LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Tutorial Sheet 12
Algebra

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Exercise 1. Let G be a finite group of order 35.

(i) Show that G has a unique Sylow 5-subgroup, and that this subgroup is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. Moreover, show that G has a unique Sylow 7-subgroup, and that this subgroup is isomorphic to $\mathbb{Z}/7\mathbb{Z}$.

(ii) Deduce that G is abelian.

Hint: You may find Exercise Sheet 5 helpful.

Exercise 2.
Answer each question with "Yes" or "No". (Each correct answer is worth one point. Incorrect answers will not be counted.)

(i) The quotient ring $\mathbb{Z}[i]/(3)$ is a field.

(ii) The quotient ring $\mathbb{Z}[i]/(2)$ is a field.

(iii) $\mathbb{R}[X]/(X^2-2)$ is isomorphic to $\mathbb{C}$.

(iv) In $\mathbb{Z}$, one has $(9,21)=(3)$.

(v) In a factorial ring, every nonzero prime ideal is maximal.

(vi) The element X is prime in $\mathbb{Z}[X,Y]$.

(vii) The polynomial ring $K[X]$ is a principal ideal domain if and only if K is a field.

(viii) The polynomial $2X^{17}+\frac{10}{7}X^7+\frac{10}{7}$ is irreducible in $\mathbb{Q}[X]$.

(ix) Let K be a field and let $f(X), g(X) \in K[X]$ be irreducible. Then the composition $f(g(X))$ is irreducible in $K[X]$.

(x) Let K be a field. Then every nontrivial ring homomorphism $K \to S$ is injective.

(xi) Let p be a prime and let $a \in \mathbb{Z}$. If $p \nmid a$, then $a^{p-1} \equiv 1 \pmod p$.

(xii) A commutative ring R is a field if and only if $\{0\}$ is a maximal ideal of R.

(xiii) Every field extension $K/\mathbb{Q}$ of degree 3 is Galois.

(xiv) Let L/K be a finitely generated algebraic field extension. Then $[L:K]<\infty$.

(xv) There exists a finite algebraically closed field.

Exercise 3. Let p, q be prime numbers. Determine the degree of the field extension $K := \mathbb{Q}\big(\zeta_p, \sqrt[p]{q}\big)$ over $\mathbb{Q}$, and prove that $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\sqrt[p]{q}) = \mathbb{Q}$.

Exercise 4. Let $K := \mathbb{Q}(\zeta_5, \sqrt[5]{3})$.

(i) Show that $K/\mathbb{Q}$ is a Galois extension.

(ii) Determine whether $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.

Exercise 5. Let $K := \mathbb{F}_{16}(Y)$ and let $f \in K[X]$ be an irreducible polynomial of degree 3. Let L be the splitting field of f over K. Show that L/K is a Galois extension, and prove that $\mathrm{Gal}(L/K) \cong S_3$ or $\mathrm{Gal}(L/K) \cong A_3$.

Exercise 6. Let $a, b \in \mathbb{Q}$ and assume that the polynomial $f(X) = X^4 + aX^2 + b \in \mathbb{Q}[X]$ is irreducible. Let K be the splitting field of f over $\mathbb{Q}$.

(i) Let $\alpha \in K$ be a root of f. Show that $[K : \mathbb{Q}(\alpha)] \leq 2$.

(ii) Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to one of the following groups:

$$D_4, \quad \mathbb{Z}/4\mathbb{Z}, \text{ or } \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$