

Algebra 2025/2026: Exercise sheet 2

Exercise 1.

Let $n \geq 3$ be an integer. Show that the center $Z(\mathcal{S}_n)$ of the n -th symmetric group is trivial.

Exercise 2.

Let $n \geq 1$ be an integer, and let $\mathbb{F}$ be a finite field with $q = |\mathbb{F}|$ elements. Prove that $GL_n(\mathbb{F})$ is a finite group of order

$$(q^n - 1) \times (q^n - q) \times \dots (q^n - q^{(n-1)}) = \prod_{i=0}^{n-1} (q^n - q^i)$$

is Compute the index of $SL_n(\mathbb{F})$ in $GL_n(\mathbb{F})$ and conclude that the order of $SL_n(\mathbb{F})$ is

$$\frac{1}{q-1} \prod_{i=0}^{n-1} (q^n - q^i) = (q^{n-1} + q^{n-2} + \dots + q + 1) \prod_{i=1}^{n-1} (q^n - q^i)$$

Exercise 3.

Let $n \geq 5$ be an integer. Show that in $\mathcal{A}_n$, a 3-cycle is a commutateur. Conclude that for $n \geq 5$, $[\mathcal{A}_n, \mathcal{A}_n] = \mathcal{A}_n$.

Deduce that for $n \geq 5$ the group $\mathcal{A}_n$ is not solvable, and consequently $\mathcal{S}_n$ is not solvable either.

Exercise 4.

Let G be a group, and $\mathbb{Z}[G]$ be the group ring of G : it is the free abelian group with basis $([g])_{g \in G}$, where $[g] \in \mathbb{Z}[G]$, and the product is induced by the bilinear map

$$\mathbb{Z}[G] \times \mathbb{Z}[G] \rightarrow \mathbb{Z}[G] \quad , \quad ([g], [h]) \mapsto [g.h]$$

We denote $[e]$ by $1 \in \mathbb{Z}[G]$, with $e \in G$ the neutral element; observe indeed that $[e]$ is the neutral ring element for the product in $\mathbb{Z}[G]$. The canonical homomorphism $\epsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$, $[g] \mapsto 1$ is a ring homomorphism called the augmentation. The kernel $I(G)$ is a 2-sided ideal in $\mathbb{Z}[G]$. We denote by $I(G)^2$ the 2-sided ideal product of $I(G)$ by itself. Clearly $I(G)^2 \subset I(G)$.

1) Show that $I(G)$ as an abelian group is the free abelian group with basis $([g] - [e])_{g \in G \setminus \{e\}}$, with $e \in G$ the neutral element. Conclude that $I(G)^2$ as abelian group is generated by the family $([g.h] - [g] - [h] + 1)_{(g,h) \in (G \setminus \{e\})^2}$.

2) (*) Let G_{ab} be the abelianisation of G . Show that the morphism of abelian groups $I(G) \rightarrow G_{ab}$, $[g] - 1 \mapsto \bar{g}$, where $\bar{g}$ is the class of g in $G_{ab} = G/[G, G]$, is trivial on $I(G)^2$ and induces an isomorphism of abelian groups

$$I(G)/I(G)^2 \cong G_{ab}$$

[Hint: define a group homomorphism $G_{ab} \rightarrow I(G)/(I(G)^2)$ which is inverse...]