Exercise sheet 7

Exercise 1.

Let as usual $\mathcal{C} := Sch_S^{ft}$.

1) We consider the following collection $\mathcal{T}_{sm}$ of families of morphism $\{f_i : U_i \to X\}_i$ whith each f_i smooth and $X = \bigcup_i f_i(U_i)$ (recall that a smooth morphism in $\mathcal{C}$ is open). Show that $\mathcal{T}_{sm}$ defines a topology on $\mathcal{C}$, called the smooth topology. Let $Shv_{sm}(\mathcal{C})$ be the corresponding category of sheaves of sets for the smooth topology.

2) Show that $Shv_{sm}(\mathcal{C}) = Shv_{et}(\mathcal{C})$.

Exercise 2.

Let $\mathcal{C}$ be an essentially small category. An object $P \in Pshv(\mathcal{C})$ (the category of presheaves of sets on $\mathcal{C}$) is called projective if for any epimorphism $A \twoheadrightarrow B$ in $Pshv(\mathcal{C})$ the map $Hom_{Pshv(\mathcal{C})}(P, A) \to Hom_{Pshv(\mathcal{C})}(P, B)$ is surjective.

Show that the category $Pshv(\mathcal{C})$ on $\mathcal{C}$ admits enough projective objects in the following sense: there is a family of objects $\{P_i\}_i$ in $Pshv(\mathcal{C})$ such that for any i, P_i is a projective object and for any $A \in Pshv(\mathcal{C})$ there is an epimoprhism from a disjoint sum (in $Pshv(\mathcal{C})$) of objects of $\{P_i\}_i$ to A.

Deduce that $\mathcal{A}b(\mathcal{C})$ admits enough projectives.

Exercise 3.

(**) Let k be a perfect field and let Sch_k be the category of separated finite type k-schemes.

1) For $f : Y \to X$ a morphism in Sch_k, define the quasi-coherent sheaf $\Omega(Y/X)$ on Y of Kähler differentials of Y relative to X, and prove that there is an exact sequence:

$$f^*(\Omega(X/k)) \to \Omega(Y/k) \to \Omega(Y/X) \to 0$$

[Hint: use the description (locally) of $\Omega(Y/X)$ in terms of universal derivation]

2) Let $X \in Sm_k$. Show that the quasi coherent sheaf $\Omega(X/k)$ is locally free of rank $dim(X)$. [Hint: show that $\Omega(\mathbb{A}_k^n/k)$ is free of rank n and use for $f : X \to \mathbb{A}_k^n$ étale the above exact sequence $f^*(\Omega(\mathbb{A}_k^n/k)) \to \Omega(X/k) \to \Omega(X/\mathbb{A}_k^n) \to 0$, which here happens to be short exact...]

3) Let $f : Y \to X$ be a finite (dominant) morphism between regular projective curves over k. Assume f is separable (that is to say the field extension of the functions fields is finite separable). Show that $\Omega(Y/X)$ is a torsion sheaf (generic fiber is 0) with support exactly the closed subset where f is not unramified (called the ramification locus).