Exercise sheet 5

Exercise 1. (Functoriality

Let $f : S' \to S$ be a morphism of noetherian separated schemes. The pull-back through f induces an obvious functor $f^{-1} : \mathcal{C} := Sch_S^{ft} \to Sch_{S'}^{ft} =: \mathcal{C}'$, where Sch_S^{ft} is the category of finite type separated S-schemes (and same for S').

1) Let $\alpha \in \{Zar, Nis, et, flat\}$. Show that the functor

$$f_*^P : Pshv(\mathcal{C}') \to Pshv(\mathcal{C})$$

obtained by composition with $(f^{-1})^{op} : \mathcal{C}^{op} \to \mathcal{C}'^{op}$ induces functors $f_*^\alpha : Shv_{\mathcal{T}_\alpha}(\mathcal{C}') \to Shv_{\mathcal{T}_\alpha}(\mathcal{C})$ (called the direct images functors).

2) Show for each α, f_*^α admits a left adjoint $f_\alpha^* : Shv_{\mathcal{T}_\alpha}(\mathcal{C}) \to Shv_{\mathcal{T}_\alpha}(\mathcal{C}')$ (called inverse image) which induces by restriction on the full subcategory $\mathcal{C} \subset Shv_{\mathcal{T}_\alpha}(\mathcal{C})$ the above functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$.

3) If $f : S' \to S$ is of finite type (so that $S' \in \mathcal{C}$), who that the functor $f_\sharp : \mathcal{C}' \to \mathcal{C}$ taking $X \to S'$ to the composition with f: $X \to S$ is left adjoint to the functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$.

4) If $f : S' \to S$ is of finite type, show that f_α^* admits a left adjoint $f_\sharp$ which restricts on $\mathcal{C}'$ to the functor $f_\sharp : \mathcal{C}' \to \mathcal{C}$. Conclude that if $f : S' \to S$ is of finite type, f_α^* is an exact functor.

Exercise 2.

Let $\mathcal{C} := Sch_S^{ft}$ endowed with the topology $\alpha \in \{Zar, Nis, et\}$. Recall that $\mathcal{C} \subset Shv_{\mathcal{T}_\alpha}(\mathcal{C})$ can be identified with a full subcategory.

Let $X \in \mathcal{C}$. Let $p : F \to X$ be a morphism of sheaves. Show the equivalence :

(f is an epimorphism) $\Leftrightarrow$ (there exists a covering family $\{f_i : U_i \to X\}_i$ in $\mathcal{T}$ in $\mathcal{T}_\alpha$ such that for any i, there is $\alpha_i \in F(U_i)$ such that $p(\alpha_i) = f_i \in Hom_{\mathcal{C}}(U_i, X)$.

Exercise 3.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a topology. Let G be a sheaf of groups on $(\mathcal{C}, \mathcal{T})$.

1) Let $G \Rrightarrow Y \to Y = X/G$ be G-torsor on $X \in Shv_{\mathcal{T}}(\mathcal{C})$. Let $X' \to X$ be a morphism in $Shv_{\mathcal{T}}(\mathcal{C})$. Show that the pull back morphism $Y' = Y \times_X X' \to X'$ is endowed automatically with a structure of G-torsor.

2) Let G act on Y and let $p : Y \to X$ be a G-equivariant morphism, with trivial action on X. Show that: ($G \Rrightarrow Y \to X$ is a G-torsor) $\Leftrightarrow$ (There is an epimorphism $X' \to X$ such that the pull back $Y' := Y \times_X X' \to X'$ isomorphic over X' to $G \times X'$ with its G-action).

3) Show that the collection $H^1(X; G)$ of isomorphism classes of G-torsors over X is a set.

4) (**) If $\mathcal{C} = Sch_S^{ft}$ and $\mathcal{T} = \mathcal{T}_{Zar}$ describe a bijection for $X \in \mathcal{C}$ between $H^1_{Zar}(X; GL_n)$ and the isomorphism classes of rank n vector bundles over X.