Exercise sheet 6

Exercise 1.

Let k be a perfect field. Let $P \in k[T]$ be a monic polynomial of degree ≥ 1. Let

$$f_P : \mathbb{A}^1_k \to \mathbb{A}^1_k$$

be the morphism corresponding to P, that is to say the morphism of k-algebras $k[T] \to k[T]$, $T \mapsto P(T)$.

1) Assume $char(k) = 0$. Show that if $deg(P) > 1$, then there is an non empty closed subset $Z \subset \mathbb{A}^1_k$ sucht that for any $y \in Z$, f is not unramified at y and for any $y \in \mathbb{A}^1_k - Z$, f is unramified at y.

2) Give the list of étale morphisms $\mathbb{A}^1_k \to \mathbb{A}^1_k$ if $char(k) = 0$.

3) Give the list of étale morphisms $(\mathbb{G}_m)_k \to (\mathbb{G}_m)_k$ if $char(k) = 0$.

4) Now assume $char(k) = p > 0$. Show that the morphism

$$f_P : \mathbb{A}^1_k \to \mathbb{A}^1_k$$

corresponding to the polynomial $P = T^p - T$ is étale.

5) Show that the previous morphism is in fact a torsor under the group $\mathbb{Z}/p$ (discrete considered as a k-scheme) in the étale topology.

Exercise 2.

Recall that a morphism of schemes (in Sch^{ft}_S) : $f : Y \to X$ is said to be smooth at $y \in Y$ if there is an open subscheme $U \subset X$ containing $x = f(y)$, an open subschem $V \subset Y$ containing y such that $f(V) \subset U$ and there exist an $n \in \mathbb{N}$, and étale morphism $gV \to \mathbb{A}^n_U$ such that $f|V = p \circ g$ with $p : \mathbb{A}^n_U \to U$ the projection.

1) Show that a composition of two smooth morphisms is smooth.

2) Let Sm_S be the category of finite type smooth S-schemes. Show that Sm_S admits finite products.

3) Does Sm_S admits all the fiber products ?

Exercise 3.

(*) Let k be a perfect field. Show that a regular curve X over k is a smooth k-scheme. [Hint: for $x \in X$, choose a local parameter $t \in \mathcal{M}_x$ (a generator); then show that there is an open neighborhood of x and an étale morphism $U \to Spec(k[T])$ "induced" by t.]