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Tutorial Sheet 10
Algebra 2

Summer Semester 2026

Exercise 1. Let $A \subseteq B$ be an integral ring extension and let $S \subseteq A$ be a multiplicatively closed subset. Show that $S^{-1}A \subseteq S^{-1}B$ is an integral ring extension. Is the converse true?

Exercise 2. Prove or disprove the following statements.

(i) Every UFD is integrally closed.

(ii) $\mathbb{Q}$ is an integral extension of $\mathbb{Z}$.

(iii) $\mathbb{Z}[i]$ is integral over $\mathbb{Z}$.

(iv) $\alpha = \frac{3+\sqrt{5}}{2-\sqrt{5}} \in \mathbb{C}$ is integral over $\mathbb{Z}$.

(v) $k[x, y]$ is integrally closed.

(vi) $\mathbb{Z}$ is integrally closed in $\mathbb{Q}(\sqrt{2})$.

(vii) If $A \subseteq B \subseteq C$ and C is integral over A, then C is integral over B.

(viii) If $A \subseteq B \subseteq C$ and B is integral over A and C is integral over B, then C is integral over A.

(ix) Let A be an integral domain and let G be a finite group of automorphisms of A. Then A is integral over A^G.

Definition. Let K be a field and let Γ be a totally ordered abelian group, written additively. A *valuation* on K is a group homomorphism $v : K^\times \longrightarrow \Gamma$ such that

$$v(x + y) \geq \min\{v(x), v(y)\}$$

whenever $x, y \in K^\times$ and $x + y \neq 0$. We extend v to all of K by setting $v(0) = \infty$, where $\infty > \gamma$ for all $\gamma \in \Gamma$.

Exercise 3. (a) Let A be an integral domain with field of fractions K. Show that the following are equivalent.

(i) Any two ideals of A are comparable with respect to inclusion.

(ii) For every $x \in K^\times$ we have $x \in A$ or $x^{-1} \in A$.

(iii) There exists a totally ordered abelian group Γ and a valuation $v : K^\times \longrightarrow \Gamma$ such that $A = \{x \in K \mid v(x) \geq 0\}$.
Hint: Let $\Gamma := K^\times/A^\times$. For $a, b \in K^\times$ define $aA^\times \leq bA^\times :\Longleftrightarrow \frac{b}{a} \in A$. Show that this defines a well-defined total order on Γ which is compatible with the group structure. Then define the obvious valuation.

A ring A satisfying the equivalent conditions from part (a) is called a *valuation ring*.

(b) Conclude that every valuation ring is a local ring.

(c) Conclude that every valuation ring is integrally closed.

(d) Conclude that every finitely generated ideal of a valuation ring is principal. In particular, every noetherian valuation ring which is not a field is a discrete valuation ring.