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Tutorial sheet 2
Algebra
Suggested Solutions

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Exercise 1. (a) Let G be an abelian group. Show that any subgroup of G is normal.

Suggested solution. Let $U \leq G$ be an arbitrary subgroup and let $a \in U$ and $g \in G$ be arbitrary elements. Since G is abelian, we have

$$gag^{-1} = gg^{-1}a = a \in U.$$

Hence U is a normal subgroup of G.

□

(b) Let $f\colon G \to H$ be a group homomorphism. Show that $\ker(f) \trianglelefteq G$ is normal.

Note: You have already seen in the lecture that the same does not hold for $\mathrm{im}(f)$.

Suggested solution. Let $a \in \ker(f)$ be arbitrary. Then

$$f(gag^{-1}) = f(g)\, f(a)\, f(g^{-1}) = f(g)\, e_H\, f(g)^{-1} = e_H.$$

In particular, $gag^{-1} \in \ker(f)$ for all $g \in G$; i.e. $\ker(f)$ is a normal subgroup of G.

□

(c) Let k be a field. Show that $\mathrm{SL}_n(k) \trianglelefteq \mathrm{GL}_n(k)$ is normal.

Suggested solution. Let $A \in SL_n(K)$ and $B \in GL_n(K)$ be arbitrary matrices. Then

$$\det(BAB^{-1}) = \det(B)\ \det(A)\ \det(B^{-1}) = \det(B)\ \det(A)\ \det(B)^{-1} = \det(A) = 1.$$

Hence $BAB^{-1} \in SL_n(K)$, which shows that $SL_n(K)$ is a normal subgroup of $GL_n(K)$.

□

(d) Let G be a group and $[G, G]$ be the commutator subgroup. Show that $[G, G] \trianglelefteq G$ is normal.

Hint: Note that it suffices to show that $g[a, b]g^{-1} \in [G, G]$.

Suggested solution. By definition, the commutator subgroup is generated by the elements $[a, b]$ with $a, b \in G$. Since $(ab)^{-1} = b^{-1}a^{-1}$, we have

$$[a, b]^{-1} = (aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1} = [b, a].$$

Thus every element of $[G, G]$ is a finite product of commutators, i.e.

$$[G, G] \;=\; \big\langle\, [a, b] \;\big|\; a, b \in G \,\big\rangle \;=\; \Big\{ \prod_{i=1}^{n} [a_i, b_i] \;\Big|\; n \in \mathbb{N}_0,\ a_i, b_i \in G \Big\}.$$

We now follow the hint and observe that it suffices to show

$$g\,[a, b]\,g^{-1} \in [G, G] \quad \text{for all } g, a, b \in G.$$

Indeed, if $x = \prod_{i=1}^{n}[a_i, b_i] \in [G, G]$, then

$$gxg^{-1} = \prod_{i=1}^{n} g[a_i, b_i]g^{-1} \in [G, G],$$

once we know $g[a, b]g^{-1} \in [G, G]$.
So let $g, a, b \in G$. Then

$$\begin{aligned} g[a, b]g^{-1} &= g\,(aba^{-1}b^{-1})\,g^{-1} \\ &= (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) \\ &= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} \\ &= [\,gag^{-1},\ gbg^{-1}\,] \in [G, G]. \end{aligned}$$

Hence $[G, G] \trianglelefteq G$.

□

Exercise 2. Let G be a group of prime order p. Show that for every nontrivial element $g \in G$, we have $\langle g \rangle = G$.

Suggested solution. Let g be an arbitrary nontrivial element of G. By Lagrange's Theorem, we have

$$\operatorname{ord}(g) \mid |G|,$$

and therefore $\operatorname{ord}(g) \in \{1, p\}$. Since g is nontrivial, its order must be p. Moreover, since $\langle g \rangle \subseteq G$ and $\operatorname{ord}(g) = |G|$, it follows that

$$\langle g \rangle = G.$$

□

Exercise 3. Show that $[S_n, S_n] = A_n$ for all $n \geq 3$.

Suggested solution. First recall

$$A_n = \{\sigma \in S_n \mid \operatorname{sgn}(\sigma) = 1\}.$$

Furthermore, from linear algebra we know that the map

$$\operatorname{sgn} : S_n \longrightarrow \{\pm 1\}$$

is a group homomorphism, and hence $S_n/A_n \cong \{\pm 1\}$.
In particular, S_n/A_n is commutative, and by Exercise 4 (a) we have

$$[S_n, S_n] \subseteq A_n.$$

It remains to show that $[S_n, S_n] \supseteq A_n$.
Recall that for $n \geq 3$, every element of A_n can be written as a finite product of 3-cycles. So let (x_1, x_2, x_3) be a 3-cycle. Then

$$\begin{aligned}(x_1, x_2, x_3) &= (x_1, x_3)(x_2, x_3)(x_3, x_1)(x_3, x_2) \\ &= (x_1, x_3)(x_2, x_3)(x_1, x_3)^{-1}(x_2, x_3)^{-1} \in [S_n, S_n].\end{aligned}$$

Consequently, we have

$$[S_n, S_n] = A_n.$$

□

Exercise 4 (Abelianization)**.** (a) Show that $G^{\mathrm{ab}} := G/[G, G]$ is abelian and that $[G, G]$ is the smallest normal subgroup of G satisfying this property.

G^{ab} is called the *abelianization* of G.

Suggested solution. By definition

$$e_G[G, G] = (aba^{-1}b^{-1})[G, G]$$

and hence

$$(ba)[G, G] = (ab)[G, G].$$

This shows that $G/[G, G]$ is abelian.

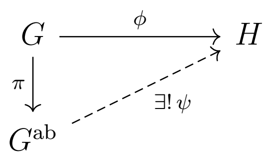
Now let G/N be a commutative group. In particular,

$$(ba)N = (ab)N.$$

Thus, it follows $e_G N = aba^{-1}b^{-1}N$ and therefore $aba^{-1}b^{-1} \in N$. Consequently, we obtain $[G, G] \subseteq N$.

□

(b) Show that the abelianization G^{ab}, together with the projection $\pi\colon G \to G^{\text{ab}}$, satisfies the following universal property: For every abelian group H and every group homomorphism $\phi\colon G \to H$, there exists a unique group homomorphism $\psi\colon G^{\text{ab}} \to H$ such that the following diagram commutes:



Note: It also follows directly from the universal property that $[G, G]$ is the smallest normal subgroup of G such that the corresponding quotient is abelian.

Suggested solution. Let H be an abelian group and $\phi : G \to H$ a group homomorphism. We now construct a group homomorphism

$$\psi : G^{\text{ab}} \longrightarrow H,$$

such that $\psi \circ \pi = \phi$, where

$$\pi : G \longrightarrow G^{\text{ab}}, \quad g \mapsto g[G, G]$$

is the canonical projection onto the quotient.

First, we note that there is only one possible choice for ψ. Indeed, since $\psi \circ \pi = \phi$, we must have

$$\psi(g[G, G]) = \phi(g).$$

So it remains to show that this choice is well defined.

Let $g[G, G] = g'[G, G]$. Then $g'^{-1}g \in [G, G]$. Without loss of generality, we may assume that

$$g'^{-1}g = aba^{-1}b^{-1}$$

(for otherwise, we would have a finite product of commutators, but the proof works in the same way) for some $a, b \in G$. Since H is abelian, we get

$$\phi(g'^{-1}g) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} = e_H.$$

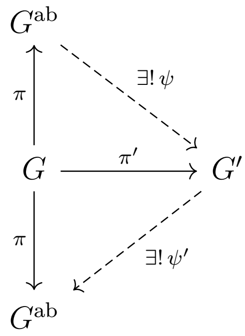
Hence $\phi(g') = \phi(g)$, and therefore ψ is well defined. □

(c) Use the universal property to show that the abelianization G^{ab}, together with π, is unique up to canonical isomorphism.

Suggested solution. Assume there is another abelian group G' together with a group homomorphism

$$\pi' : G \longrightarrow G'$$

satisfying this universal property. Then we have the following commutative diagram:



This leads to

$$\pi' = \psi \circ \pi = \psi \circ \psi' \circ \pi'.$$

Again using universal property and the uniqueness, it follows that

$$\psi' \circ \psi = \mathrm{id}_{G^{\text{ab}}}\,.$$

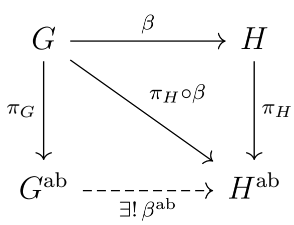
An analogous argument shows that $\psi \circ \psi' = \mathrm{id}_{G'}$. Hence, $G^{\text{ab}} \cong G'$.

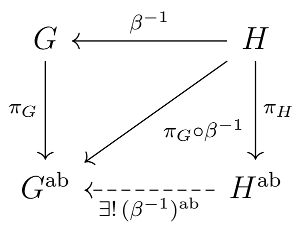
□

(d) Use the universal property to prove the following statement:

$$G \cong H \;\Rightarrow\; G^{\text{ab}} \cong H^{\text{ab}}.$$

Suggested solution. Let $G \cong H$ and consider the diagrams.





Then we have

$$\begin{aligned}(\beta^{-1})^{\text{ab}} \circ \beta^{\text{ab}} \circ \pi_G &= (\beta^{-1})^{\text{ab}} \circ \pi_H \circ \beta \\ &= \pi_G \circ \beta^{-1} \circ \beta \\ &= \pi_G.\end{aligned}$$

By the universal property and uniqueness, we obtain $(\beta^{-1})^{\text{ab}} \circ \beta^{\text{ab}} = \mathrm{id}_{G^{\text{ab}}}$. Analogously one shows $\beta^{\text{ab}} \circ (\beta^{-1})^{\text{ab}} = \mathrm{id}_{H^{\text{ab}}}$. Therefore $G^{\text{ab}} \cong H^{\text{ab}}$.

□