Exercise sheet 4

Exercise 1.

Let k be a field, and Sch_k^{ft} be the category of finite type separated k-schemes. We considere the Zariski topology $\mathcal{T}_{Zar}$ on Sch_k^{ft} formed by the open coverings. Let $n \geq 2$ be an integer. Show that the morphism of k-schemes

$$(-)^n : \mathbb{A}_k^1 \to \mathbb{A}_k^1$$

corresponding to the morphims of k-algebras $k[T] \to k[T]$, $T \mapsto T^n$ (in the obvious way) doesn't induce an epimorphism of sheaves of sets in $Shv_{\mathcal{T}_{Zar}}(Sch_k^{ft})$. [Use that the in the Zariski topology the local rings of points of schemes correspond to points]

Give an example of short exact sequence of sheaves of abelian groups in $Shv_{\mathcal{T}_{flat}}(Sch_k^{ft})$ which is not an exact sequence of sheaves of abelian groups in $Shv_{\mathcal{T}_{Zar}}(Sch_k^{ft})$.

Exercise 2.

Let k be a field, and Sch_k^{ft} be the category of finite type separated k-schemes. We considere the Zariski topology $\mathcal{T}_{Zar}$ on Sch_k^{ft} formed by the open coverings and also the flat topology $\mathcal{T}_{flat}$ defined in the lecture. Let $n \geq 0$ be an integer. We consider the operation of the k-group scheme $(\mathbb{G}_m)_k$ on the punctured affine space $\mathbb{A}_k^{n+1} - \{0\}$ by "multiplication on the coordinates".

Show that in both topologies $\mathcal{T}_{Zar}$ and $\mathcal{T}_{flat}$, the quotient sheaf of sets of $\mathbb{A}^{n+1} - \{0\}$ by the action of $(\mathbb{G}_m)_k$ is in both cases represented by the n-th projective space $\mathbb{P}_k^n$.

(***) Is the quotient of $\mathbb{A}_k^{n+1}$ by the action of $(\mathbb{G}_m)_k$ represented by a scheme ? [Hint: prove that yes the quotient is a k-scheme, in fact isomorphic to $Spec(k)$.... Reduce to the case $n = 0$...]

Exercise 3.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a topology. Let G be a sheaf of groups on $(\mathcal{C}, \mathcal{T})$, equivalently a group object in $Shv_{\mathcal{T}}(\mathcal{C})$. Let $X \in Shv_{\mathcal{T}}(\mathcal{C})$ be a sheaf of sets endowed with an action of G (in the obvious sense):

$$\mu : G \times X \to X$$

1) For any point $x = (x_*, x^*)$ of $(\mathcal{C}, \mathcal{T})$ and a sheaf $Y \in Shv_{\mathcal{T}}(\mathcal{C})$ simply denote by $Y_x = x^*(Y)$ the stalk of Y at x. Show that G_x is a group, and that it acts on X_x.

2) Define the quotient X/G of X by G in $Shv_{\mathcal{T}}(\mathcal{C})$ as a coequalizer. For any point $x = (x_*, x^*)$ of $(\mathcal{C}, \mathcal{T})$ show that the stalk $(X/G)_x$ at x of the quotient is (canonically isomorphic) to X_x/G_x .

3) We say that the action of G on X is free if the morphism in $Shv_{\mathcal{T}}(\mathcal{C}) : G \times X \to X \times X$, equal to μ on factor and the projection to X on the other is a monomorphism. Show that if G acts freely on X, for any point $x = (x_*, x^*)$ of $(\mathcal{C}, \mathcal{T})$ the group G_x acts freely on X_x, and that this characterizes free action if the site has enough points. The corresponding diagram $G \rightrightarrows X \to Y = X/G$ is called a G-torsor over Y.

4) Give an example of G torsor in $Shv_{Zar}(Sch_k^{tf})$ (one is already on this sheet...).