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Tutorial Sheet 3
Algebra 2

Summer Semester 2026

All rings in this tutorial sheet are assumed to be commutative and unital.

Exercise 1. 1) Let R be a ring and M an R-module. Show that the following statements are equivalent:

(i) Every submodule of M is finitely generated over R.

(ii) Every ascending chain of submodules of M becomes stationary.

(iii) Every nonempty set of R-submodules of M contains a maximal element with respect to inclusion.

An R-module satisfying one of these equivalent properties is called noetherian.

2) Let R be a ring and let M be a finitely generated R-module. Is M necessarily a noetherian R-module?

3) Let M be a noetherian R-module and let $f : M \to M$ be an epimorphism in R-**Mod**. Show that f is injective, and hence an isomorphism.

Hint: Consider the ascending chain

$$\ker(f) \subseteq \ker(f^2) \subseteq \cdots \subseteq \ker(f^n) \subseteq \ldots$$

Exercise 2. 1) Let R be a ring and

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Prove that M is noetherian if and only if M' and M'' are noetherian.

Hint: Use property (ii) from Exercise 1.

2) Conclude that M_i, $1 \leq i \leq n$, are noetherian R-modules if and only if $\bigoplus_{i=1}^n M_i$ is a noetherian R-module.

3) Conclude that if R is noetherian and M is a finitely generated R-module, then M is noetherian.

Remark: Note that quotients of noetherian R-modules are again noetherian. This can be shown in precisely the same way as in the previous tutorial sheet, so you do not have to prove this again.

Exercise 3. 1) Let R be a ring and let M be an R-module. Show that the functor $\mathrm{Hom}_{R\text{-}\mathbf{Mod}}(M, -) = \mathrm{Hom}_R(M, -)$ is left exact, i.e. if

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

is a short exact sequence, then

$$0 \longrightarrow \mathrm{Hom}_R(M, N') \xrightarrow{f_*} \mathrm{Hom}_R(M, N) \xrightarrow{g_*} \mathrm{Hom}_R(M, N'')$$

is exact, where $f_*(h) := f \circ h$, and analogously for g_*.

2) Prove that if M is a free R-module, then $\mathrm{Hom}_R(M, -)$ is even exact.

Remark: Modules for which the functor $\mathrm{Hom}_R(M, -)$ is exact are called projective. We will characterize them in the next tutorial.