LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Tutorial sheet 3
Algebra

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Exercise 1. (a) Let $C := \{z \in \mathbb{C}^\times \mid |z| = 1\}$. Consider the map

$$\phi\colon \mathbb{R} \to \mathbb{C}^\times, \quad \phi(x) := e^{2\pi i x}.$$

Prove that $\mathbb{R}/\mathbb{Z} \cong C$.

(b) We denote, as usual, by $\zeta_p := \exp(2\pi i/p)$ a primitive p-th root of unity. Consider the map

$$\psi\colon \mathbb{Z} \longrightarrow \mathbb{C}^\times, \qquad \psi(k) := \zeta_p^k.$$

Show that $\mathbb{Z}/p\mathbb{Z} \;\cong\; \{\,1, \zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1}\,\}$.

(c) Let k be a field. Consider the map

$$\det\colon \mathrm{GL}_n(k) \to k^\times.$$

Show that $\mathrm{GL}_n(k)/\mathrm{SL}_n(k) \cong k^\times$.

Exercise 2. (a) Let $f\colon G \to H$ be a homomorphism of finite groups such that $|G|$ and $|H|$ are coprime. Show that f is trivial, i.e. $\mathrm{im}(f) = \{e_H\}$.

(b) Let G be a finite group and let $N \trianglelefteq G$ be a normal subgroup. Show that if $n := |N|$ and $|G/N|$ are coprime, then N is the only subgroup of G of order n.

Please don't worry about Exercise 3. In the tutorial, we will mainly focus on Exercises 1 and 2. If time permits, I will also explain Exercise 3 to help you to gain a better understanding of Exercise 4 from Exercise Sheet 2.

Exercise 3. In this exercise, we become familiar with *(integral) group rings*.
Let $\mathbb{Z}$ be the ring of integers and G a group. The *(integral) group ring* $\mathbb{Z}[G]$ is the set of all finite formal linear combinations

$$\sum_{g\in G} a_g g, \quad a_g \in \mathbb{Z},$$

i.e. $a_g = 0$ for all but finitely many $g \in G$ (*note that this is the same definition as given in Exercise Sheet 2*).
The set $\mathbb{Z}[G]$ becomes a ring with the operations

$$+_{\mathbb{Z}[G]}\colon \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G], \quad \sum_{g\in G} a_g g \;+_{\mathbb{Z}[G]}\; \sum_{g\in G} b_g g := \sum_{g\in G} (a_g +_{\mathbb{Z}} b_g) g,$$

$$\cdot_{\mathbb{Z}[G]}\colon \mathbb{Z}[G] \times \mathbb{Z}[G] \to \mathbb{Z}[G], \quad \sum_{g\in G} a_g g \;\cdot_{\mathbb{Z}[G]}\; \sum_{g\in G} b_g g := \sum_{g\in G} c_g g,$$

where

$$c_g := \sum_{\substack{g_1,g_2\in G\\ g_1\cdot_G g_2=g}} a_{g_1}\cdot_{\mathbb{Z}} b_{g_2}.$$

The additive identity in $\mathbb{Z}[G]$ is the formal linear combination in which all coefficients a_g are zero, and the multiplicative identity is given by $1_{\mathbb{Z}[G]} = 1_{\mathbb{Z}} \cdot e_G$.
Note that there is no need to restrict ourselves to $\mathbb{Z}$; we can define the group ring $R[G]$ in an analogous way for any commutative ring R.

(a) Let G be the cyclic group of order 3, say $G = \{e_G, g, g^2\}$. Write down some elements in the ring $\mathbb{Z}[G]$ and compute:

$$(5 + 2g + 7g^2) +_{\mathbb{Z}[G]} (3 + 4g - 479g^2),$$
$$(4 + 3g) \cdot_{\mathbb{Z}[G]} (11 + 4g + 15g^2).$$

(b) Show that $\mathbb{Z}[G]$ is a commutative ring if and only if G is abelian. It suffices to prove that multiplication in $\mathbb{Z}[G]$ is commutative if and only if G is abelian.

From now on, we will use some basic concepts from ring theory, such as ideals or the fundamental theorem on homomorphisms. If you are already familiar with these notions, the following exercises are a good opportunity to practise working with group rings. If not, there is no need to worry, these concepts will be covered in detail later in the lecture, and we can return to these exercises once they have been discussed. You may also view them as optional "fun exercises" for those who already know this material.

(c) Let G be the cyclic group of order m. Show that

$$\mathbb{C}[G] \cong \mathbb{C}[X]/(X^m - 1).$$

Hint: Consider the homomorphism $\mathbb{C}[X] \to \mathbb{C}[G]$, $\quad \sum_{i=0}^n a_i X^i \longmapsto \sum_{i=0}^n a_i g^i$.

(d) Let G be the cyclic group of order 2. Show that

$$\mathbb{C}[G] \cong \mathbb{C} \times \mathbb{C}.$$

Hint: You may use the Chinese Remainder Theorem.

(e) Let R be a (not necessarily commutative) ring and $f\colon R \to S$ a ring homomorphism. Show that $\ker(f) \subseteq R$ is a two-sided ideal.

Hence, we can conclude that the kernel of the *augmentation map*

$$\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}, \quad \sum_{g\in G} a_g g \longmapsto \sum_{g\in G} a_g,$$

which we denote by $I(G)$, is a two-sided ideal. We call $I(G)$ the *augmentation ideal*.