

Algebra 2025/2026: Exercise sheet 9

Exercise 1.

Let R be a commutative ring, M and N be R -modules. We denote by $\text{Hom}_{R\text{-Mod}}(M, N)$ the set of R -linear maps from M to N . For R -linear maps f and $g : M \rightarrow N$, we denote by $f + g$ the map $M \rightarrow N$, $x \mapsto f(x) + g(x)$ and for $\lambda \in R$ we denote by $\lambda \cdot f$ the map $M \rightarrow N$, $x \mapsto \lambda \cdot f(x)$. Show that $f + g$ and $\lambda \cdot f$ are R -linear maps and that in this way $(\text{Hom}_{R\text{-Mod}}(M, N), +, \cdot)$ is an R -module. Moreover, show that for R -modules L, M, N the composition map

$$\text{Hom}_{R\text{-Mod}}(L, M) \times \text{Hom}_{R\text{-Mod}}(M, N) \rightarrow \text{Hom}_{R\text{-Mod}}(L, N)$$

is an R -bilinear map.

Exercise 2.

Let S be a countable set, that is to say in bijection with a subset of \mathbb{N} . Let R be a P.I.D. Show that any sub- R -module of the free R -module on S , $R[S] = \bigoplus_{s \in S} R$ is a free R -module on a countable set. [Hint: assume $S = \mathbb{N}$, and consider $R[\mathbb{N}]$ as the union of the sub- R -modules $F_n = R[\{0, \dots, n\}]$, free on the subset $\{0, \dots, n\} \subset \mathbb{N}$.]

Exercise 3.

Let R be an integral domain, K its field of fractions.

- 1) Show that if R is not a field, K is never a finite type R -module. [Hint: consider a finite set of fractions $\frac{a_i}{q_i}$ which generate K as an R -module. Set $q = \prod_i q_i$; show then that $R_q = K$ and conclude...]
- 2) Show that K contains no free sub- R -module of rank ≥ 2 .
- 3) If R is a P.I.D, show that any non zero finite type sub- R -module of K is free of rank 1.

Exercise 4.

Let R be a P.I.D. and K be its field of fractions.

- 1) Let $I \subset R$ be a non zero ideal. Show that the R -module $\text{Hom}_{R\text{-Mod}}(R/I, K/R)$ is isomorphic to R/I . [Hint: let $a \in R \setminus \{0\}$, show that the kernel of $a : K/R \rightarrow K/R$ is the sub- R -module generated by $\frac{1}{a}$ and is isomorphic to $R/(a)$.]
- 2) Let M be a finite type torsion R -module. Show that the R -module $\text{Hom}_{R\text{-Mod}}(M, K/R)$ is isomorphic to M .