LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof. Dr. Fabien Morel
Anton Zakrewski

Tutorial Sheet 9
Algebra 2

Summer Semester 2026

Exercise 1. Let M, N be A-modules over a local ring $(A, \mathfrak{m})$, $f\colon M \to N$ a module homomorphism. Suppose furthermore, that N, M are finitely generated modules.

(i) Assume that $f \otimes_A A/\mathfrak{m}\colon M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$ is surjective. Prove or disprove that f is in general surjective.
(ii) Assume that $f \otimes_A A/\mathfrak{m}\colon M \otimes_A A/\mathfrak{m} \to N \otimes_A A/\mathfrak{m}$ is injective. Prove or disprove that f is in general injective.
(iii) Where (if possible) can we weaken the finite generation assumptions ?

Exercise 2. Let K be a field and A a (proper) subring, which is not a field. Show that K is not finitely generated as module over A.

Exercise 3. Let P, Q be finitely generated projective modules over a local ring $(A, \mathfrak{m})$. Suppose there exists an isomorphism $\tilde{f}\colon P \otimes_A A/\mathfrak{m} \cong Q \otimes_A A/\mathfrak{m}$.

(i) Show that there exists a morphism $f\colon P \to Q$ such that $f \otimes_A A/\mathfrak{m} = \tilde{f}$,
(ii) Show f is an epimorphism,
(iii) Show f is an isomorphism,
(iv) Conclude that $K_0(A) \cong \mathbb{Z}$.

Hint: You may use the Splitting Lemma

Lemma 1 (Splitting Lemma (Ex. Sheet 4, Exercise 1))**.**
Let $0 \to M_1 \xhookrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$ be a short exact sequence of A-modules. Then the following are equivalent:

(i) The map $M_1 \xhookrightarrow{f_1} M_2$ admits an A-linear left inverse $p : M_2 \to M_1$, i.e., $p \circ f_1 = \mathrm{id}_{M_1}$,
(ii) The map $M_2 \xrightarrow{f_2} M_3$ admits an A-linear right inverse $\iota : M_3 \to M_2$, i.e., $f_2 \circ \iota = \mathrm{id}_{M_3}$,
(iii) There exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xhookrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\ & & \| & & \downarrow{\cong} & & \| & & \\ 0 & \longrightarrow & M_1 & \xhookrightarrow{(\mathrm{id},0)} & M_1 \oplus M_3 & \xrightarrow{(0,\mathrm{id})} & M_3 & \longrightarrow & 0 \end{array}$$

Exercise 4 (Bonus[1] Exercise[2])**.** Let A be an abelian group and $0 \to \bigoplus_I \mathbb{Z} \hookrightarrow \bigoplus_J \mathbb{Z} \twoheadrightarrow A \to 0$ a short exact sequence[3].

[1]this exercise is not relevant for the exam
[2]this exercise essentially shows that the derived Hom functor $R\mathrm{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$ is conservative
[3]one can show that over a PID, submodules of free Modules are again free. Hence each abelian group admits such a short exact sequence.

Suppose that the projection $\mathrm{pr}\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ induces an isomorphism $\mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}) \to \mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z})$.

Show that A is the zero group.

Hint: Show that $\mathrm{Hom}_{\mathrm{Ab}}(A, \mathbb{Q}/\mathbb{Z}) \cong 0$ *if and only if* $A = 0$.

Hint: Recall that both $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ *are injective modules over* $\mathbb{Z}$ *and recall/prove that an* A*-module* M *is injective if and only if* $\mathrm{Hom}_{\mathrm{Mod}_A}(-, M)$ *is exact.*

Alternative Hint: Recall/Prove that an abelian group A *admits a (unique)* $\mathbb{Q}$*-vector space structure if and only if* A *is uniquely divisible, that is, multiplication with* p *is an isomorphism for all primes* p.