Best constants for Lieb-Thirring like inequalities in cylinders and their relation to symmetry properties of extremals for the Caffarelli-Kohn-Nirenberg inequalities

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Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, and $a \neq \frac{d-2}{2}$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{\, |x|^{-b}\,u \in L^p(\mathbb{R}^d, dx)\ :\ |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^d, dx)\right\}$$

$$b - a \;\to\; 0 \iff p \to \tfrac{2d}{d-2}$$

$$b-(a+1) \;\to\; 0 \iff p \;\to\; 2_+$$

$$\frac{1}{C_{a,b}} = \inf_{\mathcal{D}_{a,b}} \quad \frac{\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx}{\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p}}$$

The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\;dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\;dx \qquad \forall\; u \in \mathcal{D}_{a,b}$$

$\mathsf{C}_{a,b} =$ best constant for general functions u
$\mathsf{C}^*_{a,b} =$ best constant for radially symmetric functions u

$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

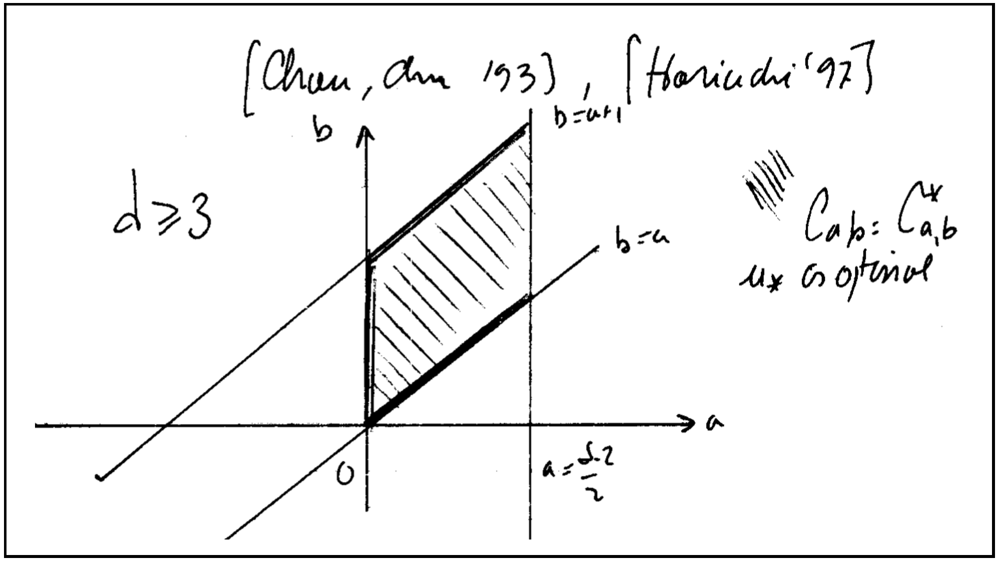
Up to scalar multiplication and dilation, the optimal radial function is

$$u^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

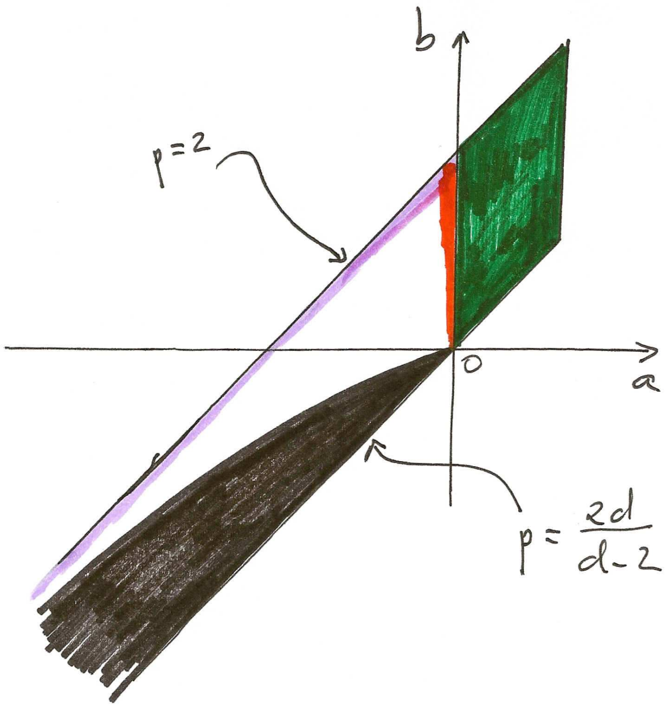
Questions: is optimality (equality) achieved ? do we have $u_{a,b} = u^*_{a,b}$?

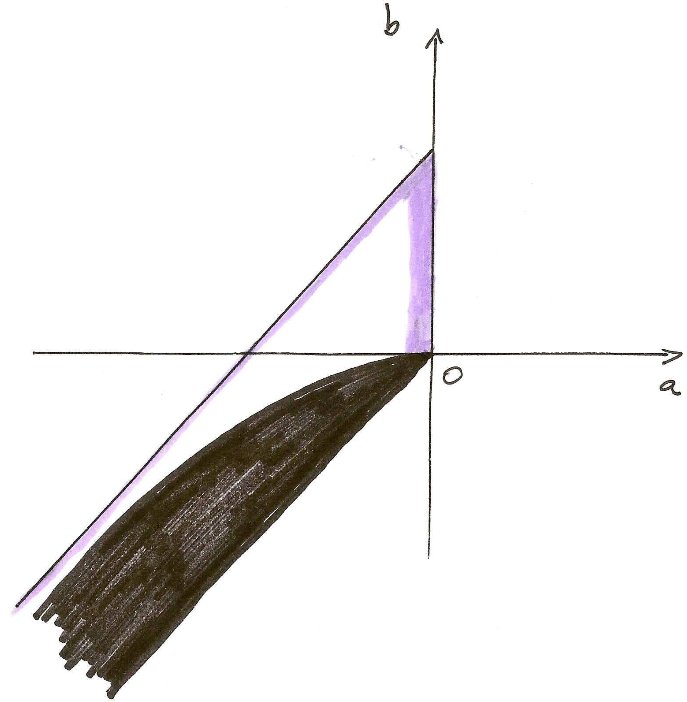
Known results (Aubin, Talenti, Lieb, Chou-Chu, Lions, ...)

- Existence inside the half-strip $a < b < a + 1\,,\ a < \frac{d-2}{2}$
- Symmetry (and existence) in the zone $a \le b < a + 1\,,\ 0 < a < \frac{d-2}{2}$
- Nonexistence for $a < 0$ and $b = a$ or $b = a + 1$.



Symmetry and symmetry breaking





SYMMETRY BREAKING: Catrina-Wang, Felli-Schneider.

Aubin, Talenti, Horiuchi, Lieb, Chou-Chu,...

Lin, Wang; Dolbeault, E., Tarantello (d=2)

Dolbeault, E., Loss, Tarantello

$$b^{FS}(a) = \frac{d(d-2-2a)}{2\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{1}{2}(d-2-2a)$$

Emden-Fowler transformation and the cylinder $\mathcal{C} = \mathbb{R} \times S^{d-1}$

$$t = \log|x| \;, \quad \omega = \frac{x}{|x|} \in S^{d-1} \;, \quad v(t,\omega) = |x|^{-a}\, u(x) \;, \quad \Lambda = \frac{1}{4}\,(d-2-2a)^2$$

• Caffarelli-Kohn-Nirenberg inequalities rewritten on the cylinder become standard interpolation inequalities of Gagliardo-Nirenberg type

$$\|v\|^2_{L^p(\mathcal{C})} \leq C_{\Lambda,p} \left[\|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\, \|v\|^2_{L^2(\mathcal{C})}\right]$$

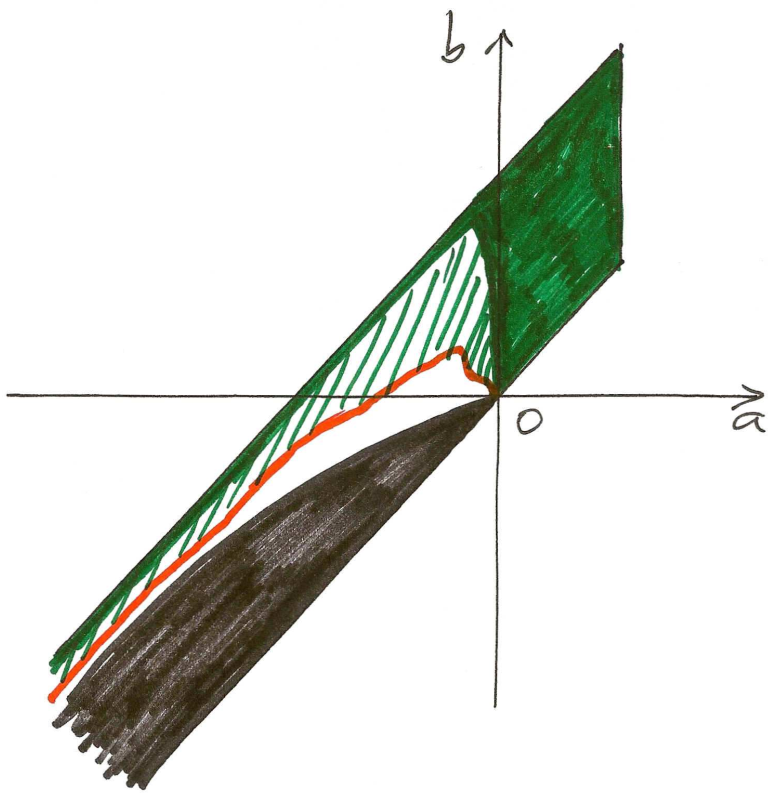
$$\mathcal{E}_\Lambda[v] := \|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\, \|v\|^2_{L^2(\mathcal{C})}$$

$${C_{\Lambda,p}}^{-1} := \mathsf{C}_{a,b}^{-1} = \inf\left\{\mathcal{E}_\Lambda(v) \;:\; \|v\|^2_{L^p(\mathcal{C})} = 1\right\}$$

$$a < \frac{d-2}{2} \implies \Lambda > 0 \;, \qquad a < 0 \implies \Lambda > \tfrac{1}{4}\,(d-2)^2$$

Two simply connected regions separated by a continuous curve

The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve.



Open question. Do the curves obtained by Felli-Schneider and ours coincide ?

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $d \geq 3$. For any $p \in [2, p(\theta,d) := \frac{2d}{d-2\theta}]$, there exists a positive constant $\mathsf{C}(\theta,p,a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a-a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$ and (see [Del Pino, Dolbeault, Filippas, Tertikas]):

$$\mathsf{C}_{\mathrm{CKN}}(\theta,p,a) \geq \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\mathrm{CKN}}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}} \left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}} \left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

New symmetry breaking results

- Gagliardo-Nirenberg interpolation inequalities: if $p \in (2, 2^*)$,

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)$$

If u is a radial minimizer for $1/\mathsf{C}_{\mathrm{GN}}(p)$ and $u_n(x) := u(x + n\,\mathsf{e})$, $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d),p,a)} \le \frac{\||x|^{-a}\,\nabla u_n\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\||x|^{-(a+1)}\,u_n\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)}}{\||x|^{-b}\,u_n\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}$$

$$= \frac{1}{\mathsf{C}_{\mathrm{GN}}(p)}\,\left(1 + \mathcal{R}\,n^{-2} + O(n^{-4})\right)$$

So, $\frac{1}{\mathsf{C}_{\mathrm{CKN}}} \le \frac{1}{\mathsf{C}_{\mathrm{GN}}}$.

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$$= \frac{1}{\mathsf{C}_{\mathrm{GN}}(p)}\, \left(1 + \mathcal{R}\,n^{-2} + O(n^{-4})\right)$$

So, $\frac{1}{\mathsf{C}_{\mathrm{CKN}}} \le \frac{1}{\mathsf{C}_{\mathrm{GN}}}$.

If we are able to find a positive Λ and a function g such that

$$\frac{1}{\mathsf{C}_{\mathrm{GN}}} \le \mathcal{E}_{GN}[g] < \frac{1}{\mathsf{C}^*_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)}\,, \quad \text{then,}$$

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)} \le \frac{1}{\mathsf{C}_{\mathrm{GN}}} < \frac{1}{\mathsf{C}^*_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda)}$$

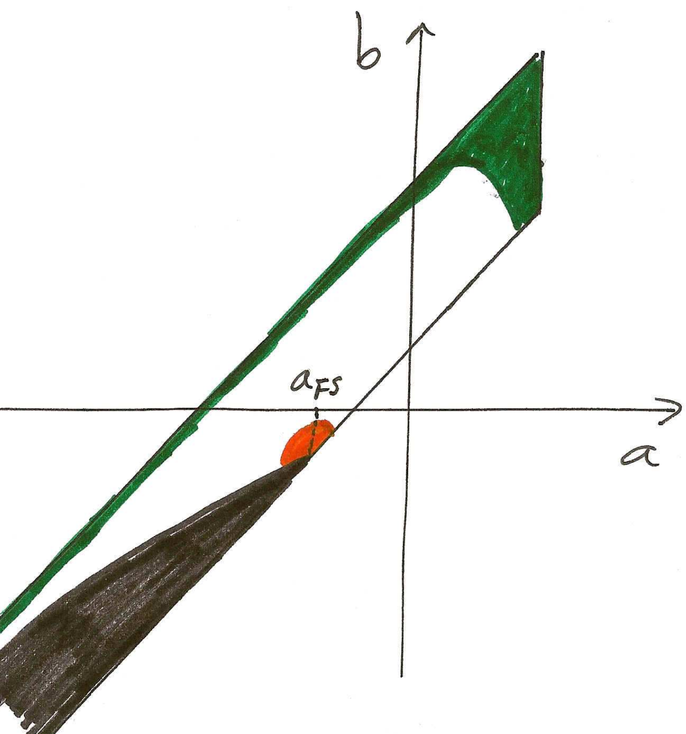
A new symmetry breaking result (2010, Dolbeault, E., Tarantello, Tertikas)

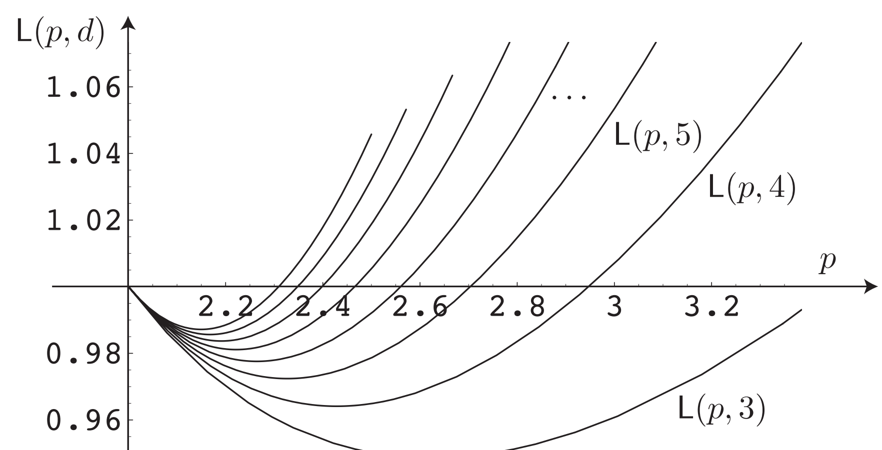
Let $\mathsf{g}(x) := (2\,\pi)^{-d/4}\,\exp(-|x|^2/4)$. Choose $\Lambda = \Lambda_{FS}(p(\theta,d),d)$

Symmetry breaking occurs if

$$\mathsf{L}(p,d) := \frac{\mathcal{E}_{GN}[g]}{\frac{1}{\mathsf{C}^*_{\rm CKN}(\vartheta(p,d),p,\Lambda)}} < 1$$

We have the following result:





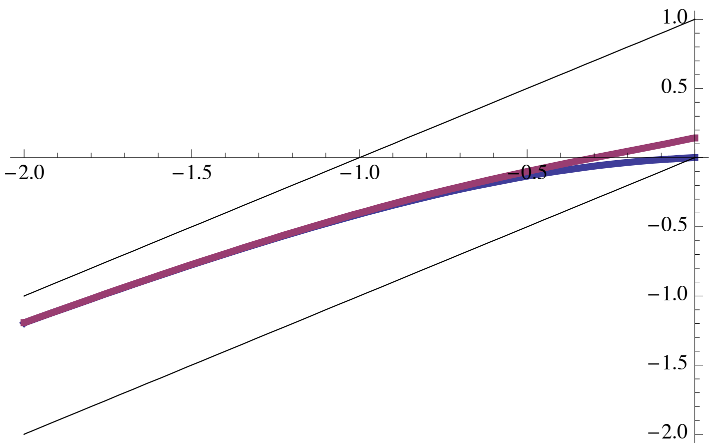
A new symmetry result (June 2011, Dolbeault, E., Loss)

For $\theta = 1$ and $d \geq 2$,

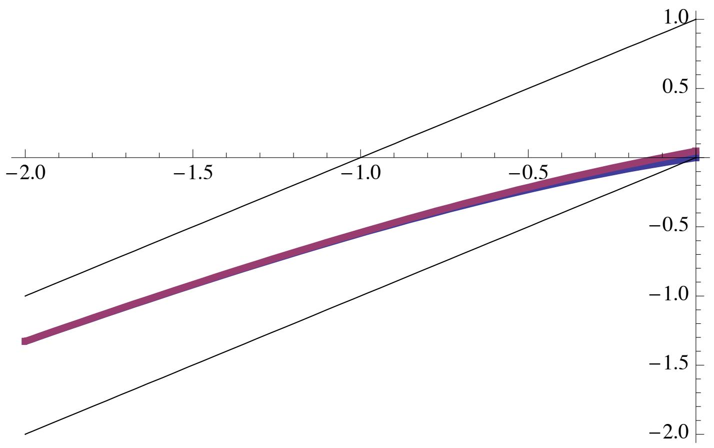
there exists a unique minimizer for the (CKN) problem, and it is symmetric, for all $\Lambda \leq \tilde{\Lambda}(p)$, for all $p \in (2, \frac{2d}{d-2})$.

$$\tilde{\Lambda}(p) := \frac{(d-1)(6-p)}{4(p-2)} < \Lambda_{FS}(p)\,.$$

$d = 3$



$d = 5$



Strategy of the proofs

Let L^2 be the Laplace-Beltrami operator on S^{d-1}. So that $-\Delta$ on the cylinder becomes $-\partial_s^2 - L^2$.

THEOREM. Let $d \geq 2$ and let u be a non-negative function on $\mathcal{C} = \mathbb{R} \times S^{d-1}$ that satisfies

$$-\partial_s^2 v - L^2 v + \Lambda\, v = v^{p-1}$$

and consider the symmetric solution v_* . Assume that

$$\int_{\mathcal{C}} |v(s,\omega)|^p \, ds\, d\omega \leq \int_{\mathbb{R}} |v_*(s)|^p \, ds$$

for some $2 < p < 6$ satisfying $p \leq \frac{2\,d}{d-2}$. If $\Lambda \leq \tilde{\Lambda}(p)$, then for a.e. $\omega \in S^{d-1}$ and $s \in \mathbb{R}$, we have $v(s,\omega) = v_*(s - C)$ for some constant C .

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REMARK 1. With the above normalization, we have

$$\frac{1}{C_{\Lambda,p}} = \inf \quad \frac{\int_{\mathcal{C}} |\nabla v|^2 + \Lambda\, v^2\ dx}{\left(\int_{\mathcal{C}} |v|^p\ dx\right)^{2/p}} = \left(\int_{\mathcal{C}} |v(s,\omega)|^p\, ds\, d\omega\right)^{\frac{p-2}{p}} \quad .$$

REMARK 2. We choose $d\omega$ to be a probability mesaure on S^{d-1}.

(Keller) - Lieb-Thirring in 1-d

LEMMA. Let $V = V(s)$ be a non-negative real valued potential in $\mathrm{L}^{\gamma+\frac{1}{2}}(\mathbb{R})$ for some $\gamma > 1/2$ and let $-\lambda_1(V)$ be the lowest eigenvalue of the Schrödinger operator $-\frac{d^2}{ds^2} - V$. Define

$$c_{\mathrm{LT}}(\gamma) = \frac{\pi^{-1/2}}{\gamma - 1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma+1/2} .$$

Then

$$\lambda_1(V)^\gamma \leq c_{\mathrm{LT}}(\gamma) \int_{\mathbb{R}} V^{\gamma+1/2}(s)\, ds$$

with equality if and only if, up to scalings and translations,

$$V(s) = \frac{\gamma^2 - 1/4}{\cosh^2(s)} =: V_0(s)$$

in which case

$$\lambda_1(V_0) = (\gamma - 1/2)^2 .$$

Furthermore, the corresponding ground state eigenfunction is given by

$$\psi_\gamma(s) = \pi^{-1/4} \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - 1/2)}\right)^{1/2} \left[\cosh(s)\right]^{-\gamma+1/2} .$$

With $V = v^{p-2}$, the equation $-\Delta v + \Lambda v = v^{p-1}$ can be seen as the “linear" equation $-\Delta v - V v = -\Lambda v$.

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Let us define $f(\omega) := \sqrt{\int_{\mathbb{R}} |v(s,\omega)|^2 \; ds}$. By the Lieb-Thirring Lemma, we find that a.e. in ω,

$$-\Lambda \int_{\mathcal{C}} |v(s,\omega)|^2 \, ds \, d\omega = \int_{S^{d-1}} \int_{\mathbb{R}} \left(v_s^2 - v^p \right) \, ds \, d\omega + \int_{\mathcal{C}} |Lv|^2 \, ds \, d\omega$$

$$\int_{S^{d-1}} \int_{\mathbb{R}} \left(v_s^2 - V \, v^2 \right) \, ds \, d\omega + \int_{\mathcal{C}} |Lv|^2 \, ds \, d\omega =: \mathcal{F}[v] \; .$$

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$$\mathcal{F}[v] \geq -\, c_{\mathrm{LT}}(\gamma)^{1/\gamma} \int_{S^{d-1}} \left(\int_{\mathbb{R}} |v(s,\omega)|^p \, ds \right)^{1/\gamma} |f|^2 + \int_{S^{d-1}} |L f|^2 \, d\omega \, .$$

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$$\mathcal{F}[v] \geq - c_{\mathrm{LT}}(\gamma)^{1/\gamma} \int_{S^{d-1}} \left(\int_{\mathbb{R}} |v(s,\omega)|^p \, ds \right)^{1/\gamma} |f|^2 + \int_{S^{d-1}} |Lf|^2 \, d\omega \, .$$

Now, setting $D := c_{\mathrm{LT}}(\gamma)^{1/\gamma} \left(\int_{\mathcal{C}} v^p \; ds \, d\omega \right)^{\frac{1}{\gamma}}$, by using Hölders's inequality, we obtain

$$\mathcal{F}[v] \geq \int_{S^{d-1}} (Lf)^2 \, d\omega - D \left(\int_{S^{d-1}} f^{\frac{2\,\gamma}{\gamma-1}} \, d\omega \right)^{\frac{\gamma-1}{\gamma}} =: \mathcal{E}[f] \; .$$

The generalized Poincaré inequality on the sphere states that for all $q \in \left(1, \frac{d+1}{d-3}\right]$,

$$\frac{q-1}{d-1} \int_{S^{d-1}} (Lf)^2 \, d\omega \geq \left(\int_{S^{d-1}} f^{q+1} \, d\omega \right)^{\frac{2}{q+1}} - \int_{S^{d-1}} f^2 \, d\omega$$

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Choosing $q + 1 = \frac{2\,\gamma}{\gamma - 1} = 2\,\frac{p+2}{6-p}$,

$$\mathcal{E}[f] \geq \left(\frac{d-1}{q-1} - D \right) \left(\int_{S^{d-1}} f^{q+1} \, d\omega \right)^{\frac{2}{q+1}} - \frac{d-1}{q-1} \int_{S^{d-1}} f^2 \, d\omega \; .$$

To justify this step, we notice that $q \leq \frac{d+1}{d-3}$ is equivalent to $p \leq \frac{2\,d}{d-2}$.

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$$\left(\int_{S^{d-1}} f^{q+1} \, d\omega \right)^{\frac{2}{q+1}} \geq \int_{S^{d-1}} f^2 \, d\omega \; .$$

Thus, if $D \leq \frac{d-1}{q-1}$, and if $\Lambda \leq \tilde{\Lambda}(p)$, we get

$$-\Lambda \int_{S^{d-1}} f^2 \, d\omega \geq \mathcal{E}[f] \geq -D \int_{S^{d-1}} f^2 \, d\omega \geq -\Lambda \int_{S^{d-1}} f^2 \, d\omega \; .$$

Consequence of the proof of the above theorem

COROLLARY. Let $d \geq 2$. Fix $\gamma > 1$ such that $\gamma \geq \frac{d-1}{2}$ if $d \geq 4$ and let $q = \frac{\gamma+1}{\gamma-1}$. Further fix $D \leq \frac{d-1}{q-1}$. Among all potentials $V = V(s,\omega)$ with

$$c_{\rm LT}(\gamma)^{\frac{1}{\gamma}} \left(\int_{\mathcal{C}} V^{\gamma+\frac{1}{2}}\, ds\, d\omega \right)^{\frac{1}{\gamma}} = D \;,$$

the potential $V = V_*$ that minimizes the first eigenvalue of $-\partial_s^2 - L^2 - V$ on $L^2(\mathcal{C}, ds\, d\omega)$ does not depend on ω . Moreover, $u_* = V_*^{(2\,\gamma-1)/4}$ is extremal for the CKN inequality in the cylinder.

Remark. $V = v^{p-2}$ and $V^{\gamma+\frac{1}{2}} = v^p$ implies $\gamma = \frac{1}{2}\,\frac{p+2}{p-2}$.

and with $\gamma = \frac{1}{2}\,\frac{p+2}{p-2}$, $u_* = V_*^{(2\,\gamma-1)/4}$ is equivalent to $V_* = u_*^{p-2}$.

The d-dimensional case I

Both $C(\Lambda,p,d)$ and $C^*(\Lambda,p,d)$ are monotone non-increasing functions of Λ and

$$C(\Lambda,p,d) \geq C^*(\Lambda,p,d)\ .$$

$$C^*(\Lambda,p,d) = C^*(1,p,d)\,\Lambda^{-\frac{p+2}{2\,p}}\ ,$$

so that $\lim_{\Lambda\to 0_+} C^*(\Lambda,p,d) = \infty$.

For any $p\in(2,\frac{2\,d}{d-2})$ if $d\geq 3$, and any $p>2$ if $d=2$,

$$\lim_{\Lambda\to\infty} \frac{\Lambda^{\frac{d-2}{2}-\frac{d}{p}}}{C(\Lambda,p,d)} = \inf_{w\in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\int_{\mathbb{R}^d}\left(|\nabla u|^2+|u|^2\right)\,dx}{\left(\int_{\mathbb{R}^d}|u|^p\,dx\right)^{2/p}} \quad\Longrightarrow\quad \lim_{\Lambda\to\infty} C(\Lambda,p,d) = 0\ .$$

With these observations in hand and $\gamma = \frac{1}{2}\,\frac{p+2}{p-2}$, we can define

$$\Lambda^d_\gamma(\mu) := \inf\left\{\Lambda>0\,:\,\mu^{\frac{2\,\gamma}{2\,\gamma+1}} = 1/C(\Lambda,p,d)\right\}\ .$$

If $d=1$, we observe that $C(\Lambda,p,1) = C^*(\Lambda,p,1)$, so that $\Lambda^1_\gamma(\mu) = \Lambda^1_\gamma(1)\,\mu$ and $\Lambda^1_\gamma(1) = C^*(1,p,d)^{\frac{2\,p}{p+2}}$.

The d-dimensional case II

$$\Lambda^d_\gamma(\mu) := \inf\left\{\Lambda > 0 \,:\, \mu^{\frac{2\,\gamma}{2\,\gamma+1}} = 1/C(\Lambda,p,d)\right\}\;.$$

Next important point: $\lambda_1(V)$ can be estimated using $\Lambda^d_\gamma(\mu)$ provided V is controlled in terms of μ . The CKN inequality in the cylinder is equivalent to the following version of the Keller - Lieb-Thirring inequality.

Theorem. For any $\gamma \in (2,\infty)$ if $d=1$, or for any $\gamma \in (1,\infty)$ such that $\gamma \geq \frac{d-1}{2}$ if $d \geq 2$, if V is a non-negative potential in $\mathrm{L}^{\gamma+\frac{1}{2}}(\mathcal{C})$, then the operator $-\partial^2 - L^2 - V$ has at least one negative eigenvalue, and its lowest eigenvalue, $-\lambda_1(V)$, satisfies

$$\lambda_1(V) \leq \Lambda^d_\gamma(\mu) \quad \text{with} \quad \mu = \mu(V) := \left(\int_{\mathcal{C}} V^{\gamma+\frac{1}{2}}\,ds\,d\omega\right)^{\frac{1}{\gamma}}\;.$$

Moreover, equality is achieved if and only if the eigenfunction u corresponding to $\lambda_1(V)$ satisfies $u = V^{(2\,\gamma-1)/4}$ and u is optimal for CKN inequalities in the cylinder.

Remark. $V = v^{p-2}$ and $V^{\gamma+\frac{1}{2}} = v^p$ implies $\gamma = \frac{1}{2}\,\frac{p+2}{p-2}$.

and with $\gamma = \frac{1}{2}\,\frac{p+2}{p-2}$, $u = V^{(2\,\gamma-1)/4}$ is equivalent to $V = u^{p-2}$.