every $k \in \mathbb{N}$, $t \in \mathbb{R}$, there exist constants $C = C(k)$ and $K = K(k)$ such that

$$\text{Tr}\left|\gamma_{N,t}^{(k)} - |\varphi_t\rangle\langle\varphi_t|^{\otimes k}\right| \leq \frac{C e^{K|t|}}{N}$$

A probabilistic setting: For a self-adjoint J on $L^2(\mathbb{R}^3)$, let

$$\mathcal{J} = \sum_{i=1}^{N} J^{(i)} \qquad \text{with } J^{(i)} = 1 \otimes \cdots \otimes J \otimes \cdots \otimes 1$$

For example, if $J = \chi_A(x)$, for $A \subset \mathbb{R}^3$, $\mathcal{J}$ measures the number of particles in A.

At time $t = 0$, $\psi_N = \varphi^{\otimes N}$, and $\mathcal{J}$ is a sum of iid random variables. Hence, we have a law of large numbers:

$$\mathbb{P}_{\varphi^{\otimes N}} \left(\left| \frac{1}{N} \sum_{i=1}^{N} (J^{(i)} - \langle \varphi, J\varphi \rangle) \right| \geq \delta \right) \to 0 \qquad \text{as } N \to \infty$$

and a central limit theorem:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (J^{(i)} - \langle \varphi, J\varphi \rangle) \to N(0, \sigma^2), \quad \text{with } \sigma^2 = \langle \varphi, J^2\varphi \rangle - \langle \varphi, J\varphi \rangle^2$$

What happens at time $t \neq 0$?

The law of large number is still correct. In fact, with

$$\widetilde{J} = J - \langle \varphi_t, J\varphi_t \rangle,$$

we find

$$\begin{aligned}
\mathbb{P}_{\psi_{N,t}}\left(\left| \frac{1}{N} \sum_{i=1}^{N} \widetilde{J}^{(i)} \right| \geq \delta \right) &\leq \frac{1}{\delta^2 N^2} \left\langle \psi_{N,t}, \left(\sum_{i=1}^{N} \widetilde{J}^{(i)} \right)^2 \psi_{N,t} \right\rangle \\
&= \frac{1}{\delta^2} \operatorname{Tr} \gamma_{N,t}^{(2)} (\widetilde{J} \otimes \widetilde{J}) + \frac{1}{\delta^2 N} \operatorname{Tr} \gamma_{N,t}^{(1)} \widetilde{J}^2 \\
&\to \frac{1}{\delta^2} \operatorname{Tr} |\varphi_t\rangle\langle\varphi_t|^2 (\widetilde{J} \otimes \widetilde{J}) = 0
\end{aligned}$$

as $N \to \infty$.

Natural question: does a central limit theorem hold w.r.t. $\psi_{N,t}$?

Theorem [Ben Arous, Kirkpatrick, S. (2011)]: W.r.t. the wave function $\psi_{N,t}$ the random variable

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(J^{(i)}-\langle\varphi_t, J\varphi_t\rangle\right)$$

converges in distribution, as $N\to\infty$ to a centered Gaussian random variable with variance

$$\sigma_t^2=\left[\left\langle\Theta_t\left(J\varphi_t,\overline{J\varphi_t}\right),\Theta_t\left(J\varphi_t,\overline{J\varphi_t}\right)\right\rangle-\left|\left\langle\Theta_t\left(J\varphi_t,\overline{J\varphi_t}\right),\frac{1}{\sqrt{2}}\left(\varphi,\overline{\varphi}\right)\right\rangle\right|^2\right]$$

Equivalently,

$$\sigma_t^2=\|U_tJ\varphi_t+\overline{V}_tJ\varphi_t\|^2-|\langle\varphi, U_tJ\varphi_t+\overline{V}_tJ\varphi_t\rangle|^2\geq 0$$

So, w.r.t. $\psi_{N,t}$ central limit theorem still holds true, but the variance changes.

Ideas from proof: compute moments in the limit $N \to \infty$.

For example,

$$\mathbb{E}_{\psi_{N,t}} \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(J^{(i)} - \langle \varphi_t, J \varphi_t \rangle \right) \right)^2 = \text{Tr}\, \gamma_{N,t}^{(1)}\, \widetilde{J}^2 + N\, \text{Tr}\, \gamma_{N,t}^{(2)} \left(\widetilde{J} \otimes \widetilde{J} \right)$$

First term gives $\|\widetilde{J}\varphi_t\|^2$, the result we would find for factorized wave function $\varphi_t^{\otimes N}$.

Second term gives contribution from correlations. It can be computed writing

$$N\, \text{Tr}\, \gamma_{N,t}^{(2)} \left(\widetilde{J} \otimes \widetilde{J} \right) = N \int \widetilde{J}(x_1, x_1') \widetilde{J}(x_2, x_2')\, \gamma_{N,t}^{(2)}(x_1', x_2'; x_1, x_2)$$

and

$$\gamma_{N,t}^{(2)}(x_1', x_2'; x_1, x_2) = \frac{1}{N^2} \left\langle \psi_{N,t}, a_{x_1}^* a_{x_2}^* a_{x_1'} a_{x_2'} \psi_{N,t} \right\rangle$$

As before, we put

$$\xi = d_N \, W^*(\sqrt{N}\varphi)\frac{a^*(\varphi)^N}{\sqrt{N!}}\Omega$$

Then

$$\begin{aligned}
& N \,\mathrm{Tr}\, \gamma_{N,t}^{(2)} \left(\widetilde{J} \otimes \widetilde{J}\right) \\
& \quad = \frac{1}{N} \int \widetilde{J}(x_1, x_1')\widetilde{J}(x_2, x_2') \\
& \qquad \times \Big\langle \xi, \mathcal{U}^*(t) \, (a_{x_1}^* + \sqrt{N}\bar{\varphi}_t(x_1))(a_{x_2}^* + \sqrt{N}\bar{\varphi}_t(x_2)) \\
& \qquad\qquad \times (a_{x_1'} + \sqrt{N}\varphi_t(x_1'))(a_{x_2'} + \sqrt{N}\varphi_t(x_2'))\mathcal{U}(t)\Omega \Big\rangle
\end{aligned}$$

Counting ξ as order one, only terms with at least 2 factors φ_t survive the limit $N \to \infty$.

On other hand, all terms with more than 2 φ_t factors vanish, because $\langle \varphi_t, \widetilde{J}\varphi_t \rangle = 0$.

We are left with

$$\begin{aligned} N\,\mathrm{Tr}\,\gamma_{N,t}^{(2)}\left(\tilde{J}\otimes\tilde{J}\right) &= \langle \xi, \mathcal{U}^*(t) : \left(a^*(\tilde{J}\varphi_t) + a(\tilde{J}\varphi_t)\right)^2 : \mathcal{U}(t)\Omega\rangle \\ &\simeq \langle \xi, \mathcal{U}_\infty^*(t) A(\tilde{J}\varphi_t, \overline{\tilde{J}\varphi_t})^2 \mathcal{U}_\infty(t)\Omega\rangle - \|\tilde{J}\varphi_t\|^2 \\ &= \langle \xi, A(\Theta_t(\tilde{J}\varphi_t, \overline{\tilde{J}\varphi_t}))^2\Omega\rangle - \|\tilde{J}\varphi_t\|^2 \end{aligned}$$

Since $\xi \simeq \Omega - \frac{1}{2}a^*(\varphi)^2\Omega + \dots$, we conclude

$$\begin{aligned} N\,\mathrm{Tr}\,\gamma_{N,t}^{(2)}\left(\tilde{J}\otimes\tilde{J}\right) = &\langle \Omega, A(\Theta_t(\tilde{J}\varphi_t, \overline{\tilde{J}\varphi_t}))^2\Omega\rangle \\ &- \frac{1}{2}\langle a^*(\varphi)^2\Omega, A(\Theta_t(\tilde{J}\varphi_t, \overline{\tilde{J}\varphi_t}))^2\Omega\rangle - \|\tilde{J}\varphi_t\|^2 \end{aligned}$$

and therefore

$$\begin{aligned} &\mathbb{E}_{\psi_{N,t}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(J^{(i)} - \langle\varphi_t, J\varphi_t\rangle\right)\right)^2 \\ &\to \left[\left\langle\Theta_t\left(J\varphi_t, \overline{J\varphi_t}\right), \Theta_t\left(J\varphi_t, \overline{J\varphi_t}\right)\right\rangle - \left|\left\langle\Theta_t\left(J\varphi_t, \overline{J\varphi_t}\right), \frac{1}{\sqrt{2}}\,(\varphi, \overline{\varphi})\right\rangle\right|^2\right] \end{aligned}$$