Exercise sheet 9

Exercise 1.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. We denote by $Shv_{\mathcal{T}}(\mathcal{C})$ the category of sheaves of sets on $(\mathcal{C}, \mathcal{T})$. For G a sheaf of groups on $(\mathcal{C}, \mathcal{T})$, or equivalently a group object in $Shv_{\mathcal{T}}(\mathcal{C})$, and an object $X \in Shv_{\mathcal{T}}(\mathcal{C})$, recall that we denote by $H^1_{\mathcal{T}}(X; G)$ the set of isomorphism classes of G-torsors over X. It is pointed by the class of the trivial G-torsor $G \times X \to X$.

1) Let $E \to X$ be a G-torsor. Show that it is trivial if and only if the morphism $E \to X$ admits a section.

2) For a morphism $\phi : G \to H$ of sheaves of groups, show that if $E \to X$ is a G-torsor, $H \times_G EtoX$ is an H-torsor (where $H \times_G E$ means the quotient of $H \times E$ by the action of G given by $g(h, e) = (h.\phi(g)^{-1}, g.e)$) with the obvious left H operation. We denote by E^ϕ this H-torsor. Show that this induces a pointed map $()^\phi : H^1_{\mathcal{T}}(X; G) \to H^1_{\mathcal{T}}(X; H)$.

3) If $\phi : G \to H$ is an epimorphism, with kernel K, show that if $E \to X$ is a G-torsor, the H-torsor E^ϕ is isomorhic to the quotient E/K (by the left action of K !!!) endowed with the obvious action of $H = G/K$.

4) Conclude that under the assumptions of 3), if $E \to X$ is a G-torsor such that E^ϕ is trivial, then it is in the image of $H^1_{\mathcal{T}}(X; K) \to H^1_{\mathcal{T}}(X; G)$.

5) Deduce a small “exact sequence” of groups (sometimes) and pointed sets

$$* \to K(X) \to G(X) \to H(X) \to H^1_{\mathcal{T}}(X; K) \to H^1_{\mathcal{T}}(X; G) \to H^1_{\mathcal{T}}(X; H)$$

6) Prove that for any field k and any integer $n \geq 1$, $H^1_{et}(k; SL_n) = *$.

Exercise 2. Nisnevich Squares

Let $Y \to X$ be an étale morphism in $\mathcal{C} = Sch^{ft}_S$ (as usual). Let $Z \subset X$ be a closed subscheme such that the induced (étale) morphism

$$Z \times_X Y \to Z$$

is an isomorphism.

Show that the diagram

$$\begin{array}{ccc} Y - Z & \subset & Y \\ \downarrow & & \downarrow \\ X - Z & \subset & X \end{array}$$

is a cartesian and a cocartesian square in $\mathcal{C}$ and in $Svh_{Nis}(\mathcal{C})$. [Hint: you might use points...]

Conclude that the induced morphism of sheaves $Y/(Y - Z) \to X/(X - Z)$ is an isomorphism.

Exercise 3.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. Let $\mathcal{M}$ be a sheaf of abelian groups on $(\mathcal{C}, \mathcal{T})$. For any $n > 0$, show that the sheaf associated to the presheaf $X \mapsto H^n_{\mathcal{T}}(X; \mathcal{M})$ is 0.