WP9 Category Theory

Comprehensive notes by

Lina Weitzenböck

Lectures by

Prof. Fabien Morel

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Course description. This lecture will serve as a complement to the Algebra 2 course. It will meet for two hours every two weeks. It is an introduction to category theory and is meant to help students understand Algebra 2 (and its sequel, Algebraic Geometry) from a more global perspective. This course is not mandatory, but it is inherently interesting for any student wishing to pursue pure mathematics in the master's program. **Course page.** https://www.mathematik.uni-muenchen.de/~morel/Teaching/AAteaching.html.

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1 Categories and Isomorphisms

Note. All Hom-sets are sets! (not classes). This holds throughout the lecture notes.

1.1 General Definition of Categories

Definition (Category). A *category* $\mathcal{C}$ consists of the following data:

1. A class of objects, denoted $\mathrm{Ob}(\mathcal{C})$.
2. A set $\mathrm{Hom}_{\mathcal{C}}(C, D)$ of morphisms for all pairs $(C, D) \in \mathrm{Ob}(\mathcal{C})^2$.
3. A composition operation: for all $(C, D, E) \in \mathrm{Ob}(\mathcal{C})^3$, there is a map

$$\begin{aligned} \circ : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(D, E) &\to \mathrm{Hom}_{\mathcal{C}}(C, E) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

Such that the following axioms hold:

- Identity: For all $C \in \mathrm{Ob}(\mathcal{C})$, there exists a unique element $\mathrm{Id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ such that for all $f : C \to D$, we have $f \circ \mathrm{Id}_C = \mathrm{Id}_D \circ f = f$.
- Associativity: For all composable morphisms $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} F$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Main Example. Set is the category of sets.

For any $C \in \mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(C, C)$ equipped with the composition $\circ$ and identity Id_C forms an associative monoid with a unit. The group of invertible elements,

$$\mathrm{Aut}_{\mathcal{C}}(C) = \mathrm{Hom}_{\mathcal{C}}(C, C)^{\times},$$

is called the *automorphism group*.

Alternative Perspective. A category $\mathcal{C}$ can also be defined by its class of objects $\mathrm{Ob}(\mathcal{C})$ and the disjoint union of all morphism sets:

$$\mathrm{Hom}(\mathcal{C}) = \coprod_{(C,D) \in \mathrm{Ob}(\mathcal{C})^2} \mathrm{Hom}_{\mathcal{C}}(C, D).$$

We have two maps $s, t : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$, where s denotes the source and t denotes the target, such that:

$$\mathrm{Hom}_{\mathcal{C}}(C, D) = \{f \in \mathrm{Hom}(\mathcal{C}) \mid s(f) = C, t(f) = D\}.$$

Composition is then given by a map defined on the fiber product of composable morphisms:

$$\mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}).$$

Examples (Categories).

- **Gr** = category of groups.
- **Ab** = category of abelian groups.

- **Ring** = category of rings.
- **Mon** = category of monoids.
- **Top** = category of topological spaces.

1.2 Subcategories and Special Categories

Definition (Full Subcategory). Let $\mathcal{C}$ be a category and $\Omega \subset \mathrm{Ob}(\mathcal{C})$ a subclass. We define the *full subcategory* $\mathcal{C}_\Omega$ as the category with $\mathrm{Ob}(\mathcal{C}_\Omega) = \Omega$, and for all $(C, D) \in \Omega^2$:

$$\mathrm{Hom}_{\mathcal{C}_\Omega}(C, D) = \mathrm{Hom}_{\mathcal{C}}(C, D).$$

Example. Let $K-$**Vect** be the category of vector spaces over K. Then $K-$**finiteVect** is the full subcategory of finite-dimensional K-vector spaces.

Definition (Category with One Object). A category with one object is the "same thing" as an associative monoid with a neutral element M. We write $\mathcal{C}_M$ where $\mathrm{Ob}(\mathcal{C}_M) = \{*\}$ and $\mathrm{Hom}_{\mathcal{C}_M}(*, *) = M$. A group defines a category in this way as well.

Definition (Category of Matrices). Let R be a commutative ring. The category of matrices $\mathrm{Mat}(R)$ is defined by $\mathrm{Ob}(\mathrm{Mat}(R)) = \mathbb{N}$. For all $(n, m) \in \mathbb{N}^2$, the morphisms are $\mathrm{Hom}_{\mathrm{Mat}(R)}(n, m) = M_{m\times n}(R)$, and composition $\circ$ is the standard product of matrices.
If K is a field, $\mathrm{Mat}(K)$ can be seen as a full subcategory of $K-$**Vect** via the assignment $n \mapsto K^n$.

Definition (Small Category). A *small category* is a category for which $\mathrm{Ob}(\mathcal{C})$ is a set.

1.3 Isomorphisms

Definition (Isomorphism). One says that a morphism $f : C \to D$ is an *isomorphism* if there exists a morphism $g : D \to C$ such that $g \circ f = \mathrm{Id}_C$ and $f \circ g = \mathrm{Id}_D$. In this case, we write $f : C \cong D$.

If $C \cong D$, then for all $E \in \mathrm{Ob}(\mathcal{C})$, there are bijections:

$$\mathrm{Hom}_{\mathcal{C}}(C, E) \cong \mathrm{Hom}_{\mathcal{C}}(D, E)$$
$$\mathrm{Hom}_{\mathcal{C}}(E, C) \cong \mathrm{Hom}_{\mathcal{C}}(E, D)$$

To be isomorphic is an equivalence relation on $\mathrm{Ob}(\mathcal{C})$. It is important to understand $\mathrm{Ob}(\mathcal{C})$ up to isomorphism, denoted $\mathrm{Ob}(\mathcal{C})/\cong$.

Definition (Essentially Small Category). A category $\mathcal{C}$ is *essentially small* if $\mathrm{Ob}(\mathcal{C})/\cong$ is a set. Equivalently, there exists a family $(C_i)_{i\in I}$ indexed by a set I such that for all $C \in \mathrm{Ob}(\mathcal{C})$, there is an $i \in I$ with $C_i \cong C$.

2 Functors and Natural Transformations

2.1 Functors

Definition (Functor). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a pair of data:

1. A map on objects: $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, mapping $C \mapsto F(C)$.
2. For all $(C, D) \in \mathrm{Ob}(\mathcal{C})^2$, a map on morphisms: $F : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(D))$.

Such that composition and identities are preserved:

$$F(g) \circ F(f) = F(g \circ f)$$
$$F(\mathrm{Id}_C) = \mathrm{Id}_{F(C)}$$

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(D) \\ & {\scriptstyle F(g\circ f)} \searrow \quad \swarrow {\scriptstyle F(g)} & \\ & F(E) & \end{array}$$

Remark (Simplicial Class). We can view the data of a category as $\mathrm{Ob}(\mathcal{C}) \rightleftarrows \mathrm{Hom}(\mathcal{C})$. Defining $\mathrm{Hom}_n(\mathcal{C}) \subset \mathrm{Hom}(\mathcal{C}) \times \cdots \times \mathrm{Hom}(\mathcal{C})$, one obtains a sequence:

$$\mathrm{Ob}(\mathcal{C}) \leftarrow \mathrm{Hom}_1(\mathcal{C}) \leftarrow \mathrm{Hom}_2(\mathcal{C}) \ldots$$

which forms a simplicial object. A functor is then a morphism of these simplicial objects.

Examples of Functors.

- The identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.
- For a subclass $\Omega \subset \mathrm{Ob}(\mathcal{C})$, the inclusion $\mathcal{C}_\Omega \hookrightarrow \mathcal{C}$ is a functor (a "full embedding").
- A morphism of monoids $M \to N$ induces a functor $\mathcal{C}_M \to \mathcal{C}_N$.
- The "Forgetful functor", such as $\mathbf{Gr} \to \mathbf{Set}$ or $\mathbf{Ab} \subset \mathbf{Gr} \to \mathbf{Set}$.
- Composition of functors: If $F : \mathbf{Ab} \to \mathbf{Mon}$ and $G : \mathbf{Mon} \to \mathbf{Set}$, then the composition $G \circ F : \mathbf{Ab} \to \mathbf{Set}$ is a functor.

Problem. In general, the collection of all functors $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is too large to be a set. However, if $\mathcal{C}$ and $\mathcal{D}$ are (essentially) small, then $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is a set. We let **Cat** denote the category of (essentially) small categories.

2.2 Natural Transformations and Equivalences

Definition (Natural Transformation). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\theta : F \to G$ consists of a morphism $\theta_C : F(C) \to G(C)$ in $\mathcal{D}$ for each $C \in \mathrm{Ob}(\mathcal{C})$, such that for all pairs $(C, D) \in \mathrm{Ob}(\mathcal{C})^2$ and all morphisms $f : C \to D$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{\theta_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(D) & \xrightarrow[\theta_D]{} & G(D) \end{array}$$

Examples of Natural Transformations.

- The identity natural transformation $\mathrm{Id}_F : F \to F$.
- Let $F = \mathrm{Id}_{\mathbf{Gr}}$ and $G = (\cdot)^{ab}$ (the abelianization functor) from $\mathbf{Gr} \to \mathbf{Gr}$. The canonical projection $\pi : G \to G^{ab}$ is a natural transformation.
- Consider functors $\mathbf{Set} \to \mathbf{Ab}$. Let $F = \mathrm{Id}_{\mathbf{Set}}$ and $G(A) = \bigoplus_{x \in A} \mathbb{Z} = \mathbb{Z}[A]$. The assignment $A \to \mathbb{Z}[A]$ via $x \mapsto 1 \cdot x$ is a natural transformation.

Definition (Equivalence of Categories). A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural transformations:

$$\theta_{\mathcal{C}} : G \circ F \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}$$
$$\theta_{\mathcal{D}} : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$$

Note: this is the "correct" notion to compare categories rather than requiring an exact isomorphism of categories.

Proof Sketch. If F is an equivalence of categories, then for all $(C, D) \in \mathrm{Ob}(\mathcal{C})^2$, $F(C) \cong F(D)$ in $\mathcal{D}$ implies C and D are isomorphic in $\mathcal{C}$.

This follows since $G \circ F(C) \cong G \circ F(D)$, and we have the natural isomorphisms $C \xrightarrow{\sim} G \circ F(C)$ and $D \xrightarrow{\sim} G \circ F(D)$. Furthermore, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(C, D) \cong \mathrm{Hom}_{\mathcal{D}}(F(C), F(D))$ because the composition

$$\mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(D)) \to \mathrm{Hom}_{\mathcal{C}}(G \circ F(C), G \circ F(D))$$

commutes with the isomorphisms θ_C and θ_D.

□

Theorem (Equivalence for Matrices). Let K be a field. The inclusion functor

$$\mathrm{Mat}(K) \hookrightarrow \text{finite dim. } K\text{−}\mathbf{Vect}$$

is an equivalence of categories.
To define the inverse functor G : finite dim. K−**Vect** $\to \mathrm{Mat}(K)$ mapping $V \mapsto K^n$, one must choose a basis. For each choice of basis, one obtains a different functor G.

End of Lec 01.

2.2.1 Examples of Equivalences of Categories

Equivalence of Finite Sets. Let **FinSet** $\subset$ **Set** be the full subcategory of finite sets. For all $n \in \mathbb{N}$, we define $\underline{n} := \{1, \ldots, n\}$. Let $\underline{\mathbb{N}} \subset$ **FinSet** be the full subcategory whose objects are precisely the sets $\underline{n}$ for $n \in \mathbb{N}$.
The inclusion functor $\iota : \underline{\mathbb{N}} \hookrightarrow$ **FinSet** is an equivalence of categories.

Proof Sketch. To prove that ι is an equivalence, we must construct a quasi-inverse functor $\Phi : \mathbf{FinSet} \to \underline{\mathbb{N}}$.
First, for every $C \in \mathrm{Ob}(\mathbf{FinSet})$, choose a bijection $\varphi_C : C \xrightarrow{\cong} \underline{n}$, where $n = |C|$. We define the functor Φ on objects by $\Phi(C) := \underline{|C|}$. For a morphism $f : C \to D$ in **FinSet**, we define the morphism $\Phi(f)$ in $\underline{\mathbb{N}}$ via the composition:

$$\Phi(f) := \varphi_D \circ f \circ \varphi_C^{-1}$$

One can verify that Φ preserves identities and composition, hence it is a valid functor.
If we choose our bijections such that $\varphi_{\underline{n}} := \mathrm{Id}_{\underline{n}}$ for all $n \in \mathbb{N}$, we obtain $\Phi \circ \iota = \mathrm{Id}_{\underline{\mathbb{N}}}$.
Conversely, for the composition $\iota \circ \Phi : \mathbf{FinSet} \to \mathbf{FinSet}$, we have $\iota \circ \Phi(C) = \underline{|C|}$. The family of chosen bijections $(\varphi_C)_{C \in \mathrm{Ob}(\mathbf{FinSet})}$ naturally provides an isomorphism $\theta : \mathrm{Id}_{\mathbf{FinSet}} \xrightarrow{\sim} \iota \circ \Phi$. For any $f : C \to D$, the commutativity of the required square follows directly from the definition of $\Phi(f)$, ensuring that θ is indeed a natural isomorphism.

□

Equivalence of Ordered Sets. Let J be the category of totally ordered finite sets where the morphisms are strictly increasing maps. Let $\omega\underline{\mathbb{N}}$ be the full subcategory whose objects are the standard ordered sets $\{1 < 2 < \cdots < n\}$ for $n \in \mathbb{N}$.
By a similar construction mapping any finite totally ordered set to its unique order-preserving bijection with $\{1 < \cdots < n\}$, the inclusion $\omega\underline{\mathbb{N}} \hookrightarrow J$ is an equivalence of categories.

2.2.2 Category of Functors

Definition (Category of Functors). Let $\mathcal{C}$ be any category and I be a small category (i.e., $\mathrm{Ob}(I)$ is a set). The *category of functors*, denoted $\mathrm{Funct}(I, \mathcal{C})$ or $\mathcal{C}^I$, is defined by the following data:

- Objects: Functors $F : I \to \mathcal{C}$.
- Morphisms: For any two functors $F, G : I \to \mathcal{C}$, the morphisms are the natural transformations $\mathrm{Hom}_{\mathrm{Funct}(I,\mathcal{C})}(F, G) := \mathrm{Nat}(F, G)$.

We must verify that $\mathrm{Nat}(F, G)$ is indeed a set so that $\mathrm{Funct}(I, \mathcal{C})$ forms a valid category. Because I is a small category, we have an inclusion:

$$\mathrm{Nat}(F, G) \subset \prod_{i \in \mathrm{Ob}(I)} \mathrm{Hom}_{\mathcal{C}}(F(i), G(i))$$

The product on the right is a set. Specifically, $\mathrm{Nat}(F, G)$ is exactly the subset of families of morphisms $(\theta_i)_{i \in \mathrm{Ob}(I)}$ such that for all morphisms $(i \xrightarrow{u} j) \in \mathrm{Mor}(I)$, the following diagram commutes:

$$\begin{array}{ccc} F(i) & \xrightarrow{\theta_i} & G(i) \\ {\scriptstyle F(u)}\downarrow & & \downarrow{\scriptstyle G(u)} \\ F(j) & \xrightarrow[\theta_j]{} & G(j) \end{array}$$

Equivalently, $G(u) \circ \theta_i = \theta_j \circ F(u)$.

3 Basic Limits and Colimits

3.1 Products

Definition (Product). Let $\mathcal{C}$ be a category and let $(C_i)_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a set I. A *product* of the family $(C_i)_{i \in I}$ consists of an object $\Pi \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $(pr_i : \Pi \to C_i)_{i \in I}$ satisfying the following universal property:
For every object $D \in \mathrm{Ob}(\mathcal{C})$ and every family of morphisms $(f_i : D \to C_i)_{i \in I}$, there exists a unique morphism $f : D \to \Pi$ such that $pr_i \circ f = f_i$ for all $i \in I$.

$$\begin{array}{ccc} D & \overset{f}{\dashrightarrow} & \prod_{i \in I} C_i \\ & {\scriptstyle f_i}\searrow & \downarrow{\scriptstyle pr_i} \\ & & C_i \end{array}$$

Equivalently, the maps pr_i induce a bijection of sets:

$$\mathrm{Hom}_{\mathcal{C}}(D, \Pi) \xrightarrow{\cong} \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(D, C_i)$$

Proof Sketch. If a product exists, it is unique up to unique isomorphism. Let (Π, pr_i) and (Π', pr'_i) be two products of the same family $(C_i)_{i \in I}$.
Applying the universal property of Π' to the family $(pr_i : \Pi \to C_i)_{i \in I}$, we obtain a unique morphism $\varphi : \Pi \to \Pi'$ such that $pr'_i \circ \varphi = pr_i$. Symmetrically, applying the universal property of Π to $(pr'_i : \Pi' \to C_i)_{i \in I}$, we obtain a unique morphism $\psi : \Pi' \to \Pi$ such that $pr_i \circ \psi = pr'_i$.
The composition $\psi \circ \varphi : \Pi \to \Pi$ satisfies $pr_i \circ (\psi \circ \varphi) = pr'_i \circ \varphi = pr_i$. However, the identity Id_Π also satisfies $pr_i \circ \mathrm{Id}_\Pi = pr_i$. By the uniqueness condition of the universal property applied to $D = \Pi$, we must have $\psi \circ \varphi = \mathrm{Id}_\Pi$. Analogously, $\varphi \circ \psi = \mathrm{Id}_{\Pi'}$. Thus, $\Pi \cong \Pi'$.

□

Notation. If the product exists, it is denoted by $\prod_{i \in I} C_i$.

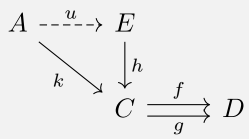
Examples of Products.

- In **Set**, all products exist and are given by the standard Cartesian product.
- In the category of finite sets F, only finite products (where I is a finite set) exist.
- If $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories, then $\mathcal{C}$ has "enough products" if and only if $\mathcal{D}$ does.
- In **Gr**, the product is the direct product of groups.
- Products similarly exist and correspond to the Cartesian product equipped with componentwise operations in **Ring**, **Ab**, and $K-$**Vect**.

3.2 Equalizers

Definition (Equalizer). Let $\mathcal{C}$ be a category and let $(f, g) \in \mathrm{Hom}_{\mathcal{C}}(C, D)^2$ be a pair of parallel morphisms $C \rightrightarrows D$. An *equalizer* of f and g is an object $E \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $h : E \to C$ such that $f \circ h = g \circ h$, which satisfies the following universal property:
For any object $A \in \mathrm{Ob}(\mathcal{C})$ and any morphism $k : A \to C$ such that $f \circ k = g \circ k$, there exists a

unique morphism $u : A \to E$ such that $h \circ u = k$.



Alternative Perspective on Equalizers. Let $\mathcal{E}q(f, g)$ be the category whose objects are pairs (A, k) where $A \in \mathrm{Ob}(\mathcal{C})$ and $k : A \to C$ satisfies $f \circ k = g \circ k$. A morphism from (A, k) to (A', k') is a morphism $u : A \to A'$ in $\mathcal{C}$ such that $k' \circ u = k$.
An equalizer of $C \rightrightarrows D$ is exactly a *final object* in the category $\mathcal{E}q(f, g)$.

Provided it exists, an equalizer is unique up to unique isomorphism. We denote the equalizer of f and g by $\mathrm{Eq}(f, g)$.

Examples of Equalizers.

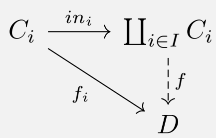
- In **Set**, $\mathrm{Eq}(f, g) = \{x \in C \mid f(x) = g(x)\}$ together with the canonical inclusion into C.
- In **Gr** and **Ring**, the equalizer is the same underlying subset as in **Set**, which forms a valid subgroup or subring.
- In **Ab**, for any morphism $f : A \to B$, the equalizer $\mathrm{Eq}(f, 0)$ is precisely the kernel $\mathrm{Ker}(f)$.

3.3 Coproducts and Coequalizers

These are the dual notions to products and equalizers, obtained by reversing the arrows.

Definition (Coproduct). Let $(C_i)_{i \in I}$ be a family of objects in $\mathcal{C}$. A *coproduct* (or sum) of this family is an object $\coprod_{i \in I} C_i$ together with a family of morphisms $(in_i : C_i \to \coprod_{i \in I} C_i)_{i \in I}$ such that for any object $D \in \mathrm{Ob}(\mathcal{C})$, the maps in_i induce a bijection:

$$\mathrm{Hom}_{\mathcal{C}}\left(\coprod_{i \in I} C_i, D\right) \xrightarrow{\cong} \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(C_i, D)$$

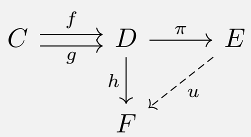


Like products, coproducts are unique up to unique isomorphism provided they exist.

Examples of Coproducts.

- In **Set**, coproducts exist and are given by the disjoint union of the sets. For a set S, defining $S_+ = S \sqcup \{*\}$ illustrates adding a disjoint basepoint.
- In **Ab**, the coproduct is the direct sum $\bigoplus_{i \in I} A_i$.
- In **Gr**, the coproduct is the free product $*_{i \in I} G_i$.

Definition (Coequalizer). Let $\mathcal{C}$ be a category and let $f, g : C \rightrightarrows D$ be a pair of parallel morphisms. A *coequalizer* of f and g is an object $E \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\pi : D \to E$ such that $\pi \circ f = \pi \circ g$, satisfying the following universal property: End of Lec 02.
For any object $F \in \mathrm{Ob}(\mathcal{C})$ and any morphism $h : D \to F$ such that $h \circ f = h \circ g$, there exists a unique morphism $u : E \to F$ such that $u \circ \pi = h$.



Equivalently, the coequalizer E ensures that the following sequence of sets is an exact equalizer sequence in **Set**:

$$\mathrm{Hom}_{\mathcal{C}}(E,F) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(D,F) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(C,F)$$

Coequalizers in Set. In the category **Set**, the notion of a subset is canonically clear, but the notion of a "quotient" relies on equivalence relations. For a surjective map $\pi : S \to T$, one naturally associates an equivalence relation $R_T \subset S \times S$ defined by $R_T = \{(s,t) \in S \times S \mid \pi(s) = \pi(t)\}$. To compute the coequalizer of $f, g : C \rightrightarrows D$ in **Set**, consider the subset $D_{f,g} \subset D \times D$ given by:

$$D_{f,g} = \{(d,d') \in D \times D \mid \exists c \in C \text{ such that } d = f(c) \text{ and } d' = g(c)\}$$

In general, $D_{f,g}$ is not an equivalence relation. Let $R_{f,g}$ be the equivalence relation on D generated by $D_{f,g}$. The coequalizer is then the canonical quotient map $\pi : D \to D/R_{f,g}$.

4 Yoneda's Lemma

Definition (Representable Functor). Let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor. One says that F is *representable* if there exists an object $C \in \mathrm{Ob}(\mathcal{C})$ and a natural isomorphism:

$$\theta : \mathrm{Hom}_{\mathcal{C}}(C,-) \xrightarrow{\sim} F$$

The pair (C,θ) is called a *representative* of the functor F. If it exists, it is unique up to unique isomorphism.

Lemma (Yoneda's Lemma). Let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor, and let $A \in \mathrm{Ob}(\mathcal{C})$. There is a canonical bijection:

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) \xrightarrow{\cong} F(A)$$

Proof. We construct a map $\varphi : \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-),F) \to F(A)$ defined by $\varphi(\theta) = \theta_A(\mathrm{Id}_A) \in F(A)$. Conversely, we construct a map $\psi : F(A) \to \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-),F)$. Let $\alpha \in F(A)$. For any object $B \in \mathrm{Ob}(\mathcal{C})$, we define a map $\theta_B : \mathrm{Hom}_{\mathcal{C}}(A,B) \to F(B)$ by sending a morphism $f : A \to B$ to $F(f)(\alpha)$. One verifies that the family $(\theta_B)_{B \in \mathrm{Ob}(\mathcal{C})}$ commutes with composition, thus forming a valid natural transformation.[1]
We now check that ψ and φ are mutually inverse. Evaluating $\varphi \circ \psi$: for $\alpha \in F(A)$, we have $\psi(\alpha)_A(\mathrm{Id}_A) = F(\mathrm{Id}_A)(\alpha) = \mathrm{Id}_{F(A)}(\alpha) = \alpha$. Thus $\varphi \circ \psi = \mathrm{Id}_{F(A)}$. Evaluating $\psi \circ \varphi$: for a natural transformation θ, applying $\psi(\varphi(\theta))$ to a morphism $f : A \to B$ yields $F(f)(\theta_A(\mathrm{Id}_A))$. By the naturality of θ, the map θ commutes with $f_* = f \circ -$, ensuring $F(f) \circ \theta_A = \theta_B \circ f_*$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(A,A) & \xrightarrow{\theta_A} & F(A) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}_{\mathcal{C}}(A,B) & \xrightarrow[\theta_B]{} & F(B) \end{array}$$

Applied to Id_A, this yields exactly $\theta_B(f \circ \mathrm{Id}_A) = \theta_B(f)$. Thus $\psi \circ \varphi = \mathrm{Id}_{\mathrm{Nat}}$.[2]

□

Corollary. Let $(A,B) \in \mathrm{Ob}(\mathcal{C})^2$. Then $A \cong B$ in $\mathcal{C}$ if and only if there is a natural isomorphism of functors $\mathrm{Hom}_{\mathcal{C}}(A,-) \cong \mathrm{Hom}_{\mathcal{C}}(B,-)$.

Examples of Representable Functors.

- Let R be a commutative ring and $\mathcal{C} = R{-}\mathbf{Mod}$. For a set $S \in \mathbf{Set}$, the functor $F(M) = M^S = \prod_{s \in S} M$ is representable by the free R-module $R[S]$.
- Let $M, N \in R{-}\mathbf{Mod}$. The functor $Q \mapsto \mathrm{Bilin}_R(M,N;Q)$ is representable by the tensor product $M \otimes_R N$.

[1] Standard convention often denotes this as $\psi(\alpha)_B$ to avoid reusing the variable θ, but we preserve the lecture's exact notation.

[2] More precisely, $\mathrm{Id}_{\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(A,-),F)}$.

- In algebraic geometry, for a variety X, the functor assigning to each scheme the set of line bundles with $n+1$ generating sections on X is representable by the projective space $\mathbb{P}^n$. Thus, such sections correspond to morphisms $\mathrm{Hom}(X, \mathbb{P}^n)$.

5 Pairs of Adjoint Functors

Definition (Adjoint Functors). Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that the pair (F, G) are *adjoint* to each other if there exists a natural bijection for all $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$:

$$\theta_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}}(F(C), D)$$

The family of bijections θ is called the *adjunction* between F and G. In this setting, F is called the *left adjoint* and G is called the *right adjoint*.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C, G(D)) & \xrightarrow[\cong]{\theta_{C,D}} & \mathrm{Hom}_{\mathcal{D}}(F(C), D) \\ {\scriptstyle u^*}\downarrow & & \downarrow{\scriptstyle F(u)^*} \\ \mathrm{Hom}_{\mathcal{C}}(C', G(D)) & \xrightarrow[\theta_{C',D}]{\cong} & \mathrm{Hom}_{\mathcal{D}}(F(C'), D) \end{array}$$

Observation. Stating that the bijection is "natural" means it constitutes a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$. Note that the assignment $(C, D) \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$ inherently defines a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$.

One says that a functor $F : \mathcal{C} \to \mathcal{D}$ *admits a right adjoint* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and an adjunction θ making (F, G) an adjoint pair.

Lemma (Uniqueness of Adjoints). If a functor $F : \mathcal{C} \to \mathcal{D}$ admits two right adjoints $G, G' : \mathcal{D} \rightrightarrows \mathcal{C}$, then G and G' are canonically isomorphic.

Proof Sketch. This is a direct consequence of Yoneda's Lemma. For any $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$, the adjunctions provide natural bijections:

$$\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \cong \mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G'(D))$$

Thus, the functors $\mathrm{Hom}_{\mathcal{C}}(-, G(D))$ and $\mathrm{Hom}_{\mathcal{C}}(-, G'(D))$ are naturally isomorphic. By the corollary to Yoneda's Lemma, this implies $G(D) \cong G'(D)$ canonically for all D, extending to a natural isomorphism $G \cong G'$.

□

Lemma (Representability Characterization of Adjoints). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F admits a right adjoint if and only if for all $D \in \mathrm{Ob}(\mathcal{D})$, the functor $\mathrm{Hom}_{\mathcal{D}}(F(-), D) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable by some object $G_D \in \mathrm{Ob}(\mathcal{C})$.

Proof. If F has a right adjoint G, then the adjunction directly establishes $\mathrm{Hom}_{\mathcal{D}}(F(-), D) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(D))$, meaning the functor is representable by $G(D)$.
Conversely, if for each $D \in \mathrm{Ob}(\mathcal{D})$, the functor $\mathrm{Hom}_{\mathcal{D}}(F(-), D)$ is representable by an object $G_D \in \mathrm{Ob}(\mathcal{C})$, the assignment $D \mapsto G_D$ can be uniquely extended to a functor $G : \mathcal{D} \to \mathcal{C}$. This canonical functor naturally satisfies the adjunction property, hence forming a right adjoint to F. An analogous characterization holds for left adjoints.

□

Examples of Adjoint Functors.

- The forgetful functor $G : \mathbf{Ab} \to \mathbf{Set}$ admits a left adjoint assigning to a set S the free abelian group $\mathbb{Z}[S] = \bigoplus_{s \in S} \mathbb{Z}$. This follows from $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}[S], A) \cong A^S \cong \mathrm{Hom}_{\mathbf{Set}}(S, G(A))$.
- The forgetful functor $G : \mathbf{Gr} \to \mathbf{Set}$ has a left adjoint mapping a set $S \mapsto F(S)$, the free group generated by S. We have $\mathrm{Hom}_{\mathbf{Gr}}(F(S), C) \cong \mathrm{Hom}_{\mathbf{Set}}(S, G(C))$.
- The forgetful functor $\mathbf{Ring} \to \mathbf{Set}$ has a left adjoint mapping a set S to the free ring $\mathbb{Z}[\{X_s\}_{s \in S}]$.

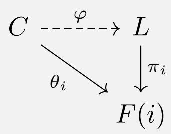
- The inclusion functor **Ab** $\hookrightarrow$ **Gr** has a left adjoint given by the abelianization functor $C \mapsto C^{\text{ab}}$. We have $\text{Hom}_{\mathbf{Ab}}(C^{\text{ab}}, A) \cong \text{Hom}_{\mathbf{Gr}}(C, A)$.
- The forgetful functor **Ring** $\to$ **Ab** has a left adjoint assigning an abelian group A to its tensor algebra $\text{Tens}(A) = \bigoplus_{n \geq 0} A^{\otimes n}$.
- The forgetful functor **CommRing** $\to$ **Ab** has a left adjoint assigning an abelian group A to its symmetric algebra $\text{Sym}(A) = \bigoplus_{n \geq 0} (A^{\otimes n})/\mathfrak{S}_n$.
- The functor of units **Ring** $\to$ **Gr** mapping $R \mapsto R^\times$ has a left adjoint mapping a group G to the group ring $\mathbb{Z}[G]$.
- In $R-$**Mod**, for any $N \in \text{Ob}(R-\mathbf{Mod})$, the functor $M \mapsto M \otimes_R N$ is left adjoint to the functor $Q \mapsto \text{Hom}_R(N, Q)$. This induces the adjunction $\text{Hom}_R(M \otimes_R N, Q) \cong \text{Hom}_R(M, \text{Hom}_R(N, Q))$.
- In **Set**, for a fixed set T, the functor $S \mapsto S \times T$ is left adjoint to the functor $U \mapsto U^T$. Thus $\text{Hom}_{\mathbf{Set}}(S \times T, U) \cong \text{Hom}_{\mathbf{Set}}(S, U^T)$.
- Let **Top** be the category of locally compact Hausdorff spaces. For a space Y, the functor $X \mapsto X \times Y$ has a right adjoint $Z \mapsto \mathcal{C}^0(Y, Z)$ (equipped with the compact-open topology). Thus $\text{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \text{Hom}_{\mathbf{Top}}(X, \mathcal{C}^0(Y, Z))$.

End of Lec 03.

6 General Limits and Colimits

6.1 Limits of Functors

Definition (Limit of a Functor). Let $\mathcal{C}$ be a category, and let I be a small category. Consider a functor $F : I \to \mathcal{C}$, which can be viewed as a "diagram" in $\mathcal{C}$. A *limit* of F is a pair $(L, (\pi_i)_{i \in \text{Ob}(I)})$, where $L \in \text{Ob}(\mathcal{C})$ and each $\pi_i : L \to F(i)$ is a morphism such that the family forms a natural transformation from the constant functor $\underline{L}$ to F. Recall that the constant functor $\underline{L} : I \to \mathcal{C}$ is defined by $\underline{L}(i) = L$ and $\underline{L}(u) = \text{Id}_L$ for all morphisms u in I.
This pair must satisfy the following universal property: For any object $C \in \text{Ob}(\mathcal{C})$ and any natural transformation $(\theta_i)_{i \in \text{Ob}(I)}$ from $\underline{C}$ to F, there exists a unique morphism $\varphi : C \to L$ such that $\theta_i = \pi_i \circ \varphi$ for all $i \in \text{Ob}(I)$.



If a limit of F exists, it is unique up to unique isomorphism. We generally denote the limit by $\lim_I F$ or $\varprojlim_I F$.

Alternative Characterizations of Limits. The definition of a limit admits several useful reformulations:

1. Representability: The universal property implies that the limit (L, π) represents the functor $C \mapsto \text{Nat}_{I,\mathcal{C}}(\underline{C}, F)$ from $\mathcal{C}^{\text{op}}$ to **Set**. We have a natural bijection:
$$\text{Hom}_{\mathcal{C}}(C, L) \cong \text{Nat}_{I,\mathcal{C}}(\underline{C}, F)$$
2. Equalizer Formulation: The set of natural transformations can be embedded into a product of Hom-sets, forming an equalizer sequence in **Set**:
$$\text{Nat}_{I,\mathcal{C}}(\underline{C}, F) \hookrightarrow \prod_{i \in \text{Ob}(I)} \text{Hom}_{\mathcal{C}}(C, F(i)) \rightrightarrows \prod_{(u: i \to j) \in \text{Mor}(I)} \text{Hom}_{\mathcal{C}}(C, F(j))$$
where the two arrows map a family (f_i) to $(F(u) \circ f_i)$ and (f_j), respectively.
3. Overcategory: Consider the category $\mathcal{C}/F$, where objects are pairs (C, θ) with $\theta \in \text{Nat}(\underline{C}, F)$, and a morphism from (C, θ) to (C', θ') is a morphism $v : C \to C'$ in $\mathcal{C}$ such that $\theta' \circ \underline{v} = \theta$. A limit of F is exactly a *final object* in the category $\mathcal{C}/F$.

6.2 Examples of Limits

Products. If I is a discrete category (having no morphisms other than identities), then $\mathrm{Hom}_I(s,t) = \emptyset$ for $s \neq t$ and $\mathrm{Hom}_I(s,s) = \{\mathrm{Id}_s\}$. A functor $F : I \to \mathcal{C}$ is equivalent to a family of objects $(F(i))_{i\in\mathrm{Ob}(I)}$. The limit of this functor, if it exists, is exactly the product $\prod_{i\in\mathrm{Ob}(I)} F(i)$.

Equalizers. Following the notation of Mac Lane, let I be the category with two objects A, B and two parallel morphisms $f, g : A \rightrightarrows B$. A functor $F : I \to \mathcal{C}$ specifies a diagram $F(A) \rightrightarrows F(B)$. The limit of F is the equalizer of $F(f)$ and $F(g)$.

Fiber Products. Let I be the category consisting of three objects and two morphisms forming the shape $B \to A \leftarrow C$. A functor $F : I \to \mathcal{C}$ yields a diagram

$$\begin{CD} F(B)\times_{F(A)} F(C) @>>> F(C) \\ @VVV @VV{g}V \\ F(B) @>{f}>> F(A) \end{CD}$$

The limit of this diagram is the fiber product (or pullback), denoted by $F(B) \times_{F(A)} F(C)$.
For example, in **Set**, the fiber product is constructed as:

$$C \times_A B = \{(c,b) \in C \times B \mid g(c) = f(b)\}$$

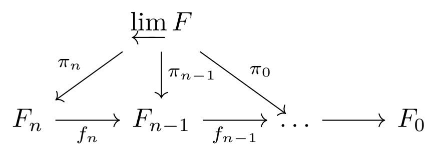
If $F(A) = *$ is a final object in $\mathcal{C}$, then the fiber product reduces to the standard product $F(B) \times F(C)$. Note that if A is empty, the fiber product is also just the standard product.

Invariants under Group Actions. Let G be a group and $\mathcal{C}_G$ be the category with one object $*$ such that $\mathrm{Hom}_{\mathcal{C}_G}(*,*) = G$. A functor $F : \mathcal{C}_G \to$ **Set** assigns the single object $*$ to a set S, and the morphisms to a group action of G on S. The limit of F in **Set** is the set of fixed points (invariants) under this action:

$$\lim_{\mathcal{C}_G} S = S^G = \{s \in S \mid g \cdot s = s \text{ for all } g \in G\}$$

Inverse Limits. Let I be the category with $\mathrm{Ob}(I) = \mathbb{N}$, and $\mathrm{Hom}_I(n,m) = \{*\}$ if $n \geq m$, and $\emptyset$ otherwise. A functor $F : I \to \mathcal{C}$ defines an inverse system or "tower" of objects and morphisms:

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$



The limit of this tower, denoted $\varprojlim F$, consists of compatible sequences:

$$\varprojlim F = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} F_n \;\middle|\; \forall n \geq 1, f_n(x_n) = x_{n-1} \right\}$$

As a concrete example, let R be a commutative ring and consider the formal power series ring $R[[X]]$. The sequence of quotients $R[[X]]/(X^n)$ forms a tower of rings:

$$\cdots \to R[[X]]/(X^n) \to R[[X]]/(X^{n-1}) \to \cdots \to R$$

One can verify that $\varprojlim_n R[[X]]/(X^n) \cong R[[X]]$. This justifies the terminology "limit", drawing a parallel to limits in analysis.

6.3 Existence of Limits

Definition (Complete Category). A category $\mathcal{C}$ is said to *admit all limits* (or be *complete*) if, for every small category I and every functor $F : I \to \mathcal{C}$, the limit $\lim_I F$ exists in $\mathcal{C}$.

Theorem (Limits in Set). The category **Set** admits all limits.

Proof. Let I be a small category and $F : I \to \mathbf{Set}$ be a functor. We construct the limit L explicitly as a subset of the product of all sets $F(i)$:

$$L = \left\{ (x_i)_{i \in \mathrm{Ob}(I)} \in \prod_{i \in \mathrm{Ob}(I)} F(i) \;\middle|\; \forall (u : i \to j) \in \mathrm{Mor}(I), F(u)(x_i) = x_j \right\}$$

The family of canonical projections $\pi_i : L \to F(i)$ forms a natural transformation from $\underline{L}$ to F. The construction exactly exhibits L as the equalizer of the two canonical maps:

$$\prod_{i \in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{(u:i \to j) \in \mathrm{Mor}(I)} F(j)$$

Thus, L satisfies the universal property of the limit in **Set**.

□

Corollary 1. Let $F : I \to \mathcal{C}$ be a functor which admits a limit in $\mathcal{C}$. For any object $C \in \mathrm{Ob}(\mathcal{C})$, the Hom-functor commutes with limits in **Set**. Specifically, we have a canonical bijection:

$$\mathrm{Hom}_{\mathcal{C}}(C, \varprojlim_I F) \cong \varprojlim_I (\mathrm{Hom}_{\mathcal{C}}(C, F(-)))$$

Corollary 2 (Existence Criterion). Let $\mathcal{C}$ be a category. If $\mathcal{C}$ admits all (small) products and equalizers, then $\mathcal{C}$ admits all limits.

Proof Sketch. Let I be a small category and $F : I \to \mathcal{C}$ be a functor. By hypothesis, the product over all objects, $\prod_{i \in \mathrm{Ob}(I)} F(i)$, and the product over all morphisms, $\prod_{(u:i \to j) \in \mathrm{Mor}(I)} F(j)$, both exist in $\mathcal{C}$. Furthermore, there are two canonical morphisms between these products: one given by evaluating the morphisms $F(u)$ and the other by canonical projections. Since $\mathcal{C}$ admits equalizers, the equalizer of these two morphisms exists. By an analogous argument to the construction in **Set**, one can verify that this equalizer satisfies the universal property of $\lim_I F$.

□

End of Lec 04.

6.4 Limits and Adjunctions

Remark (Limits via Products and Equalizers). We saw that in a category $\mathcal{C}$:

$$\text{(all limits exist)} \iff \text{(arbitrary products and equalizers exist)}$$

For any small category I and any functor $F : I \to \mathcal{C}$, the limit $\lim_I F$ is given by an equalizer:

$$\lim_I F \to \prod_{i \in \mathrm{Ob}(I)} F(i) \rightrightarrows \prod_{(u:i \to j) \in \mathrm{Mor}(I)} F(j)$$

Lemma (Limits and Right Adjoints). Let I be a small category. Let $\mathcal{C}^I$ be the category of functors $F : I \to \mathcal{C}$ (and natural transformations). The category $\mathcal{C}$ admits all limits of functors $F : I \to \mathcal{C}$ if and only if the constant diagram functor $\Delta : \mathcal{C} \to \mathcal{C}^I$, mapping $C \mapsto \underline{C}$, admits a right adjoint.

Proof. $\Rightarrow$: Assume $\mathcal{C}$ admits all limits. Consider the functor $\lim_I : \mathcal{C}^I \to \mathcal{C}$ mapping $F \mapsto \lim_I F$. For all $C \in \mathrm{Ob}(\mathcal{C})$ and all $F \in \mathrm{Ob}(\mathcal{C}^I)$, there is a natural bijection by the definition of the limit of F:

$$\mathrm{Hom}_{\mathcal{C}}(C, \lim_I F) \cong \mathrm{Hom}_{\mathcal{C}^I}(\underline{C}, F) = \mathrm{Nat}(\underline{C}, F)$$

This defines the adjunction, showing that the limit functor is a right adjoint to the constant diagram functor Δ.

$\Leftarrow$: Conversely, if $\Delta : \mathcal{C} \to \mathcal{C}^I$ has a right adjoint $R : \mathcal{C}^I \to \mathcal{C}$, then for all $F \in \mathrm{Ob}(\mathcal{C}^I)$, $R(F) \in \mathrm{Ob}(\mathcal{C})$ is a limit of F. We have a natural isomorphism:

$$\mathrm{Hom}_{\mathcal{C}}(C, R(F)) \cong \mathrm{Hom}_{\mathcal{C}^I}(\underline{C}, F)$$

We already observed that $\lim_I F$ exists if and only if the functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ given by $C \mapsto \mathrm{Hom}_{\mathcal{C}^I}(\underline{C}, F)$ is representable. Since it is represented by $R(F)$, the limit exists and is naturally isomorphic to $R(F)$.

□

Lemma (Right Adjoints Commute with Limits). Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors, where $R : \mathcal{D} \to \mathcal{C}$ is the right adjoint and $L : \mathcal{C} \to \mathcal{D}$ is the left adjoint. Let $F : I \to \mathcal{D}$ be a functor from a small category I. If F admits a limit in $\mathcal{D}$, then $R \circ F : I \to \mathcal{C}$ automatically admits a limit, which is exactly $R(\lim_I F) \in \mathrm{Ob}(\mathcal{C})$.
In short: "a right adjoint commutes with limits".

Proof. For all $C \in \mathrm{Ob}(\mathcal{C})$, we verify the universal property via canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(C, R(\lim_I F)) &\cong \mathrm{Hom}_{\mathcal{D}}(L(C), \lim_I F) && \text{(adjunction)}\\
&\cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{D}}(L(C), F(i)) && \text{(Hom commutes with limits)}\\
&\cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(C, R(F(i))) && \text{(adjunction)}\\
&\cong \mathrm{Nat}(\underline{C}, R \circ F) && \text{(definition of limit)}
\end{aligned}$$

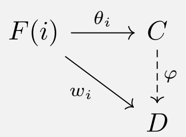
This natural bijection implies that $R(\lim_I F)$ represents the limit functor for $R \circ F$, therefore $R(\lim_I F) = \lim_I R(F(i))$.

□

6.5 Colimits

Colimits are defined equivalently as limits in the opposite category $\mathcal{C}^{\mathrm{op}}$. Let $F : I \to \mathcal{C}$ be a functor, with I being a small category.

Definition (Colimit). A *colimit* of F is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a natural transformation $\theta : F \to \underline{C}$ to the constant functor $\underline{C}$. This pair must satisfy the universal property: For all $D \in \mathrm{Ob}(\mathcal{C})$ and all natural transformations $w \in \mathrm{Nat}(F, \underline{D})$, there exists a unique morphism $\varphi : C \to D$ such that $w = \underline{\varphi} \circ \theta$.



In case it exists, we denote it by $\mathrm{colim}_I F$ or $\varinjlim_I F$.

Reformulations and Remarks.

1. The functor $\mathcal{C} \to \mathbf{Set}$ mapping $D \mapsto \mathrm{Nat}(F, \underline{D})$ is representable if and only if there exists a natural isomorphism $\Theta : \mathrm{Nat}(F, \underline{D}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_I F, D)$.
2. For all $D \in \mathrm{Ob}(\mathcal{C})$, we have the isomorphism: $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_I F, D) \cong \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(F(i), D)$.
3. If I is fixed, all functors $F : I \to \mathcal{C}$ admit a colimit if and only if the constant diagram functor $\mathcal{C} \to \mathcal{C}^I$ mapping $D \mapsto \underline{D}$ has a left adjoint.
4. In a category $\mathcal{C}$, all colimits exist if and only if arbitrary coproducts and arbitrary coequalizers exist.
5. If $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a pair of adjoint functors and $F : I \to \mathcal{C}$ admits a colimit in $\mathcal{C}$, then $L(\mathrm{colim}_I F) \cong \mathrm{colim}_I(L \circ F)$. Left adjoints commute with colimits.

Example (Pushout). Assume $\mathcal{C}$ admits arbitrary coproducts and arbitrary coequalizers. A diagram of the shape $Y \xleftarrow{f} X \xrightarrow{g} Z$ admits a colimit, which is called the pushout (or amalgamated sum) and denoted $Y \coprod_X Z$. It naturally yields a commuting square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & Y \coprod_X Z \end{array}$$

6.5.1 Construction and Examples of Colimits

Construction via Coproducts and Coequalizers. Let $F : I \to \mathcal{C}$ be a functor, with I being a small category. The colimit of F can be constructed as the coequalizer of the following two maps:

$$\coprod_{(u:i\to j)\in \mathrm{Mor}(I)} F(i) \rightrightarrows \coprod_{i\in \mathrm{Ob}(I)} F(i) \to \mathrm{colim}_I\, F$$

Example (Colimits in Set). In **Set**, all limits and colimits exist. Coproducts are given by disjoint unions ($\coprod$). The coequalizer of two maps $f, g : S \rightrightarrows T$ is given by the quotient $T/\sim$, where $\sim$ is the equivalence relation generated by $f(s) \sim g(s)$ for all $s \in S$.

Note: The forgetful functor **Ab** $\to$ **Set** is a right adjoint. Consequently, it keeps limits but forgets colimits (it does not preserve them).

Example (Pushout in Set). For a diagram $Y \xleftarrow{f} X \xrightarrow{g} Z$ in **Set**, the pushout $Y \coprod_X Z$ is the amalgamated sum of Y and Z over X. It is constructed as the coequalizer of $X \rightrightarrows Y \coprod Z$, where the two maps are induced by f and g. Thus, $Y \coprod_X Z = (Y \coprod Z)/\sim$, where the equivalence relation is generated by $(f(x), g(x))_{x\in X}$.

Example (Colimits in Groups). In the category **Gr**, all colimits exist. The coproduct $\coprod_{i\in I} G_i$ is the free product $*_{i\in I} G_i$. For example, the free product of G and H is $G * H$, consisting of alternating words formed by elements of $\mathring{G} = G \setminus \{e_G\}$ and $\mathring{H} = H \setminus \{e_H\}$.

Coequalizer in Gr. Let $f, g : H \rightrightarrows G$ be group homomorphisms. The coequalizer is given by the quotient G/N:

$$H \rightrightarrows G \to \mathrm{Coeq}(f, g) = G/N$$

where $N \subset G$ is the normal subgroup generated by elements of the form $f(h)g(h)^{-1}$ for all $h \in H$. Specifically, we take the subgroup $S = \langle f(h)g(h)^{-1}\rangle \subset G$ and its normal closure in G:

$$N_G(\langle f(h)g(h)^{-1}\rangle) = \langle g\langle f(h)g(h)^{-1}\rangle g^{-1}\rangle_{g\in G} \subset G$$

which yields the exact normal subgroup N.

Note: The "forgetful functor" (like **Gr** $\to$ **Set**) almost never commutes with colimits!

6.6 Presheaves and Yoneda

Remark (Presheaves). Let $\mathcal{C}$ be an (essentially) small category. For all $C \in \mathrm{Ob}(\mathcal{C})$, the Hom-functor yields a functor $\mathrm{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\mathrm{op}} \to$ **Set**. This is called the *presheaf on $\mathcal{C}$ represented by C*. We define the category of presheaves on $\mathcal{C}$ as:

$$\mathrm{Presheaf}(\mathcal{C}) = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$$

By Yoneda's Lemma, $\mathcal{C}$ embeds into $\mathrm{Presheaf}(\mathcal{C})$ as a full subcategory via the assignment $C \mapsto \mathrm{Hom}_{\mathcal{C}}(-, C)$.

Remark: The category $\mathrm{Presheaf}(\mathcal{C})$ has all limits and all colimits!

End of Lec 05.

7 Adjunctions, Epimorphisms, and Group Objects

7.1 Unit and Counit of an Adjunction

Definition (Unit and Counit of an Adjunction). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors, meaning there is a natural bijection for all $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$:

$$\theta_{C,D} : \mathrm{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, R(D))$$

By setting $D = L(C)$ and evaluating θ on the identity morphism $\mathrm{Id}_{L(C)} \in \mathrm{Hom}_{\mathcal{D}}(L(C), L(C))$, we obtain a natural transformation:

$$\eta : \mathrm{Id}_{\mathcal{C}} \to R \circ L$$

which is called the *unit of the adjunction*. Its components are given by $\eta_C = \theta_{C,L(C)}(\mathrm{Id}_{L(C)}) : C \to R(L(C))$.
In the same way, by setting $C = R(D)$ and applying the inverse bijection θ^{-1} to the identity morphism $\mathrm{Id}_{R(D)} \in \mathrm{Hom}_{\mathcal{C}}(R(D), R(D))$, we obtain a natural transformation:

$$\varepsilon : L \circ R \to \mathrm{Id}_{\mathcal{D}}$$

which is called the *counit of the adjunction*. Its components are given by $\varepsilon_D = \theta^{-1}_{R(D),D}(\mathrm{Id}_{R(D)}) : L(R(D)) \to D$.

Reconstruction of the Adjunction Isomorphism. The data of the natural bijection θ is equivalent to giving the pair of natural transformations (η, ε) satisfying the triangle identities.

$$\begin{array}{ccc} L(C) & \xrightarrow{L(\eta_C)} & L(R(L(C))) \\ & {\scriptstyle \mathrm{Id}_{L(C)}} \searrow & \downarrow {\scriptstyle \varepsilon_{L(C)}} \\ & & L(C) \end{array} \qquad \begin{array}{ccc} R(D) & \xrightarrow{\eta_{R(D)}} & R(L(R(D))) \\ & {\scriptstyle \mathrm{Id}_{R(D)}} \searrow & \downarrow {\scriptstyle R(\varepsilon_D)} \\ & & R(D) \end{array}$$

If (η, ε) are given, the adjunction isomorphism is recovered as follows: For any morphism $f \in \mathrm{Hom}_{\mathcal{D}}(L(C), D)$, the corresponding morphism in $\mathrm{Hom}_{\mathcal{C}}(C, R(D))$ is obtained via the composition:

$$\mathrm{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{R} \mathrm{Hom}_{\mathcal{C}}(R(L(C)), R(D)) \xrightarrow{\circ \eta_C} \mathrm{Hom}_{\mathcal{C}}(C, R(D))$$

The inverse map θ^{-1} is analogously constructed using ε. An *equivalence of categories* is precisely a pair of adjoint functors for which both the unit η and the counit ε are natural isomorphisms.

7.2 Epimorphisms

Definition (Categorical Epimorphism). Let $\mathcal{C}$ be a category. A morphism $f : C \to D$ is called a *categorical epimorphism* in $\mathcal{C}$ if it is right-cancellative. That is, for all objects $E \in \mathrm{Ob}(\mathcal{C})$, the induced map on Hom-sets:

$$\mathrm{Hom}_{\mathcal{C}}(D, E) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(C, E)$$
$$g \mapsto g \circ f$$

is injective. Equivalently, $g_1 \circ f = g_2 \circ f \implies g_1 = g_2$. We often denote an epimorphism by a two-headed arrow: $C \twoheadrightarrow D$.

Examples (Categorical Epimorphisms).

1. Category of Sets: In **Set**, the epimorphisms are exactly the surjective maps.

   Proof. Let $f : C \to D$ be an epimorphism in **Set**. Assume for the sake of contradiction that f is not surjective, so the image $\mathrm{Im}(f) \subsetneq D$. We can define two distinct functions $g_1, g_2 : D \to D$. Let $g_1 = \mathrm{Id}_D$. Let g_2 be the identity on $\mathrm{Im}(f)$, but for elements in $D \setminus \mathrm{Im}(f)$, assign them to a fixed point in $\mathrm{Im}(f)$ (or map them to any distinct value, assuming a sufficiently large codomain, such as $D \amalg \{*\}$). Thus, $g_2|_{\mathrm{Im}(f)} = \mathrm{Id}_{\mathrm{Im}(f)}$ and $g_2|_{D\setminus\mathrm{Im}(f)} \neq \mathrm{Id}_{D\setminus\mathrm{Im}(f)}$. By construction, $g_1 \circ f = g_2 \circ f$, but $g_1 \neq g_2$. This contradicts that f is an epimorphism. Hence, we must have $\mathrm{Im}(f) = D$.

□

2. Category of Groups: In **Gr**, an epimorphism $f : H \to G$ is exactly a surjective homomorphism.

   Proof. Let $f : H \to G$ be an epimorphism in **Gr**. We reduce the problem to showing that the inclusion of a proper subgroup $H \subsetneq G$ is not an epimorphism. If $H \neq G$, there exists a group K and two distinct homomorphisms $G \rightrightarrows K$ that agree on H. Specifically, consider the natural action of G on the left cosets G/H, which yields a homomorphism $p_1 : G \to S(G/H)$. By twisting this action or introducing a trivial action p_2, one can construct two maps to an appropriate permutation group (or an amalgamated product $G *_H G$) that agree precisely on H. Since f is an epimorphism, right-cancellation forces these two maps to be identical on all of G, forcing $H = G$. Thus, f is surjective.

   □

3. Category of Commutative Rings: In commutative rings, it is false that epimorphisms must be surjective! The inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a categorical epimorphism despite not being surjective.

   Proof. The inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a categorical epimorphism. For any commutative ring R, a homomorphism $\varphi \in \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Q}, R)$ is completely determined by its restriction to $\mathbb{Z}$. Since $\mathbb{Z}$ is the initial object in the category of commutative rings, there exists a unique homomorphism $\psi : \mathbb{Z} \to R$ mapping $1_{\mathbb{Z}} \mapsto 1_R$.

   Any extension of this map to $\mathbb{Q}$ is uniquely determined, because for any rational number $a/b \in \mathbb{Q}$, the homomorphism must satisfy $b \cdot \varphi(a/b) = \varphi(a) = a \cdot 1_R$. Thus, $\varphi(a/b)$ must be the unique inverse of $b{\cdot}1_R$ multiplied by $a{\cdot}1_R$ in R. Consequently, the Hom-set $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Q}, R)$ contains at most one element. In particular, the restriction map induced by the inclusion:

   $$\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Q}, R) \hookrightarrow \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}, R)$$

   is injective. Thus, i is an epimorphism.

   □

7.2.1 Effective and Regular Epimorphisms

Following Grothendieck, we refine the notion of epimorphisms to address behavior under base change and limits.

Definition (Effective, Universal, and Regular Epimorphisms). Let $\mathcal{C}$ be a category and let $f : C \to D$ be a morphism.

1. f is called an *effective epimorphism* if the fiber product $C \times_D C$ exists in $\mathcal{C}$, and the canonical diagram formed by the projections pr_1, pr_2:

   $$C \times_D C \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} C \xrightarrow{f} D$$

   is a coequalizer.

2. f is called a *universal effective epimorphism* if it is an effective epimorphism and for all morphisms $D' \to D$, the fiber product $C \times_D D'$ exists, and the induced projection $f' : C \times_D D' \to D'$ is also an effective epimorphism.

3. f is called a *regular epimorphism* if there exists an object B and a pair of parallel morphisms $f_1, f_2 : B \rightrightarrows C$ such that the sequence:

   $$B \overset{f_1}{\underset{f_2}{\rightrightarrows}} C \xrightarrow{f} D$$

   is a coequalizer.

Fact (Equivalence of Effective and Regular Epimorphisms). If all fiber products exist in $\mathcal{C}$, then f is a regular epimorphism $\iff$ f is an effective epimorphism.

Proof. The implication ($\Leftarrow$) holds directly by definition, setting $B = C \times_D C$ and $f_1 = pr_1, f_2 = pr_2$.

For ($\Rightarrow$), let $B \rightrightarrows C \xrightarrow{f} D$ be a coequalizer. We must show that $C \times_D C \rightrightarrows C \xrightarrow{f} D$ is also a coequalizer. By the universal property of the fiber product $C \times_D C$ with respect to the maps f_1, f_2 (which satisfy $f \circ f_1 = f \circ f_2$), there exists a unique morphism $h : B \to C \times_D C$ such that $pr_1 \circ h = f_1$ and $pr_2 \circ h = f_2$.
Let $g : C \to E$ be any morphism such that $g \circ pr_1 = g \circ pr_2$. Precomposing with h, we obtain:

$$\begin{aligned} & g \circ pr_1 \circ h = g \circ pr_2 \circ h \\ \implies & g \circ f_1 = g \circ f_2 \end{aligned}$$

Because f is the coequalizer of f_1 and f_2, the equality $g \circ f_1 = g \circ f_2$ guarantees the existence of a unique morphism $u : D \to E$ such that $u \circ f = g$. This unique factorization proves that f is the coequalizer of pr_1 and pr_2.

□

Example (Epimorphisms in Groups). Let $f : G \twoheadrightarrow H$ be a surjective group homomorphism with kernel $K = \mathrm{Ker}(f)$. The fiber product $G \times_H G$ is given by:

$$G \times_H G = \{(g_1, g_2) \in G \times G \mid f(g_1) = f(g_2)\} \subset G \times G$$

Notice that $f(g_1) = f(g_2)$ is equivalent to $g_1 g_2^{-1} \in K$.

Exercise: The map $K \times G \to G \times_H G$ defined by $(k, g) \mapsto (g, kg)$ is a bijection. This formally describes the equivalence relation associated to the quotient $G \twoheadrightarrow H$.

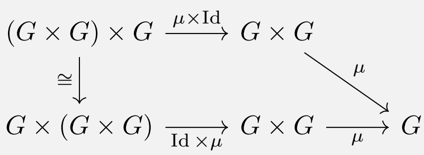
7.3 Group Objects

Definition (Group Object). Let $\mathcal{C}$ be a category where all finite products exist. In particular, $\mathcal{C}$ admits a final object $*$, which serves as the empty product. A *group object* (or "group in $\mathcal{C}$") is an object $G \in \mathrm{Ob}(\mathcal{C})$ equipped with three morphisms:

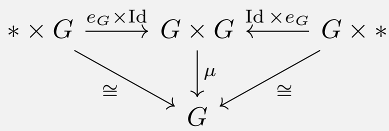
- Multiplication: $\mu : G \times G \to G$
- Unit: $e_G : * \to G$
- Inverse: $\chi : G \to G$

Such that the following three diagrams commute:

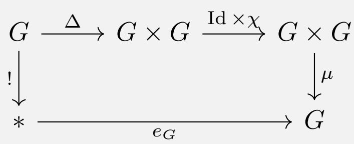
1. Associativity:



2. Identity:



3. Inverse:



where $\Delta : G \to G \times G$ is the canonical diagonal morphism and $! : G \to *$ is the unique morphism to the final object.

Remark (Representability of Group Objects). Let $G \in \mathrm{Ob}(\mathcal{C})$ and let $\mu : G \times G \to G$ be a morphism. G is a group object in $\mathcal{C}$ if and only if for all $C \in \mathrm{Ob}(\mathcal{C})$, the set $\mathrm{Hom}_{\mathcal{C}}(C, G)$ is a group in **Set**. The group operation is given by the pushforward of μ:

$$\mu_* : \mathrm{Hom}_{\mathcal{C}}(C, G) \times \mathrm{Hom}_{\mathcal{C}}(C, G) \cong \mathrm{Hom}_{\mathcal{C}}(C, G \times G) \xrightarrow{\circ \mu} \mathrm{Hom}_{\mathcal{C}}(C, G)$$

The neutral element corresponds to $\mathrm{Hom}_{\mathcal{C}}(C, *) \cong \{*\} \to \mathrm{Hom}_{\mathcal{C}}(C, G)$.

7.3.1 Quotients by Group Actions

Definition (Quotient by a Group Action). Let G be a group object in $\mathcal{C}$ and let $X \in \mathrm{Ob}(\mathcal{C})$. An *action of G on X* is a morphism $\lambda : G \times X \to X$ satisfying the standard associativity and identity axioms expressed via commuting diagrams.
The *quotient* of X by the action of G, denoted X/G, is defined as the coequalizer of the action map λ and the canonical projection pr_2:

$$G \times X \underset{pr_2}{\overset{\lambda}{\rightrightarrows}} X \dashrightarrow X/G$$

Examples (Quotients by a Group Action).

1. Action of $\mathbb{C}^\times$ on $\mathbb{C}$: Let the multiplicative group $\mathbb{C}^\times$ act on $\mathbb{C}$ by standard multiplication. In **Set**, the orbits are exactly $\mathbb{C}^\times$ and $\{0\}$, so the categorical quotient is $\mathbb{C}/\mathbb{C}^\times = \{\overline{\mathbb{C}^\times}, \overline{0}\}$. However, in the category of Hausdorff topological spaces or manifolds, the only continuous invariant functions to a separated space are constant. Thus, the quotient collapses to a single point: $\mathbb{C}/\mathbb{C}^\times = *$. This phenomenon motivates *Geometric Invariant Theory (GIT)*.

2. Action of the Symmetric Group S_n on $\mathbb{C}^n$: The symmetric group acts on $\mathbb{C}^n$ by permuting coordinates. By the fundamental theorem of symmetric polynomials, the quotient is isomorphic to $\mathbb{C}^n$ itself: $\mathbb{C}^n/S_n \cong \mathbb{C}^n$. The map is given by assigning the roots $(z_1, \ldots, z_n)$ to the coefficients of the monic polynomial $\prod(X - z_i)$. For example, when $n = 2$, the map $\mathbb{C}^2 \to \mathbb{C}^2/S_2 \cong \mathbb{C}^2$ is given by $(z_1, z_2) \mapsto (a, b)$, where $a = z_1 + z_2$ and $b = z_1 z_2$, corresponding to the polynomial $X^2 - aX + b$. For the real case, the quotient $\mathbb{R}^2/S_2$ is restricted by the condition that the polynomial must have real roots, hence the discriminant must be non-negative: $\mathbb{R}^2/S_2 = \{(a, b) \in \mathbb{R}^2 \mid a^2 - 4b \geq 0\}$.

7.4 Grothendieck Topologies and Sheaves

Remark (Yoneda Embedding and Grothendieck Topologies). Recall the Yoneda embedding, which provides a full embedding of any small category $\mathcal{C}$ into the category of presheaves $\widehat{\mathcal{C}} = \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$$\begin{aligned} \mathcal{C} &\hookrightarrow \widehat{\mathcal{C}} \\ C &\mapsto \mathrm{Hom}_{\mathcal{C}}(-, C) \end{aligned}$$

A *Grothendieck topology* τ on $\mathcal{C}$ consists of specifying for each object $X \in \mathrm{Ob}(\mathcal{C})$ a collection of *covering families* $(U_\alpha \to X)_\alpha$, satisfying stability under base change and local character properties. This specification allows one to define the category of sheaves on $\mathcal{C}$ with respect to τ, denoted $\mathrm{Sh}_\tau(\mathcal{C})$. We have a sequence of inclusions:

$$\mathcal{C} \subset \mathrm{Sh}_\tau(\mathcal{C}) \subset \widehat{\mathcal{C}}$$

Importantly, both the category of presheaves and the category of sheaves $\mathrm{Sh}_\tau(\mathcal{C})$ admit all limits and colimits.

End of Lec 06.

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