Algebra 2025/2026: Exercise sheet 5

Exercise 1.

Let A be a finite **abelian** group of order $n \geq 2$. Let $\{p_1, \ldots, p_s\} \subset \mathcal{P}$ be the distinct prime divisors of n. For each $i \in \{1, \ldots, s\}$ let $A_{p_i} \subset A$ be the (unique) p_i-Sylow subgroup of A. Show that the group homomorphism

$$\Pi_{i=1}^{s} A_{p_i} \to A$$

$$(x_1, \ldots, x_s) \mapsto \Sigma_{i=1}^{s} x_i$$

is an isomorphism of abelian groups.

Exercise 2.

Let G be a finite group, n its order. We assume that there are only 2 distincts prime divisors p and q of n. We assume further that there is only one p-Sylow G_p and one q-Sylow G_q. Show that G is isomorphic to the product (of groups) $G_p \times G_q$. (Hint: show first that it is a semi direct product).

Exercise 3.

1) Let $V \subset S_4$ be the subset consisting of the Identity, and the permutations which are product of two transpositions with disjoint support. Show V is a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2) Prove that there is a group epimorphism $\pi : S_4 \twoheadrightarrow S_3$ whose kernel is V. Conclude that V is a normal subgroup of S_4, contained in A_4, equal to $[A_4, A_4]$ and that S_4 is solvable. (Hint: let P be the set of partitions of $\{1, 2, 3, 4\}$ in two subsets with two elements. Observe that P has 3 elements...)

Exercise 4.

Let $\pi : G \twoheadrightarrow H$ an epimorphism of groups, $K = Ker(\pi)$ its kernel.

1) Show that (G is solvable) $\Leftrightarrow$ (H and K are solvable).

2) Show that if G is nilpotent, H and K are nilpotent. Give an example where H and K are nilpotent but not G.