Algebra 2025/2026: Exercise sheet 1

Exercise 1.

Let G be a finite group. Show that if the order $|G|$ of G is a prime number p, then G is isomorphic to $\mathbb{Z}/p$.

Exercise 2.

Let G be a group. Show that the map $\chi : G \to G$, $g \mapsto g^{-1}$ is a group homomorphism if and only if G is abelian.

Exercise 3.

For G a group, we denote by $Z(G) = \{g \in G | \forall h \in G, g.h = h.g\}$ the center of G. Show that $Z(G)$ is an abelian subgroup of G.

Let K be a field and $n \geq 1$ be an integer. Show that $Z(GL_n(K))$ is the subgroup of homotheties, consisting of diagonal matrices with entry $\lambda \in K^\times$ on the diagonal.

Using an analog method, show that the center $Z(SL_n(K))$ is canonically isomorphic to the subgroup $\mu_n(K) \subset K^\times$ of the multiplicative group of K consisting of n-th root of unity in K.

Exercise 4.

1) (*) Let $H \subset G$ be be a subgroup of the group G. If the index $[G : H]$ is 2, show that H is normal.

2) Consider $\mathcal{S}_4$ as the group of permutations of the 4 vertices of a square in $\mathbb{R}^2$. Show that the subgroup $D_4 \subset \mathcal{S}_4$ consisting of isometries of the square (for the canonical euclidean structure) has 8 elements.

3) Show that if $D_4 \subset \mathcal{S}_4$ was a normal subgroup, it would be the kernel of a group homomorphism $\mathcal{S}_4 \to \mathbb{Z}/3$.

4) Conclude that $\mathcal{S}_4$ has a subroup of index 3 which is not normal.