LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Tutorial sheet 6
Algebra

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In this tutorial sheet we do some further exercises about group actions and the Sylow Theorems. There will probably be more exercises that we will discuss in the tutorial, nevertheless the ones given here are good for practice.

Exercise 1. Decide whether the following statements are true or false and justify your answers.

1) Let p be a prime number, G be a finite group and $N \trianglelefteq \mathrm{G}$ be a normal subgroup. Assume that $p \nmid [G : N]$, then all Sylow p-subgroups of G lie in N.

2) Let G be a group of order $|G| = p^r$, where p is a prime number and $r \geq 1$. Then:
   (i) There exists an element $g \in G \setminus \{1\}$ such that $hg = gh$ for all $h \in G$.
   (ii) G is solvable.
   (iii) G has precisely one Sylow p-subgroup.

3) Groups of the following order are abelian:
   (i) $|G| = 4$.
   (ii) $|G| = 6$.
   (iii) $|G| = 12$.
   (iv) $|G| = 17$.
   (v) $|G| = 121$.
   (vi) $|G| = 143$.

Exercise 2. 1) Let G be a group of order 22 acting on a set X of size 11 with no fixed points. Show that the action is transitive.

2) The canonical action from $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2 \setminus \{0\}$ is transitive.

3) Let G be a finite group. Then for $|G| \geq 3$, the action of G on $G \setminus \{1\}$ by conjugation is not transitive.

Exercise 3. 1) Show that every group of order 36 has a non-trivial normal subgroup.

Hint: Consider the action of G on the set of Sylow 3-subgroups and use the abstract definition.

2) Let G be a group of order 48. Show that G has a normal subgroup of order 8 or 16.

Exercise 4. 1) Let G be a group of order 30. Show that G has a normal subgroup N of order 15 and that $N \cong \mathbb{Z}/15\mathbb{Z}$.

2) Show that every group G of order 45 is abelian.

Bonus Exercise (Not relevant for the final exam)**.**

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(a) The functor F is called *faithful* if, for all objects $A, B \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

is injective.

(b) The functor F is called *full* if, for all objects $A, B \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

is surjective.

(c) The functor F is called *fully faithful* if it is full and faithful.

1) Show that the identity functor

$$\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

is fully faithful.

2) Let $\mathcal{C}$ be a subcategory of a category $\mathcal{D}$ and let

$$i : \mathcal{C} \hookrightarrow \mathcal{D}$$

be the inclusion functor.

(i) Show that i is faithful.

(ii) Show that i is, in general, not full. Give an explicit example (e.g. $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$).

3) Consider the forgetful functor

$$U : \mathbf{Ab} \to \mathbf{Set}.$$

(i) Show that U is faithful.

(ii) Show that U is not full.

4) Consider the abelianization functor

$$(-)^{\mathrm{ab}} : \mathbf{Grp} \to \mathbf{Ab}.$$

Show that $(-)^{\mathrm{ab}}$ is *not* faithful.

Hint: Use S_3 and an inner automorphism.