On the motivic π_0 of the sphere spectrum

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1 Introduction: homotopy theory and differential topology.

1.1 Homotopy theory

Let $d \geq 1$ and X a pointed topological space. We denote as usual by $\pi_d(X)$ the d-th homotopy group of X. One of the starting point in homotopy theory is the following result:

Theorem 1.1.1 *Let $n > 0$ be any integer.*
1) If $d < n$ then $\pi_d(S^n) = 0$;
2) If $d = n$ then $\pi_n(S^n) = \mathbb{Z}$;
2) (Serre) If $d > n$ then $\pi_d(S^n)$ is a finite group unless n is even and $d = 2n - 1$ in which case it a direct sum of $\mathbb{Z}$ and a finite group.

The first point can be roughly proven as follows: any continuous map $S^d \to S^n$ is homotopic to a $\mathcal{C}^\infty$-differentiable map $f : S^d \to S^n$. But such a map admits by Sard's theorem at least one regular value $x \in S^n$ which in the case $d < n$ means that $f^{-1}(x)$ is empty. So f avoids at least a point and is thus homotopic to a constant map.

The second point can almost be settled the same way. At least we can define a non trivial homomorphism

$$\pi_n(S^n) \to \mathbb{Z}$$

the so-called Brouwer degree. Given a $\mathcal{C}^\infty$-differentiable map $f : S^n \to S^n$ and a regular value x, this time we see that for each $y \in f^{-1}(x)$, $df_y : T_y(S^n) \to T_x(S^n)$ is an isomorphism. Thus the y are isolated, in finite number. We define a sign $\epsilon_y(f) \in \{\pm 1\}$ to be $+1$ if df_y preserves the orientation and -1 else. Then the Brouwer degree of f is

$$\sum_{y \mapsto x} \epsilon_y(f)$$

which can be shown to only depend on the homotopy class of f.

The third point is highly non-trivial and follows from Serre's thesis.

The above result implies the following one on the *stable* homotopy groups of spheres, which we denote by $\pi_i(S^0)$:

$$\pi_i(S^0) = \begin{cases} 0 \text{ if } i < 0 \\ \mathbb{Z} \text{ if } i = 0 \\ \text{finite if } i > 0 \end{cases}$$

For instance we know the Hopf map η generates $\pi_1(S^0) = \mathbb{Z}/2$, η^2 generates $\pi_2(S^0) = \mathbb{Z}/2$, $\eta^3 \neq 0 \in \pi_3(S^0) \cong \mathbb{Z}/24$ and $\eta^4 = 0$. A consequence of what I will explain in these notes is that the homotopy $\eta^4 = 0$ is not algebraic. The Hopf map is up to homotopy the obvious projection

$$\mathbb{C}^2 - \{0\} \to \mathbb{P}^1(\mathbb{C})$$

which is an algebraic morphism defined over $Spec(\mathbb{Z})$. It is clear there is no "algebraic homotopy" $\eta^4 = 0$ defined over the real numbers: this comes from the fact that the map induced by η on the real points (with the standard topology) $S^1 \cong \mathbb{R}^2 - \{0\} \to \mathbb{P}^1(\mathbb{R}) = S^1$ has degree 2, so that it can't be nilpotent.

Another closely related fact is the following. Let $f \in \mathbb{C}(X)$ be a rational function. Let us denote by the same letter the induced map $f : S^2 = \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) = S^2$. This map thus has a degree $d_{\mathbb{C}}(f) \in \mathbb{Z}$, which in fact is natural number (because of the complex structure each of the signs are $+1$ in Brouwer's formula). Now, assume that in fact $f \in \mathbb{R}(X)$ has coefficients in $\mathbb{R}$. Then the induced map $S^1 = \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R}) = S^1$ has also a degree $d_{\mathbb{R}}(f) \in \mathbb{Z}$.

Lemma 1.1.2 *For $f \in \mathbb{R}(X)$ one has*

$$d_{\mathbb{R}}(f) \in \{-d_{\mathbb{C}}(f), -d_{\mathbb{C}}(f) + 1, \ldots, d_{\mathbb{C}}(f) - 1, d_{\mathbb{C}}(f)\}$$

and

$$d_{\mathbb{R}}(f) \equiv d_{\mathbb{C}}(f) \ mod\ 2$$

This is rather easy to check: one can pick up a regular value x (both for $f(\mathbb{C})$ and $f(\mathbb{R})$) which is a real point. Then each of the signs used to compute the Brouwer degree of $f(\mathbb{C})$ is $+1$ because of the complex structure which preserves the orientation. But the degree of $f(\mathbb{R})$ differs from the degree of $f(\mathbb{C})$ by two facts: $f(\mathbb{R})^{-1}(x)$ misses the points $y \in f(\mathbb{C})^{-1}(x)$ with residue fields the complex numbers, but those appear in complex conjugate pairs. The other fact comes from the signs assigned for the $y \in f(\mathbb{R})^{-1}(x)$ with residue field $\mathbb{R}$. For the degree of $f(\mathbb{C})$ they appear with sign $+1$ as already mentioned. But for the degree of $f(\mathbb{R})$ they appear with some sign (depending on the sign of the derivative at y of $f(\mathbb{R})$): but all of these are congruent mod 2.

Now it is well known that pairs (n, σ) with $n \in \mathbb{N}$ and $\sigma \in \{-n, \dots, n\}$ are in one-to-one correspondence with quadratic forms over the real numbers. Thus, the degree of a rational fraction with real coefficients has the same "nature" as a quadratic form over the real numbers. One of our aims in these lectures will be to formulate and to give some overview of the proof of the following generalisation:

Theorem 1.1.3 *Let k be a field of characteristic $\neq 2$. Then the "motivic" homotopy group $\pi_0(S^0)$ of the "motivic" sphere spectrum over k is canonically isomorphic to the Grothendieck-Witt ring $GW(k)$ of quadratic forms over k.*

This fits very well with the results of [16] in which the Adams spectral sequence based on mod 2 motivic spectral sequence is shown, on the column where it tries to compute $\pi_0(S^0)$, to converge to the completion of the Grothendieck-Witt ring with respect to the mod 2 rank homomorphism. We observe that our proof of Theorem 1.1.3 relies on a result of Arason-Elman [1] (see also [17]) which uses the validity of Milnor's conjecture on Galois cohomology [30] and Milnor's conjecture on quadratic forms [26, 16, 18]. Nevertheless, we have a general programme to prove directly Theorem 1.1.3 by using only "elementary" results.

In view of the results on stable homotopy groups of spheres we recalled above, this settles the second fundamental point (the first one is due to Voevodsky and appears in [29]). The third one, the analogue of Serre's finitess result, is much, much more difficult and might be the main conjecture in this area. Its "classical proof" can be cut into two steps: first prove the fact that stable homotopy groups of finite C.W.-complexes are finite type abelian groups and then identify rationalized stable homotopy groups to rational homology. The following seems unreachable up to now:

Conjecture 1.1.4 *Given a finite field k (or more generally a regular scheme* *of finite type over Spec(Z)) then for each smooth k-scheme X the stable $\mathbb{A}^1$-homotopy groups*

$$[S^n, \Sigma^\infty(X_+)]$$

are all groups of finite type.

In [21] we will prove the following result, which will be stated in a slightly different form in Theorem 5.2.2 below.

Theorem 1.1.5 *Given any field k (or more generally any regular noetherian scheme of finite Krull dimension) in which -1 is a sum of squares, the unit morphism*

$$S^0 \to H\mathbb{Z}$$

of the motivic cohomology spectrum is a rational equivalence, i.e. the induced morphism of $\mathbb{P}^1$-spectra

$$S^0 \otimes \mathbb{Q} \to H\mathbb{Q}$$

is a stable $\mathbb{A}^1$-weak equivalence.

1.2 Homotopy theory and geometry

Once one has some techniques to compute motivic homotopy groups of spheres, one may wonder : what is that useful for? I would like first to recall very quickly how the very soft world of homotopy theory is very useful to classify rather rigid objects: differentiable manifolds up to diffeomorphism for instance.

First, as we know, most of the basic geometric invariants of a ($\mathcal{C}^\infty$) differentiable manifold X are representable up to homotopy. For instance, the De Rham cohomology groups $H^n_{DR}(X)$ can be identified with the set of homotopy classes $\pi(X, K(\mathbb{R}, n))$ from X to an Eilenberg-MacLane space of type $(\mathbb{R}, n)$, and given any integer n the set $\Phi^n(X)$ of isomorphism classes of real (resp. complex) vector bundles over X can be identified with the set

$$\pi(X, \mathbb{G}r_n(\mathbb{R})) \text{ (resp. } \pi(X, \mathbb{G}r_n(\mathbb{C}))\text{)}$$

of homotopy classes to the corresponding infinite grassmannian.

An other strong relationship between geometry and homotopy theory appears in Thom's cobordism theory. Assume our manifold X (which we assume now to be compact) is embedded into some sphere S^{N+d} with N large enough compared to $d = dim(X)$. Choose a *tubular neighbourhood* of i, which means an open subset $U \subset S^{N+d}$ containing X and a diffeomorphism $U \cong E(\nu_X)$ which identifies $X \subset U$ with the zero section of $E(\nu_i)$, where ν_X denotes the normal bundle of i which is also the stable normal bundle of X. Let $Th(\nu_X)$ denotes its Thom space, i.e. the one point compactification $\dot{\widehat{E(\nu_X)}}$ of the total space of ν_X. The "obvious" map :

$$S^{N+d} \to \dot{\widehat{U}} \cong Th(\nu_X)$$

only depends up to homotopy on $i : X \to S^{N+d}$ (and not on the choice of the tubular neighbourhood). This map is called the homotopy fundamental class of X; the algebraic analogue of that map is one of the fundamental tools used by Voevodsky in his proof of Milnor's conjecture.

As we know, that construction yields (by composition with the map coming from the classifying map $\nu_X : X \to \mathbb{G}r_N(\mathbb{R})$) an element

$$[X] \in \pi_d(MO)$$

the d-th stable homotopy group of the (real) Thom spectrum $M\gamma$, and using his theory of transversality Thom proved that indeed $\pi_d(MO)$ is isomorphic to the set of cobordism classes of compact differentiable manifolds of dimension d.

One of the fundamental achievement of algebraic topology in differential topology is a tremendous improvement of this idea. For X a compact, oriented, simply connected, differentiable manifold we get the following purely homotopical data:

1. The homotopy type of X together with a class $[X] \in H_d(X; \mathbb{Z})$ inducing a duality on the integral singular chain complex $C_*(X; \mathbb{Z})$ (Poincaré duality);

2. A homotopy class $\xi : X \to \mathbb{G}r_N(\mathbb{R})$ (the normal vector bundle);

3. A homotopy class $S^{N+d} \to Th(\xi)$ (the homotopy fundamental class of X).

We have just described a procedure going from manifolds to "homotopy theory" by assigning homotopy invariants. Is it possible to go the other way ?

One has "obvious" compatibilities. The data given in (1) is called a Poincaré complexe and Spivak showed that such a Poincaré complex has a "spherical" normal bundle which must be the sphere bundle of the vector bundle ξ of (2). Another compatibility when say $n = 4m$ is Hirzebruch's signature formula expressing the signature of the intersection quadratic form $\omega \mapsto \int_{[X]}(\omega \wedge \omega) \in \mathbb{R}$ on $H^{2m}_{DR}(X)$ (which comes from (1)), in terms of Pontryagin numbers of ξ.

The amazing fact, which was achieved by Browder and Novikov in the simply connected case, and Wall in the non-simply connected case (using a generalized obstruction involving L-groups of the fundamental group), is that there is basically NO other obstruction (if $d > 4$): it is possible to prove that this compatible homotopical data come from a genuine differentiable, compact, oriented manifold. This is highly non-trivial and involves a lot of mathematics: cobordism, h-cobordism, surgery.

Maybe to conclude this introduction I should say that the most optimistic aim is for us to try to realise such a programme in algebraic geometry, with the category of differentiable manifolds replaced by that of smooth varieties over a given field. We now have in hands all the homotopy theory which is needed: the datum (1) corresponds more or less to a finite type object in the $\mathbb{A}^1$-homotopy category together with a duality theorem for mixed motives over it as defined by Voevodsky, datum (2) corresponds to the $\mathbb{A}^1$-homotopy classification of algebraic vector bundles (algebraic K-theory) and (3) was used by Voevodsky in his proof of the Milnor conjecture.

The whole programme is still very mysterious: for instance what is the analogue of Smale's h-cobordism theorem or of surgery theory ? But at least, it is starting: indeed, we have developped with Marc Levine [12, 13, 11] a geometric approach to algebraic cobordism (defined by the algebraic Thom's spectrum) which might be seen as a first step towards realising Thom's programme in algebraic geometry.

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2 From smooth varieties to "spaces".

Let k denote a perfect field of $char(k) \neq 2$. We will let $\mathcal{V}$ denote the category of smooth k-schemes or smooth k-varieties. We denote by $\mathbb{A}^1$ the affine line over k.

2.1 Naive approach

For X and Y two smooth k-schemes it is rather tempting to say that two morphisms in $\mathcal{V}$

$$f, g : X \rightrightarrows Y$$

are $\mathbb{A}^1$-homotopic if there exists a morphism

$$h : X \times \mathbb{A}^1 \to Y$$

such that $h_0 = f$ and $h_1 = g$ where $h_i : X \to Y$ is the restriction of h to $X \times \{i\} \subset \mathbb{X} \times \mathbb{A}^1$, for i a rational point of $\mathbb{A}^1$. For X and Y two smooth k-schemes we let

$$\pi(X, Y)$$

denote the quotient of the set $Hom_{\mathcal{V}}(X, Y)$ of $\mathcal{V}$-morphisms from X to Y by the equivalence relation generated by the $\mathbb{A}^1$-homotopy relation. We call it the set of $\mathbb{A}^1$-homotopy classes of morphisms from X to Y. One can check that composition of morphisms is compatible with this $\mathbb{A}^1$-homotopy relation and thus one obtains this way the "naive" $\mathbb{A}^1$-homotopy category of smooth k-varieties, which we denote $\pi(\mathcal{V})$.

Example 2.1.1 Let X be a smooth k-scheme.

(1) The *point* in $\mathbb{A}^1$-homotopy theory is $Spec(k)$, and an important special homotopy set is the set

$$\pi_0(X) = \pi(Spec(k), X).$$

One should observe that this set is a quotient of the set $X(k)$ of rational points of X and thus, contrary to what happens in standard algebraic topology, it is not equal to the set X^0 of connected components of X: the map $\pi_0(X) \to X^0$ is in general neither injective nor surjective ($\pi_0(X)$ can be empty).

(2) As another example, very easy to check, the $\mathbb{A}^1$-homotopy relation on the set $Hom_{\mathcal{V}}(X, \mathbb{G}_m)$ is trivial, so that $\pi(X, \mathbb{G}_m) = \mathcal{O}(X)^\times$, the group of invertible elements in $\mathcal{O}(X)$.

(3) Other important geometrical functors are represented by smooth varieties. For instance let $\mathbb{P}^n$ denote the n dimensional projective space over k. Then the set $\pi(X, \mathbb{P}^n)$ can easily be identified with the set of isomorphism classes of line bundles over X generated by $n+1$ sections. Thus if X is affine, the set

$$\pi(X, \mathbb{P}^\infty) := colim_n \pi(X, \mathbb{P}^n)$$

becomes canonically in bijection with the Picard group $Pic(X)$ of isomorphism classes of line bundles over X (this uses the fact the Picard group for smooth varieties is $\mathbb{A}^1$-invariant, *i.e.* the morphism $Pic(X) \to Pic(X \times \mathbb{A}^1)$ is an isomorphism).

(4) In much the same way if $\mathbb{G}r_{n,r}$ denotes the Grassmannian variety of rank n quotient vector spaces of $\mathbb{A}^{n+r}$ the set $\pi(X, \mathbb{G}_{n,r})$ can be identified with the set of isomorphism classes of rank n bundles over X generated by $n + r$ sections. This uses Lindel's solution [14] of Serre's question: a vector bundles over $X \times \mathbb{A}^1$ is induced from X. One can also pass to the colimit both in n and r to get, for X affine irreducible, that the set

$$\pi(X, \mathbb{G}r) := colim_{n,r} \pi(X, \mathbb{G}_{n,r})$$

is isomorphic to the reduced K_0 of X, the kernel of the rank morphism $K_0(X) \to \mathbb{Z}$.

Back to spheres. Recall that in algebraic geometry we have the following "obvious" spheres $\mathbb{G}_m$, $\mathbb{P}^1$ and $\mathbb{A}^n - \{0\}$. One can easily perform the following computations:

- $\pi_0(\mathbb{G}_m) = \pi(Spec(k), \mathbb{G}_m) = k^\times$
- $\pi_0(\mathbb{P}^1) = \pi(Spec(k), \mathbb{P}^1) = \pi(Spec(k), \mathbb{A}^n - \{0\}) = *$ for $n \geq 2$;
- $\pi(\mathbb{G}_m, \mathbb{G}_m) = k^\times \times \mathbb{Z}$;
- $\pi(\mathbb{G}_m, \mathbb{P}^1) = *$, because any map $\mathbb{G}_m \to \mathbb{P}^1$ extends to a map $\mathbb{A}^1 \to \mathbb{P}^1$.

We may also consider, for X and Y two *pointed* smooth k-schemes, the set

$$\pi_\bullet(X, Y)$$

of *pointed* $\mathbb{A}^1$-homotopy classes of pointed morphisms $X \to Y$. Here pointed means that a rational point of the smooth k-scheme is chosen and the morphisms preserve the base point. Observe we have the computation

$$\pi_\bullet(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z},$$

which looks nice!

Quadratic phenomena appear. The surprise arises when trying to compute the next candidate

$$\pi(\mathbb{P}^1, \mathbb{P}^1)$$

because quadratic forms then enter the picture (this explains partly our assumption on the base field).

Let $\Phi(k)$ be the set of isomorphism classes of quadratic forms over k.

Then, the "motivic" Brouwer degree formula will give us a highly non-trivial map

$$\deg : \pi(\mathbb{P}^1, \mathbb{P}^1) \to \Phi(k)$$

We will give the construction only when $char(k) = 0$, because in that case the definition is very close to Brouwer's classical definition. Given any morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ we pick a regular value $x \in \mathbb{P}^1(k)$, i.e., x is a rational point such that f is étale at any closed point $y \in \mathbb{P}^1$ mapping to x. We also assume (which we always can up to some action of $\mathbb{G}l_2(k)$) that $x \neq \infty$ and $f(\infty) \neq x$.

Now we take the formula

$$\deg(f;x) := \sum_{y \longmapsto x} tr_{\kappa(y)|k}(\frac{df}{dx}(y))$$

with $\frac{df}{dx}(y) \in \kappa(y)^{\times}$. Here $\kappa(y)$ is the residue field at y, the derivative is non zero because f is étale at each y; it is considered as a quadratic form of rank one over $\kappa(y)$ and

$$tr_{\kappa(y)|k} : \Phi(\kappa(y)) \to \Phi(k)$$

is the transfer map; for a quadratic form $(q : V \to \kappa(y))$ its transfer $tr_{\kappa(y)|k}(q) : V \to k$ is to be the quadratic form $(x \mapsto \mathrm{trace}_{\kappa(y)|k}(q(x))$. One checks $\deg(f;x)$ is independent of x and is invariant under $\mathbb{A}^1$-homotopy, defining the invariant.

Example 2.1.2 As we already mentioned, for $k = \mathbb{R}$ quadratic forms are classified up to isomorphism by their rank and signature. One may check that the above degree for a rational fraction with real coefficients f agrees with the previous one: the rank of the above quadratic form is indeed the degree of $f(\mathbb{C})$ and its signature is the degree of $f(\mathbb{R})$.

Example 2.1.3 For $u \in k^{\times}$ we define

$$f_u : \mathbb{P}^1 \to \mathbb{P}^1$$

by $f_u([x,y]) = [ux, y]$ (where $[-,-]$ denote the homogeneous coordinate in $\mathbb{P}^1$). Choosing $[1,0]$ as the regular value, we compute

$$deg(f_u) =< u >\in \Phi(k)$$

where $< u >$ means the quadratic form of rank one $u.X^2$.

Since the quadratic form $< u^2 >$ is similar to 1, we see that f_{u^2} and the Identity of $\mathbb{P}^1$ have the same degree. In fact we have more:

Lemma 2.1.4 *For $u \in k^{\times}$, the map*

$$f_{u^2} : \mathbb{P}^1 \to \mathbb{P}^1$$

is $\mathbb{A}^1$-homotopic to the identity.

Since $f_{u^2}([x,y]) = [u^2x, y] = [ux, u^{-1}y]$, the map $f_{u^2} : \mathbb{P}^1 \to \mathbb{P}^1$ is given by the action of the matrix

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

on $\mathbb{P}^1$. As it is an element of $SL_2(k)$, it can be expressed as a product of elementary matrices E^λ:

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

or E_λ:

$$\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

which are each $\mathbb{A}^1$-homotopic to the identity, using a linear homotopy in λ.

Remark 2.1.5 The map

$$\pi(\mathbb{P}^1_k, \mathbb{P}^1_k) \to \Phi(k)$$

is always surjective, and in fact should be close to being a bijection. For instance Barge and Lannes have proven that it is a bijection when k is algebraically closed, or at least when the group of units $k^\times$ is 2-divisible, in which case both terms are just $\pi(\mathbb{P}^1_k, \mathbb{P}^1_k) = \Phi(k) = \mathbb{N}$.

2.2 Why is the naive approach inconvenient?

There are several facts which suggest that the naive $\mathbb{A}^1$-homotopy category of smooth k-schemes $\pi(\mathcal{V})$ is not convenient. Firstly, we want to consider a morphism of smooth varieties

$$f : X \to Y$$

to be an $\mathbb{A}^1$-weak equivalence if there is an open covering $\{U_\alpha\}$ of Y such that each of the induced morphisms

$$f|_{U_\alpha} : f^{-1}(U_\alpha) \to U_\alpha$$

is isomorphic to the projection $\mathbb{A}^{n_\alpha}_{U_\alpha} \to U_\alpha$ of some affine space over U_α. But in general such a morphism won't induce a bijection either

$$\pi(Y, Z) \to \pi(X, Z)$$

or

$$\pi(Z,X) \to \pi(Z,Y)$$

Secondly there is no way of constructing the *cone* $C(f)$ of a morphism $f : X \to Y$ in such a way that one gets the expected long exact sequences of $\mathbb{A}^1$-homotopy sets. For instance the projective line $\mathbb{P}^1$, which admits a covering by two affine lines whose intersection is $\mathbb{G}_m$ should be, up to $\mathbb{A}^1$-homotopy, equivalent to the *suspension* of $\mathbb{G}_m$. But this can't be done in any reasonable sense, for instance because the set $\pi_\bullet(\mathbb{P}^1, X)$ generally has no reasonable group structure as it would have to have. For example, in the case $X = \mathbb{P}^1$ above, it only has a monoid structure. Recall $\Phi(k)$ is just a monoid (with cancellation by Witt's theorem) but is not a group. For instance the rank of a quadratic form is always a natural number: it can't be negative. We would rather expect the associated Grothendieck group of $\Phi(k)$ to appear in the "true" $\mathbb{A}^1$-homotopy theory. This is the so-called Grothendieck-Witt ring $GW(k)$. And indeed our main aim is to show this actually holds in the genuine stable $\mathbb{A}^1$-homotopy theory.

Before realising this programme, we must enlarge the category of smooth k-schemes in some way to be able to perform the usual "homotopical" constructions such as cones, $\mathbb{A}^1$-homotopy colimits of any kind, internal function objects, etc...

2.3 Topologies and sheaves

Definition 2.3.1 *1) A finite family $\{f_\alpha : U_\alpha \to U\}$ of étale morphisms in $\mathcal{V}$ is called* a covering in the étale topology *if*

$$U = \bigcup_\alpha f_\alpha(U_\alpha).$$

2) The family is a covering in the Nisnevich topology *if it is an étale covering and in addition, for every $x \in U$ there is an α and $y \in U_\alpha$ such that*

$$f_\alpha(y) = x \text{ and } \kappa(x) = \kappa(y).$$

3) The family is a covering in the Zariski topology *if each f_α is an open immersion and*

$$U = \bigcup_\alpha f_\alpha(U_\alpha).$$

We observe that any Zariski covering is a Nisnevich covering, and any Nisnevich covering is an étale morphism. The Nisnevich covering were introduced in [25].

Definition 2.3.2 *Let τ denotes a symbol $\in \{Zar, Et, Nis\}$. A sheaf of sets on $\mathcal{V}$ in the τ-topology is a presheaf*

$$F : \mathcal{V}^{op} \to \mathcal{S}et$$

such that for every τ-covering family $\{U_\alpha \to U\}$ the diagram

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

is exact.

We now let $Shv(\mathcal{V}_{Et})$, $Shv(\mathcal{V}_{Nis})$, $Shv(\mathcal{V}_{Zar})$ and $Shv(\mathcal{V}_{\emptyset})$ respectively denote the categories of sheaves in the étale, Nisnevich and Zariski topologies, $Shv(\mathcal{V}_{\emptyset})$ denoting the category of presheaves of sets.

Example 2.3.3 For instance, given any k-scheme X the presheaf represented by X

$$\underline{X} : \mathcal{V}^{op} \to \mathcal{S}et$$

$$Y \longmapsto Hom_k(Y, X)$$

is always a sheaf in the étale topology so that it is in each of the topologies considered above.

Hence we get the following sequence of full and faithful embeddings

$$\mathcal{V} \subseteq Shv(\mathcal{V}_{Et}) \subseteq Shv(\mathcal{V}_{Nis}) \subseteq Shv(\mathcal{V}_{Zar}) \subseteq Shv(\mathcal{V}_{\emptyset}).$$

Here, each of the four categories of scheaves (including that of presheaves) contains the category $\mathcal{V}$ as a full subcategory and might serve as a "convenient category of spaces" to do homotopy theory: indeed we can perform all the colimits, all the limit we need in homotopy theory. There are internal function objects, etc....

But for our purpose, we will use the Nisnevich one, for reasons explained in [24]. We briefly recall some of these reasons.

2.4 A convenient category of “spaces”

The Nisnevich topology is between the étale and Zariski topologies. To explain some of its basic properties, let us mention it has the following two main advantages which partly explain why our convenient category of “spaces” (to refer to a well-known problem in algebraic topology) will eventually be that of simplicial sheaves of sets on $\mathcal{V}_{Nis}$.

1. For all sheaves M of abelian groups, and for all $X \in \mathcal{V}$ of Krull dimension d
$$H^n_{Nis}(X; M) = 0 \text{ for } n > d.$$
The Nisnevich topology is similar to the Zariski topology in this respect. Furthermore with notation for the covering in the definition, the Nisnevich condition applied to the generic points of X shows there is some α such that U_α is birational to X.

2. For all $X \in \mathcal{V}$ and for all $x \in X(k)$
$$X/(X \setminus \{x\}) \cong \mathbb{A}^n/(\mathbb{A}^n - \{0\})$$
This isomorphism is understood in the category of pointed sheaves of sets in the Nisnevich topology. Thus “Smooth varieties look locally like affine spaces.” This is wrong in the Zariski topology. Thus the Nisnevich topology is similar to the étale topology in this respect.

For these reasons, we will always work in the sequel in the category $\mathcal{V}$ of smooth k-schemes, endowed with the Nisnevich topology. Also, for our purpose, it is not really useful to recall the *unstable* approach of [24] where very little is known on computations of homotopy sets between smooth schemes. In the sequel we will only deal with the stable $\mathbb{A}^1$-homotopy theory as defined e.g. in [29]. Let's recall briefly some of the basic definitions.

3 Stable homotopy categories of S^1-spectra.

3.1 Simplicial S^1-spectra.

Recall from [5] for instance, that an S^1-spectrum is a collection $(E_r, \sigma_r)_{r\geq 0}$ in which for each integer r, E_r is a pointed simplicial set and σ_r is a pointed map $E_r \wedge S^1 \to E_{r+1}$. We denote for each integer $n \in \mathbb{Z}$ by $\pi_n(E)$ the n-th stable homotpoy group of the spectrum E, that is to say the colimit

$$\pi_n(E) := colim_r \pi_{n+r}(E_r).$$

Let us denote by $Sp^{S^1}(\mathcal{V}_{Nis})$ the category of sheaves on $\mathcal{V}_{Nis}$ of S^1-spectra. Such a spectrum can as well be considered as a collection $(E_n, \sigma_n)_{n \geq 0}$ with $E_n \in \Delta^{op} Shv_{\bullet}(\mathcal{V}_{Nis})$ (the category of pointed simplicial sheaves of sets on $\mathcal{V}_{Nis}$) and $\sigma_n : E_n \wedge S^1 \to E_{n+1}$ is a morphism of pointed simplicial sheaves of sets on $\mathcal{V}_{Nis}$.

Example 3.1.1 For any S^1-spectrum E in the sense of [5], we can consider the sheaf of S^1-spectra on $\mathcal{V}_{Nis}$ associated to the constant presheaf $U \mapsto E$, which we denote by the same letter E.

Example 3.1.2 Let $\mathcal{X}$ be a pointed simplicial sheaf of sets. We can construct its *suspension spectrum* $\Sigma^\infty(\mathcal{X})$ as follows: its n-term is just the smash-product

$$\mathcal{X} \wedge S^n$$

(where S^n denotes $(S^1)^{\wedge n}$) and with structure morphisms the identities.

For any smooth k-scheme U, let $E(U)$ denote the S^1-spectrum of sections of E over U; that is to say the S^1-spectrum with terms the pointed simplicial sets $E_n(U)$ and structure morphisms $E_n(U) \wedge S^1 \to E_{n+1}(U)$ the one induced by σ_n. For each $n \in \mathbb{Z}$ the correspondence

$$U \mapsto \pi_n(E(U))$$

is a presheaf of abelian groups on $\mathcal{V}$, and we will denote by

$$\pi_n(E) \in \mathcal{A}b(\mathcal{V}_{Nis})$$

the associated sheaf of abelian groups in the Nisnevich topology.

Definition 3.1.3 *A morphism $f : E \to F$ in $Sp^{S^1}(\mathcal{V}_{Nis})$ is said to be a* stable simplicial weak equivalence *if*

$$f_* : \pi_n(E) \xrightarrow{\cong} \pi_n(F)$$

for all $n \in \mathbb{Z}$. We let W_s denotes the class of stable simplicial weak equivalences and we denote by

$$\mathcal{SH}_s^{S^1}(k)$$

the category obtained from $Sp^{S^1}(\mathcal{V}_{Nis})$ by inverting the class W_s, and we call it the simplicial stable homotopy category of S^1-spectra over $\mathcal{V}_{Nis}$.

Adapting the standard method of [5] using the Brown-Gersten property in the Nisnevich topology one can prove:

Theorem 3.1.4 *(Jardine [9]) There is a simplicial closed model structure on $Sp^{S^1}(\mathcal{V}_{Nis})$ with W_s as its weak equivalences.*

Once this is done, one can apply the usual techniques to prove that $\mathcal{SH}_s^{S^1}(k)$ becomes a triangulated category in which the shift functor $E \mapsto E[1]$ is given by the smash product by the simplicial circle: $E[1]_n = E_n \wedge S^1$. In fact following [10, 7] one can put a symmetric monoidal structure on $\mathcal{SH}_s^{S^1}(k)$ whose product, denoted by $\wedge$, is indeed induced by the smash-product of pointed simplicial sheaves of sets on $\mathcal{V}_{Nis}$.

We will simply denote (when no confusion can arise) by $[E, F]$ the abelian group of morphisms in $\mathcal{SH}_s^{S^1}(k)$ between the S^1-spectra E and F.

Also to simplify the notations, we will usually simply denote for a simplicial sheaf $\mathcal{X}$

$$[\mathcal{X}] := \Sigma^\infty(\mathcal{X}_+)$$

and if moreover we assume that $\mathcal{X}$ is pointed, we set

$$(X) := \Sigma^\infty(X).$$

For instance the sphere S^1-spectrum S^0 is $[Spec(k)]$ and in general we have a decomposition $[\mathcal{X}] = (\mathcal{X}) \vee S^0$ when $\mathcal{X}$ is pointed (where $\vee$ denotes the sum in $\mathcal{SH}_s^{S^1}(k)$). We also recall among other things the following properties:

- $(\mathcal{X} \wedge \mathcal{Y}) \cong (\mathcal{X}) \wedge (\mathcal{Y})$ for two pointed simplicial sheaves of sets; thus $[\mathcal{X}] \wedge [\mathcal{Y}] \cong [\mathcal{X} \times \mathcal{Y}]$.
- $S^0 = [Spec(k)] = \Sigma^\infty(Spec(k)_+)$ is the unit for the monoidal structure.
- $S^1 := (S^1)$ is invertible: the shift is an equivalence of categories.
- If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of pointed simplicial sheaves of sets, there is a triangle
$$(\mathcal{X}) \to (\mathcal{Y}) \to (C(f)) \to (\mathcal{X})[1]$$
where $C(f)$ is the usual cone construction on simplicial objects.

3.2 $\mathbb{A}^1$-localization

Definition 3.2.1 *(1) An S^1-spectrum E is said to be* $\mathbb{A}^1$-local *if, for all $X \in \mathcal{V}$ and all integers n,*

$$[[X][n], E] \xrightarrow{\cong} [[X \times \mathbb{A}^1][n], E]$$

is an isomorphism.

(2) A morphism $f : E \to F$ in $Sp^{S^1}(\mathcal{V}_{Nis})$ is a stable $\mathbb{A}^1$-weak equivalence *if for all $\mathbb{A}^1$-local S^1-spectra G*

$$[F, G] \xrightarrow{\cong} [E, G]$$

is an isomorphism.

(3) We define the stable $\mathbb{A}^1$-homotopy category of S^1-spectra as the category

$$\mathcal{SH}^{S^1}(k) := \mathcal{SH}_s^{S^1}(k)[W_{\mathbb{A}^1}^{-1}] = Sp^{S^1}(\mathcal{V}_{Nis})[W_{\mathbb{A}^1}^{-1}]$$

obtained by inverting in $Sp^{S^1}(\mathcal{V}_{Nis})$ the class $W_{\mathbb{A}^1}$ of stable $\mathbb{A}^1$-weak equivalences.

Again by [10] the standard techniques can be used to endow $Sp^{S^1}(\mathcal{V}_{Nis})$ with a model category structure in which the weak equivalences are stable $\mathbb{A}^1$-weak equivalences. Let us denote the full subcategory of $\mathcal{SH}_s^{S^1}(k)$ consisting of $\mathbb{A}^1$-local S^1-spectra by $\mathcal{SH}_{\mathbb{A}^1}^{S^1}(\mathcal{V}_{Nis})$.

Proposition 3.2.2 *The inclusion*

$$\mathcal{SH}_{\mathbb{A}^1}^{S^1}(\mathcal{V}_{Nis}) \subset \mathcal{SH}_s^{S^1}(k)$$

admits a left adjoint $L_{\mathbb{A}^1} : \mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_{\mathbb{A}^1}^{S^1}(\mathcal{V}_{Nis})$ called the $\mathbb{A}^1$-localization functor, and which induces an equivalence

$$\mathcal{SH}^{S^1}(k) \xrightarrow{\cong} \mathcal{SH}_{\mathbb{A}^1}^{S^1}(\mathcal{V}_{Nis})$$

To construct the $\mathbb{A}^1$-localization we use the internal function object

$$\mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1}(k), E \mapsto E^{(\mathbb{A}^1)}$$

which is by definition the right adjoint to $F \mapsto F \wedge (\mathbb{A}^1)$, where we consider here $\mathbb{A}^1$ as pointed by the zero section. We also denote by $C(f)$ "a" cone in the triangulated category $\mathcal{SH}_s^{S^1}(k)$ of a morphism $f : E \to F$ so that we have an exact triangle $E \xrightarrow{f} F \to C(f) \to E[1]$. For all E we define

$$L_{\mathbb{A}^1}(E) = hocolim(E = E^{(0)} \to E^{(1)} \to E^{(2)} \to \cdots)$$

where the spectra $E^{(n)}$ are defined recursively by the formula

$$E^{(n)} := C\big((E^{(n-1)})^{(\mathbb{A}^1)} \xrightarrow{ev_1} E^{(n-1)} \big)$$

where ev_1 is the evaluation at 1 morphism. The fact that we need only take the colimit of a tower relies on the fact that smooth k-schemes have finite cohomological dimension in the Nisnevich topology. The fact that the homotopy colimit is indeed $\mathbb{A}^1$-local uses the structure of ring object of the affine line $\mathbb{A}^1$ (the fact that the product $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ induces an $\mathbb{A}^1$-naive homotopy between the identity and the zero morphism).

As a consequence of the proposition and this description of $\mathcal{SH}^{S^1}(k)$, we easily deduce on $\mathcal{SH}^{S^1}(k)$ a triangulated and symmetric monoidal structures induced from $\mathcal{SH}_s^{S^1}(k)$.

3.3 Thom spaces and homotopy purity

We should mention that the suspension spectrum functor induces a functor, which commutes with the smash-product, $\mathcal{H}_\bullet(k) \to \mathcal{SH}^{S^1}(k)$ from the homotopy category of pointed simplicial sheaves considered in [24] to the stable homotopy category $\mathcal{SH}^{S^1}(k)$. Thus any of the results proven there hold a fortiori in $\mathcal{SH}^{S^1}(k)$. Here are the two main examples of [24].

1) A *distinguished square* in $\mathcal{V}$ is a cartesian square of the form

$$\begin{array}{ccc} W & \to & V \\ \downarrow & & \downarrow \\ U & \to & X \end{array}$$

where $U \subset X$ is an open subscheme, $f : V \to X$ an étale morphism which induces an isomorphism of schemes $f^{-1}((X-U)_{red}) \cong (X-U)_{red}$.

For instance, if f is itself an open immersion, this exactly means that U and V are two open subschemes of X that cover X and W is the intersection. Then one has a (Mayer-Vietoris type) triangle

$$[W] \to [U] \vee [V] \to [X] \to [W][1]$$

2) Let $i : Z \to X$ be a closed immersion in $\mathcal{V}$. Let ν_i be its normal vector bundle and let $Th(\nu_i) = E(\nu_i)/E(\nu_i)^{\times}$ be its Thom space (a pointed Nisnevich sheaf of sets). For instance the Thom space of the trivial vector bundle of rank n over X, Θ^n_X is isomorphic to $(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \wedge (X_+)$. The (stable version of the) homotopy purity theorem then states:

Theorem 3.3.1 *[24]: There is a canonical triangle (in $\mathcal{SH}^{S^1}(k)$)*

$$[X - Z] \to [X] \to (Th(\nu_i))$$

This is a direct consequence of the unstable homotopy purity result of [24]. It is essential here to work $\mathbb{A}^1$-locally: it is not true in $\mathcal{SH}^{S^1}_s(k)$. The proof uses deformation to the normal cone. For example we find

$$(\mathbb{G}_m)[1] \cong \Sigma^{\infty}(\,\mathbb{A}^1/(\mathbb{G}_m)\,) \cong (\mathbb{P}^1).$$

where $\mathbb{P}^1$ and $\mathbb{G}_m \subset \mathbb{P}^1$ are pointed by 1. We will set $T := (\mathbb{A}^1/(\mathbb{G}_m)) \in Sp^{S^1}(\mathcal{V}_{Nis})$. We get more generally isomorphisms in $\mathcal{SH}^{S^1}(k)$

$$(\mathbb{A}^n/(\mathbb{A}^n - \{0\})) \cong (\mathbb{P}^1)^{\wedge n} \cong T^n.$$

Remark 3.3.2 Let $k \subseteq L$ be a finite separable field extension and $x \in L$ a generator defining the closed immersions $Spec(L) \to \mathbb{A}^1_k$ or $i : Spec(L) \to \mathbb{P}^1$. The homotopy purity triangle reads

$$[\mathbb{P}^1 - Spec(L)] \to [\mathbb{P}^1] \to (Th(\nu_i)) \cong T \wedge [Spec(L)].$$

Thus one gets a

$$(\mathbb{P}^1) \to T \wedge [Spec(L)] \cong \mathbb{P}^1 \wedge [Spec(L)]$$

One recognizes the first step to obtain "stable" transfers for finite étale covering and more generally a first step to get S^0-duality. This observation justifies that in the "genuine" motivic stable homotopy theory we should invert the smash product by $(\mathbb{P}^1)$ (or T) also to be able to get a stable transfer morphism

$$S^0 = [Spec(k)] \to [Spec(L)]$$

This will be accomplished in Section 5

4 The $\mathbb{A}^1$-homotopy t-structure and the stable Hurewicz theorem.

4.1 Eilenberg-MacLane S^1-spectra and the simplicial t-structure

For any sheaf of abelian groups $M \in \mathcal{A}b(\mathcal{V}_{Nis})$ we may define its Eilenberg-MacLane S^1-spectrum

$$HM := (M, K(M,1), K(M,2), \ldots, K(M,n), \ldots),$$

where $K(M,n)$ is an explicit [24] simplicial sheaf of abelian groups (considered as a simplicial sheaf of sets) which has only one non-trivial homotopy sheaf in degree n, where it is isomorphic to M. One can show that for any $X \in \mathcal{V}$ and any integer $n \in \mathbb{Z}$ the group

$$[[X], HM[n]]$$

of morphisms in $\mathcal{SH}_s^{S^1}(k)$ from the suspension S^1-spectrum of X_+ to the n-th shift of HM is canonically isomorphic to

$$H^n_{Nis}(X; M)$$

the n-th group of sheaf cohomology of X with coefficients in M.

Recall (from classical topology) that any S^1-spectrum E admits a connective cover $E_{\geq 0} \in Sp^{S^1}(\mathcal{V}_{Nis})$ and a map

$$E_{\geq 0} \to E$$

which induces an isomorphism of $\pi_n(E_{\geq 0}) \cong \pi_n(E)$ for $n \geq 0$ and with $\pi_n(E_{\geq 0}) = 0$ for $n < 0$. One defines

$$E_{\geq n} := (E[-n]_{\geq 0})[n]$$

and the cone of each map

$$E_{\geq (n+1)} \to E_{\geq n}$$

is identified with the Eilenberg-MacLane spectrum

$$H(\pi_n(E))[n]$$

Indeed, the cone of $E_{\geq(n+1)} \to E$ is the usual Postnikov n-truncation of E, which we denote here by $E_{\leq n}$. This can be "sheafified" on $\mathcal{V}_{Nis}$ to define for any (sheaf of) S^1-spectra E on $\mathcal{V}_{Nis}$ its Postnikov tower $\dots \to E_{\leq n} \to E_{\leq n-1} \to \dots$ and its co-Postnikov tower $\dots \to E_{\geq n-1} \to E_{\geq n} \to \dots$. These notions can be shown to define on $\mathcal{SH}_s^{S^1}(k)$ a t-structure in the sense of [4] whose heart is identified, using π_0 in one direction and $K(-,0)$ in the other, with the abelian category $\mathcal{A}b(\mathcal{V}_{Nis})$.

This will be referred to as the *simplicial t-structure*.

4.2 Strictly $\mathbb{A}^1$-invariant sheaves and the connectivity theorem

Definition 4.2.1 *A sheaf of abelian groups M on $\mathcal{V}_{Nis}$ is said to be* strictly $\mathbb{A}^1$-invariant *if for all $X \in \mathcal{V}$ and all $n \geq 0$ the canonical morphism*

$$H^n_{Nis}(X;M) \xrightarrow{\cong} H^n_{Nis}(X \times \mathbb{A}^1;M)$$

is an isomorphism. We let

$$\Pi^{S^1}(k) \subseteq \mathcal{A}b(\mathcal{V}_{Nis})$$

denote the full subcategory consisting of strictly $\mathbb{A}^1$-invariant sheaves.

This definition is justified by the following lemma.

Lemma 4.2.2 For an abelian sheaf M, we have the following equivalence

$$M \text{ is strictly } \mathbb{A}^1\text{-invariant} \;\Leftrightarrow\; HM \text{ is } \mathbb{A}^1\text{-local.}$$

Indeed, this follows at once from the identifications

$$[[X], HM[n]] \cong H^n_{Nis}(X;M)$$

mentioned above.

Corollary 4.2.3 *Let M be a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on $\mathcal{V}_{Nis}$. We have the following equivalence*

$$M = 0 \;\Leftrightarrow\; M(F) = 0 \textit{ for any finite type field extension } F/k$$

The implication $\Rightarrow$ is trivial. To prove the other implication it suffices to show that for any irreducible smooth k-variety X, with function field F the obvious map $M(X) \to M(F)$ is injective for any strictly $\mathbb{A}^1$-invariant sheaf M. This fact then follows from the the purity triangle in $\mathcal{SH}^{S^1}(k)$, 3.3.1, (by passing to the colimit as Z increases) and the fact that $[(Th(\xi)), HM] = 0]$, for any vector bundle of positive dimension ξ, which one can prove using the previous lemma and the observation that $M(E(\xi)) \to M(E(\xi)^{\times})$ is injective.

Example 4.2.4 (1) (Voevodsky [32, Cohomological Theory of presheaves with transfers]) Any homotopy invariant sheaf with transfers over a perfect field k is strictly $\mathbb{A}^1$-invariant.

(2) Any constant sheaf associated to an abelian group $M \in \mathrm{Ab}$ is strictly $\mathbb{A}^1$-invariant.

(3) Any abelian variety A, or the multiplicative groupe $\mathbb{G}_m$, or more generally any semi-abelian group-scheme S, when considered as a sheaf of abelian groups, is strictly $\mathbb{A}^1$-invariant: in fact its cohomology is trivial.

Here is one of our main theorems [19, 23], which says that the $\mathbb{A}^1$-localization functor preserves the connectivity:

Theorem 4.2.5 *Let E be any S^1-spectrum over $\mathcal{V}_{Nis}$. Then*

(1) If E is non-negative in the sense that $\pi_n(E) = 0$ for $n < 0$, one then also has

$$\pi_n(L_{\mathbb{A}^1}E) = 0$$

for $n < 0$.

(2) If E is $\mathbb{A}^1$-local then for all integers n, the truncation $E_{\geq n}$ is $\mathbb{A}^1$-local.

(3) E is $\mathbb{A}^1$-local $\iff$ for all $n \in \mathbb{Z}$, $\pi_n E$ is strictly $\mathbb{A}^1$-invariant.

A first application of this result is that for all $X \in \mathcal{V}$, all $i < 0$

$$\pi_i^{\mathbb{A}^1}([X])(k) = [S^i, [X]]_{\mathbb{P}^1} = 0$$

which was first proven by Voevodsky [29].

Example 4.2.6 For any "classical" S^1-spectrum E let us still denote by $E \in$ the associated sheaf of S^1-spectra over k. For any $n \in \mathbb{Z}$ the sheaf $\pi_n(E)$ is just the (constant) associated sheaf to the abelian group $\pi_n(E)$. So that E is $\mathbb{A}^1$-local, as we have seen previously. Moreover, it follows that in $\mathcal{SH}_s^{S^1}(k)$, the stable homotopy groups $\pi_n(S^0)$ coincide with the classical ones. As a consequence the induced functor

$$\mathcal{SH} \to \mathcal{SH}_s^{S^1}(k)$$

is a fully faithful embedding. Indeed, to prove this statement, using a skeletal filtration argument, one can reduce to proving it for two spheres.

For any S^1-spectrum E and any integer $n \in \mathbb{Z}$, we will denote by $\pi_n^{\mathbb{A}^1}(E)$ the n-th homotopy sheaf $\pi_n(L_{\mathbb{A}^1}(E))$ of its $\mathbb{A}^1$-localization. One can easily deduce that the simplicial t-structure on $\mathcal{SH}_s^{S^1}(k)$ induces a t-structure on $\mathcal{SH}^{S^1}(k)$ whose heart is equivalent (through $\pi_0^{\mathbb{A}^1}$) to the abelian category $\Pi^{S^1}(k)$.

Corollary 4.2.7 *1) The category $\Pi^{S^1}(k)$ is an abelian $\otimes$-category and the functor $\Pi^{S^1}(k) \to \mathcal{A}b(\mathcal{V}_{Nis})$ is exact (but doesn't preserve the tensor product).*

2) For all $\mathbb{A}^1$-local S^1-spectra E the sheaf associated to the presheaf

$$X \to [[X], E]$$

is strictly $\mathbb{A}^1$-invariant, and is isomorphic to $\pi_0(E)$.

3) An S^1-spectrum E is $\mathbb{A}^1$-trivial if and only if for all finite type field extension F/k and all integers $n \in \mathbb{Z}$ the group $\pi_n^{\mathbb{A}^1}(E)(F)$ vanishes. Equivalently, a morphism in $\mathcal{SH}^{S^1}(k)$ is an isomorphism if and only if for all finite type field extension F/k and all integers $n \in \mathbb{Z}$ it induces isomorphisms on $\pi_n^{\mathbb{A}^1}(-)(F)$.

The first point follows from [4]. For the second point, just use the fact that the associated sheaf is $\pi_0(E)$. The third point follows from 4.2.3.

Remark 4.2.8 It is easy to prove that the sheaves $\pi_0^{\mathbb{A}^1}([X])$, for $X \in \mathcal{V}$, generate the abelian category $\Pi^{S^1}(k)$ and that for the induced $\otimes$-tensor structure we have $\pi_0^{\mathbb{A}^1}([X]) \otimes \pi_0^{\mathbb{A}^1}([Y]) = \pi_0^{\mathbb{A}^1}([X \times Y])$.

4.3 The stable Hurewicz theorem

The Hurewicz theorem for S^1-spectra up to stable simplicial weak equivalence is elementary. Let's recall its meaning. Let $D(\mathcal{A}b(\mathcal{V}_{Nis}))$ denote the derived category of the abelian category $\mathcal{A}b(\mathcal{V}_{Nis})$: it is thus obtained from the category $\mathcal{C}_*\mathcal{A}b(\mathcal{V}_{Nis})$ of (unbounded !) chain complexes in $\mathcal{A}b(\mathcal{V}_{Nis})$ by inverting quasi-isomorphisms. One can in fact show (because it is a Grothendieck abelian category[1]) that there is a model category structure with the quasi-isomorphisms as weak equivalences and the monomorphisms as cofibrations.

For any sheaf of sets F on $\mathcal{V}_{Nis}$, let us denote by $\mathbb{Z}[F]$ the free sheaf of abelian groups on F: thus to give a morphism of sheaves of abelian groups $\mathbb{Z}[F] \to A$ to any sheaf of abelian groups A is exactly the same thing as to give a morphism of sheaves of sets $F \to A$. Let $\mathcal{X}$ be a simplicial sheaf of sets. We extend the previous construction degreewise and denote by $\mathbb{Z}[\mathcal{X}]$ the free simplicial sheaf of abelian groups on $\mathcal{X}$. Now, if $\mathcal{X}$ is a *pointed* simplicial sheaf of sets, we let $\mathbb{Z}(\mathcal{X})$ denote the quotient of $\mathbb{Z}[\mathcal{X}]$ by the relation: base point $= 0$. We denote by $C_*^N(\mathcal{X})$ the *normalized chain complex* (in $\mathcal{A}b(\mathcal{V}_{Nis})$) of the simplicial abelian sheaf $\mathbb{Z}(\mathcal{X})$.

For any S^1-spectrum E, one then constructs its normalized chain complex $C_*^N(E)$ as the colimit of the diagram (induced by the structure morphisms σ_n of E)

$$C_*^N(E_0) \to C_*^N(E_1)[-1] \to \dots \to C_*^N(E_n)[-n] \to \dots$$

The functor $Sp^{S^1}(\mathcal{V}_{Nis}) \to \mathcal{C}_*\mathcal{A}b(\mathcal{V}_{Nis}), E \mapsto C_*(E)$ thus constructed maps stable simplicial weak equivalences to quasi-isomorphisms and thus induces a functor

$$C_*^N : \mathcal{SH}_s^{S^1}(k) \to D(\mathcal{A}b(\mathcal{V}_{Nis}))$$

It admits as right adjoint the generalized Eilenberg-MacLane spectrum functor

$$H : D(\mathcal{A}b(\mathcal{V}_{Nis})) \to \mathcal{SH}_s^{S^1}(k), C_* \mapsto HC_*$$

defined in much the same way as in the case C_* is an abelian sheaf in degree 0 we already considered: we take for $H(C_*)_n$ the generalized Eilenberg-MacLane simplicial sheaf of sets associated to the non-negative truncation of $C_*[n]$.

[1] i.e., it has exact filtering colimits and a set of generators

Definition 4.3.1 *1) An $\mathbb{A}^1$-local complex C_* is a complex whose homology sheaves $\mathbb{H}_n C_* \in \mathcal{A}b(\mathcal{V}_{Nis})$ are all strictly $\mathbb{A}^1$-invariant; this is equivalent to say that HC_* is an $\mathbb{A}^1$-local S^1-spectrum. A morphism $f_* : C_* \to D_*$ is said to be an $\mathbb{A}^1$-quasi-isomorphism if and only if for any $\mathbb{A}^1$-local complex E_* the homomorphism $Hom_{D(\mathcal{A}b(\mathcal{V}_{Nis})}(D_*, E_*) \to Hom_{D(\mathcal{A}b(\mathcal{V}_{Nis})}(C_*, E_*)$ is an isomorphism.*

2) The category $\widetilde{DM}^{eff}(k)$ is the triangulated category obtained from $D(\mathcal{A}b(\mathcal{V}_{Nis}))$ by inverting the $\mathbb{A}^1$-quasi-isomorphisms; it is called the $\mathbb{A}^1$-derived category of $\mathcal{A}b(\mathcal{V}_{Nis})$.

As for S^1-spectra, on can show the existence of an $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$ which identifies $\widetilde{DM}^{eff}(k)$ with the full subcategory category

$$D_{\mathbb{A}^1}(\mathcal{A}b(\mathcal{V}_{Nis}))$$

of $D(\mathcal{A}b(\mathcal{V}_{Nis}))$ consisting of $\mathbb{A}^1$-local chain complexes, i.e., complexes C_* with strictly $\mathbb{A}^1$-invariant homology sheaves. The functor $C_* : \mathcal{SH}_s^{S^1}(k) \to D(\mathcal{A}b(\mathcal{V}_{Nis}))$, $E \mapsto C_*(E)$ and maps $\mathbb{A}^1$-weak equivalences to $\mathbb{A}^1$-quasi-isomorphisms and thus induces a functor

$$C_* : \mathcal{SH}^{S^1}(k) \to \widetilde{DM}^{eff}(k)$$

It admits a right adjoint $H^{\mathbb{A}^1} : \widetilde{DM}^{eff}(k) \to \mathcal{SH}^{S^1}(k)$, $C_* \mapsto H^{\mathbb{A}^1} C_*$ induced by H.

For $E \in Sp^{S^1}(\mathcal{V}_{Nis})$ and $n \in \mathbb{Z}$, we denote by $\tilde{\mathbb{H}}_n(E) := \mathbb{H}_n C_*(E)$ the n-homology sheaf of the chain complex $C_*(E)$ and call it the n-th homology sheaf of E. In much the same way we denote by $\tilde{\mathbb{H}}_n^{\mathbb{A}^1}(E) := \tilde{\mathbb{H}}_n^{\mathbb{A}^1} C_*(E) = \mathbb{H}_n L_{\mathbb{A}^1}(C_*(E))$. For $\mathcal{X}$ a simplicial sheaf of sets, we set $\mathbb{H}_*^{\mathbb{A}^1}(\mathcal{X}) := \tilde{\mathbb{H}}_*^{\mathbb{A}^1}([\mathcal{X}])$.

Observe that in general, the canonical morphism in $\widetilde{DM}^{eff}(k)$

$$C_*(L_{\mathbb{A}^1}(E)) \to L_{\mathbb{A}^1}(C_* E)$$

is not an isomorphism. The Hurewicz theorem now holds in the the following sense:

Theorem 4.3.2 *Let E be any non negative S^1-spectrum. Then, in the above morphism both term are non-negative (from Theorem 4.2.5 and its analogue in $D(\mathcal{A}b(\mathcal{V}_{Nis}))$) and moreover the morphism of sheaves:*

$$\pi_0^{\mathbb{A}^1}(E) \to \tilde{\mathbb{H}}_0^{\mathbb{A}^1}(E)$$

is an isomorphism.

Remark 4.3.3 Let E be an S^1-spectrum over $\mathcal{V}_{Nis}$. We shall say it is $\mathbb{Q}$-*local* if its homotopy sheaves $\pi_n(E)$ are each sheaves of $\mathbb{Q}$-vector spaces. A *stable rational weak equivalences* is a morphism $f : E \to F$ of S^1-spectra which induces isomorphisms

$$\pi_*(E) \otimes \mathbb{Q} \cong \pi_*(F) \otimes \mathbb{Q}$$

Denote by $\mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}}$ the triangulated, symmetric monoidal category obtained from $\mathcal{SH}_s^{S^1}(k)$ by inverting the stable rational weak equivalences. Then the functor $\mathcal{SH}_s^{S^1}(k) \to \mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}}$ admits a left adjoint $\mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}} \to \mathcal{SH}_s^{S^1}(k), E \mapsto E_{\mathbb{Q}}$, called the rational localization functor which induces an equivalence between $\mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}}$ and the full subcategory $\mathcal{SH}_{s,\mathbb{Q}}^{S^1}(\mathcal{V}_{Nis})$ of $\mathcal{SH}_s^{S^1}(k)$ consisting of $\mathbb{Q}$-local spectra. Let $D(\mathcal{A}b(\mathcal{V}_{Nis}))_{\mathbb{Q}}$ be the category obtained from $D(\mathcal{A}b(\mathcal{V}_{Nis}))$ in the same way $\mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}}$ is obtained from $\mathcal{SH}_s^{S^1}(k)$. The finiteness result of Serre on stable homotopy groups of spheres implies that the rationalized chain complex functor induces an equivalence of categories:

$$\mathcal{SH}_s^{S^1}(k)_{\mathbb{Q}} \cong D(\mathcal{A}b(\mathcal{V}_{Nis}))_{\mathbb{Q}}.$$

Clearly this implies, because the $\mathbb{A}^1$-localization process commutes with $\mathbb{Q}$-localization, that the induced functor between the categories $\mathcal{SH}^{S^1}(k)_{\mathbb{Q}}$ and $\widetilde{DM}^{eff}(k)_{\mathbb{Q}}$ obtained by an analoguous $\mathbb{Q}$-localization process from $\mathcal{SH}^{S^1}(k)$ and $\widetilde{DM}^{eff}(k)$ respectively is an equivalence of categories

$$\mathcal{SH}^{S^1}(k)_{\mathbb{Q}} \cong \widetilde{DM}^{eff}(k)_{\mathbb{Q}}$$

We mention the following "topological" conjecture:

Conjecture 4.3.4 *Let X be a smooth k-scheme of Krull dimension d. Then*

$$\mathbb{H}_i(X;\mathbb{Z}) = 0 \textit{ for } i > d.$$

We observe that the connectivity theorem implies that $\mathbb{H}_i(X;\mathbb{Z}) = 0$ for $i < 0$ and that the above conjecture can be shown to imply the Beilinson-Soulé vanishing conjecture.

Remark 4.3.5 *The homotopy sheaf of a smooth variety.* For any $X \in \mathcal{V}$, the above Hurewicz theorem (4.3.2) implies that the obvious morphism of sheaves

$$\pi_0^{\mathbb{A}^1}([X]) = \mathbb{H}_0^{\mathbb{A}^1}(X)$$

is an isomorphism. The computation itself of the sheaves $\pi_0^{\mathbb{A}^1}([X]) = \mathbb{H}_0^{\mathbb{A}^1}(X)$ is another story. Our main theorem (see Section 6) is the "explicit" computation of $\pi_0^{\mathbb{A}^1}({\mathbb{G}_m}^{\wedge n})$, for each n. We observe that there is a canonical morphism of sheaves

$$\pi_0^{\mathbb{A}^1}([X]) \to h_0(X)$$

where $h_0(X)$ is the homotopy invariant sheaf with transfers attached to X by Voevodsky [32]; recall it is the associated Nisnevich sheaf to the presheaf

$$U \longmapsto [M(U), M(X)]_{DM^{eff}(k)}.$$

where $DM^{eff}(k)$ is the triangulated category of motives over the field k, and $M : \mathcal{V} \to DM^{eff}(k), X \mapsto M(X)$ the functor which assigns to X its motive $M(X)$, as defined by Voevodsky [32].

One can show that this homomorphism displays $h_0(X)$ as the homotopy invariant sheaf with transfers freely generated by the strictly $\mathbb{A}^1$-invariant sheaf $\pi_0^{\mathbb{A}^1}([X])$.

5 Inverting $\mathbb{P}^1$.

5.1 $\mathbb{P}^1$-spectra

Definition 5.1.1 *A $\mathbb{P}^1$-spectrum in $\mathcal{V}$ is a collection $(E_n, \sigma_n)_{n\in\mathbb{N}}$ consisting for each integer $n \in \mathbb{N}$, of a pointed simplicial sheaf of sets $E_n \in \Delta^{op}Shv_\bullet(\mathcal{V}_{Nis})$ and a morphism $\sigma_n : E_n \wedge (\mathbb{P}^1) \to E_{n+1}$. The category of $\mathbb{P}^1$-spectra is denoted by $Sp(\mathcal{V}_{Nis})$.*

For each $U \in \mathcal{V}$, any pair $(n, m) \in \mathbb{Z}^2$ of integers and any $\mathbb{P}^1$-spectrum E one defines the group

$$\tilde{\pi}_n(E)_m(U)$$

to be the colimit of the obvious diagram

$$\dots \to Hom_{\mathcal{H}_\bullet(k)}((U_+) \wedge S^{n+m} \wedge (\mathbb{P}^1)^{\wedge r-m}, E_r) \to \dots$$

(which is defined for r large enough so that $2r + n - m \geq 0$ and $r - m \geq 0$, in which case $S^{n+m} \wedge (\mathbb{P}^1)^{\wedge r-m}$ is an "unstable sphere").

Definition 5.1.2 *A morphism $f : E \to F$ in $Sp(\mathcal{V}_{Nis})$ is said to be a* stable $\mathbb{A}^1$-weak equivalence *if and only if for each $U \in \mathcal{V}$, and each pair $(n, m) \in \mathbb{Z}^2$ of integers the group homomorphism*

$$f_* : \tilde{\pi}_n(E)_m(U) \to \tilde{\pi}_n(F)_m(U)$$

is an isomorphism. We let $W_{\mathbb{A}^1}$ denote the class of stable $\mathbb{A}^1$-weak equivalences and we denote by

$$\mathcal{SH}(k)$$

the category obtained from $Sp(\mathcal{V}_{Nis})$ by inverting the class $W_{\mathbb{A}^1}$. We call it the stable homotopy category *over $\mathcal{V}_{Nis}$.*

To avoid any confusion we will always denote by $[-,-]$ the groups of morphisms in $\mathcal{SH}^{S^1}(k)$ (or sometimes in $\mathcal{SH}_s^{S^1}(k)$) and by $[-,-]_{\mathbb{P}^1}$ the groups of morphisms in $\mathcal{SH}(k)$ to insist on the fact that $\mathbb{P}^1$ is invertible there.

Remark 5.1.3 Once we have defined $\mathcal{SH}(k)$ we can "explain" our notations for the groups used in the definition of stable $\mathbb{A}^1$-stable weak equivalences. The group $\tilde{\pi}_n(E)_m(U)$ is isomorphic to the group $[[U] \wedge S^n, E \wedge (\mathbb{G}_m)^{\wedge m}]_{\mathbb{P}^1}$. By this observation, it is clear that the $\mathbb{P}^1$-spectra of the form $[U] \wedge (\mathbb{G}_m)^{\wedge -m}$, $m \in \mathbb{Z}$, are generators[2] for the triangulated category $\mathcal{SH}(k)$.

One can obviously define for any "reasonable" pointed simplicial sheaf $\mathbb{S}$ a corresponding category of $\mathbb{S}$-spectra and a corresponding stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}^{\mathbb{S}}(\mathcal{V}_{Nis})$ [10]. One can show that a morphism $f : \mathbb{S}' \to \mathbb{S}$ induces a functor $\mathcal{SH}^{\mathbb{S}'}(\mathcal{V}_{Nis}) \to \mathcal{SH}^{\mathbb{S}}(\mathcal{V}_{Nis})$ and that if f is an $\mathbb{A}^1$-weak equivalence [24] this functor is an equivalence of categories. For instance we already did that for $\mathbb{S} = S^1$ and $\mathbb{P}^1$. These are indeed reasonable in the sense that they are "finite type" simplicial sheaves, that they are suspensions in $\mathcal{H}_\bullet(k)$, which guarantees that the resulting stable $\mathbb{A}^1$-homotopy category will be triangulated. Moreover, some suspension (by $\mathbb{S}$) of the cyclic permutation $\mathbb{S}^{\wedge 3} \cong \mathbb{S}^{\wedge 3}$ is homotopic to the identity, this guarantees that the resulting stable $\mathbb{A}^1$-homotopy category becomes symmetric monoidal [10].

To define $\mathcal{SH}(k)$, we could have started from the following objects $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ or $S^1 \wedge \mathbb{G}_m$, because they both can be joined by $\mathbb{A}^1$-weak equivalences to $\mathbb{P}^1$. In the sequel we will usually "identify" these three types of $\mathbb{S}$-spectra.

[2] In fact $m \geq 0$ suffices

Remark 5.1.4 A $\mathbb{G}_m$-S^1-spectrum in $\mathcal{V}$ is a collection $(E_n, \sigma_n)_{n\in\mathbb{N}}$ consisting for each integer $n \in \mathbb{N}$ of an S^1-spectrum (in the sense above) $E_n \in Sp^{S^1}(\mathcal{V}_{Nis})$ and a morphism $\sigma_n : E_n \wedge (\mathbb{G}_m) \to E_{n+1}$. The category of $\mathbb{G}_m$-S^1-spectra is denoted by $Sp^{\mathbb{G}_m, S^1}(\mathcal{V}_{Nis})$. One can associate to such a $\mathbb{G}_m$-S^1-spectrum an actual $S^1 \wedge \mathbb{G}_m$-spectrum by taking its diagonal (on the set $\mathbb{N} \times \mathbb{N}$ of indices). One can check that $\mathcal{SH}$ can as well be defined by inverting in $Sp^{\mathbb{G}_m, S^1}(\mathcal{V}_{Nis})$ the class of morphisms which becomes stable $\mathbb{A}^1$-stable weak equivalences in $\mathcal{SH}$. We will also often identify in the sequel this notion of spectrum to the previous one.

As an example, any S^1-spectrum E defines an $\mathbb{G}_m \wedge S^1$-spectrum denoted by $\sigma_{\mathbb{G}_m}(E)$ and called its $\mathbb{G}_m$-supension spectrum (which we will be very often considering as a $\mathbb{P}^1$-spectrum) as follows: its n-th term is $E_n \wedge (\mathbb{G}_m)^{\wedge n}$ and in which the structure morphisms

$$(E_n \wedge (\mathbb{G}_m)^{\wedge n}) \wedge (\mathbb{G}_m \wedge S^1) \to E_{n+1} \wedge (\mathbb{G}_m)^{\wedge (n+1)}$$

are the "obvious" ones.

This construction can be shown using [10] to induce an exact and symmetric monoidal functor

$$\sigma_{\mathbb{G}_m} : \mathcal{SH}^{S^1}(k) \to \mathcal{SH}(k)$$

which admits a right adjoint

$$\omega_{\mathbb{G}_m} : \mathcal{SH}(k) \to \mathcal{SH}^{S^1}(k)$$

S^1-Stable homotopy groups ***versus*** **$\mathbb{P}^1$-Stable homotopy groups**

Given a *finite S^1-spectrum* over $\mathcal{V}_{Nis}$ such as $[X]$ for $X \in \mathcal{V}$ then it is not hard to show [10] that for any $\mathbb{G}_m$-S^1-spectrum E, the canonical homomorphism

$$colim_{i\to\infty}[[X] \wedge {\mathbb{G}_m}^{\wedge i}, E_i] \to [\sigma_{\mathbb{G}_m}([X]), \sigma_{\mathbb{G}_m} E]_{\mathbb{P}^1}$$

is an isomorphism. For example, if E is in fact an S^1-spectrum the homomorphism

$$colim_{i\to\infty}[[X] \wedge {\mathbb{G}_m}^{\wedge i}, E_i \wedge {\mathbb{G}_m}^{\wedge i}] \to [\sigma_{\mathbb{G}_m}([X]), \sigma_{\mathbb{G}_m} E]_{\mathbb{P}^1}$$

is an isomorphism. For us it will be important thus to observe that the endomorphism ring of the sphere $\mathbb{P}^1$-spectrum in $\mathcal{SH}(k)$ is given by

$$[S^0, S^0]_{\mathbb{P}^1} = colim_{i\to\infty}[{\mathbb{G}_m}^{\wedge i}, {\mathbb{G}_m}^{\wedge i}]$$

5.2 Inverting $\mathbb{P}^1$ in $\widetilde{DM}^{eff}(k)$: not so naive motives

Define a $\mathbb{P}^1$-*chain complex* C in $\mathcal{V}$ to be a collection $(C_n, \sigma_n)_{n\in\mathbb{N}}$ consisting for each integer $n \in \mathbb{N}$ of a chain complex $C_n \in \mathcal{C}_*\mathcal{A}b(\mathcal{V}_{Nis})$ and a morphism of chain complexes $\sigma_n : C_n \otimes \mathbb{Z}(\mathbb{P}^1) \to C_{n+1}$. The category of $\mathbb{P}^1$-chain complexes is denoted by $\mathcal{C}_*^{\mathbb{P}^1}(\mathcal{A}b(\mathcal{V}_{Nis}))$.

For any $\mathbb{P}^1$-chain complex C we define as above presheaves of homology groups by assigning to any $U \in \mathcal{V}$ and any pair $(n, m) \in \mathbb{Z}^2$ of integers the group

$$\mathbb{H}_n(C)_m(U)$$

as the colimit of the obvious diagram

$$\cdots \to Hom_{\widetilde{DM}^{eff}(k)}(\mathbb{Z}[U][n+m] \otimes (\mathbb{P}^1)^{\otimes r-m}, C_r) \to \dots$$

(which is defined for r large enough) where C_n denotes the n-th chain complex of C.

A morphism of $\mathbb{P}^1$-chain complexes is said to be a *stable* $\mathbb{A}^1$-*quasi-isomorphism* if and only if it induces isomorphisms on these homology presheaves. We let $\widetilde{DM}(k)$ denote the category obtained from $\mathcal{C}_*^{\mathbb{P}^1}(\mathcal{A}b(\mathcal{V}_{Nis}))$ by inverting these stable $\mathbb{A}^1$-quasi isomorphisms. It is triangulated and symmetric monoidal. Moreover the chain complex functor maps stable $\mathbb{A}^1$-weak equivalences between $\mathbb{P}^1$-spectra to stable $\mathbb{A}^1$-quasi isomorphisms of $\mathbb{P}^1$-chain complexes and we still denote by

$$C_* : \mathcal{SH}(k) \to \widetilde{DM}(k)$$

the induced functor. It is exact and symmetric monoidal (in any reasonable sense) and it admits a right adjoint denoted by $H : \widetilde{DM}(k) \to \mathcal{SH}(k)$.

It clearly follows from 4.3.3 that this functor induces an equivalence of categories

$$\mathcal{SH}(k)_{\mathbb{Q}} \to \widetilde{DM}(k)_{\mathbb{Q}}$$

after $\mathbb{Q}$-localization.

We conclude this section by making precise the relationship between the category $\widetilde{DM}(k)$ which we could call the triangulated category of "naive" motives and Voevodsky's approach [32, Triangulated category of motives over a field]. Voevodsky defined there a triangulated category $DM_{gm}^{eff}(k)$ of *effective geometric motives* over the field k.

In case k is perfect, his cohomological theory of presheaves with transfers [32], allowed him to embed that category into his category $DM^{eff}(k)$ of motivic chain complexes: the full subcategory of the derived category of the abelian category $\mathcal{A}b^{tr}(\mathcal{V}_{Nis})$ of abelian sheaves with transfers whose (co-)homology sheaves are $\mathbb{A}^1$-invariant.

Remark 5.2.1 Voevodsky's category $DM^{eff}(k)$ of motivic chain complexes is also obtained by applying the $\mathbb{A}^1$-localization process to $D(\mathcal{A}b^{tr}(\mathcal{V}_{Nis}))$, when k is perfect. Moreover when k is not assumed perfect, the $\mathbb{A}^1$-localization $DM^{eff}(k)$ of $D(\mathcal{A}b^{tr}(\mathcal{V}_{Nis}))$ seems to be the "correct" candidate for the triangulated category of motivic chain complexes over the base field[3]: instead of the full subcategory of $D(\mathcal{A}b^{tr}(\mathcal{V}_{Nis}))$ of chain complexes with $\mathbb{A}^1$-invariant homology sheaves (with transfers) we thus get exactly the full subcategory of chain complexes with *strictly* $\mathbb{A}^1$-invariant homology sheaves (with transfers). Voevodsky's proof of the basic embedding extends to give a fully faithful embedding

$$DM_{gm}^{eff}(k) \subset DM^{eff}(k)$$

However, we don't know whether or not the motivic cohomology defined in that category agrees with the one of Suslin-Voevodsky.

Performing the same procedure as we did above to invert $(\mathbb{P}^1)$ yields the category denoted $DM(k)$ of $\mathbb{P}^1$-motivic chain complexes. Over any perfect field k, it can be shown using Voevodsky's cancellation theorem[4] [31], that the obvious $\mathbb{G}_m$-suspension functor $DM^{eff}(k) \to DM(k)$ is a fully faithful embedding.

Now there are canonical functors

$$\widetilde{DM}^{eff}(k) \to DM^{eff}(k)$$

and

$$\widetilde{DM}(k) \to DM(k)$$

[3] In fact this construction can be performed over any regular noetherian scheme of finite Krull dimension

[4] which states, when k is perfect, that the tensor product by $\mathbb{G}_m$, $\widetilde{DM}(k)^{eff}(k) \to \widetilde{DM}(k)^{eff}(k)$, is a fully faithful embedding

but these are not equivalence of categories. We sum up the relationship between these categories ine the following diagram

$$\begin{tikzcd}
 & D(\mathcal{A}b^{tr}(\mathcal{V}_{Nis})) \arrow[dl, "L_{\mathbb{A}^1}"] & & D(\mathcal{A}b(\mathcal{V}_{Nis})) \arrow[ll] \arrow[dl, "L_{\mathbb{A}^1}"] \arrow[dd] \\
DM^{eff}(k) \arrow[ddd] & & \tilde{DM}^{eff}(k) \arrow[ll] \arrow[ddd] & \\
 & & & \\
 & & & \widetilde{DM}(k) \arrow[dl, "="] \\
DM(k) & & \widetilde{DM}(k) \arrow[ll] &
\end{tikzcd}$$

One of the main results in [22, II], however, is the fact that the bottom functor induces an equivalence

$$\widetilde{DM}(k)_{\mathbb{Q}} \to DM(k)_{\mathbb{Q}}$$

after $\mathbb{Q}$-localization when -1 is a sum of squares in k and $char(k) = 0$. More generally, at least when $char(k) = 0$, $DM(k)_{\mathbb{Q}}$ is always a direct summand of $\widetilde{DM}(k)_{\mathbb{Q}}$. This fits well with our computation of the endomorphism ring of the unit object of $\widetilde{DM}(k)$ which we will give in the last section of this paper: it is the Grothendieck-Witt ring of k and for $DM(k)$ it is just $\mathbb{Z}$. This uses the resolution of singularities in characteristic 0 as well as :

Theorem 5.2.2 *[21] Over any regular, notherian scheme S of finite Krull dimension) in which -1 is a sum of squares there is a canonical isomorphism*

$$[S^i, {\mathbb{G}_m}^{\wedge j}]_{\mathbb{P}^1} \otimes \mathbb{Q} \cong H^{j-i}(S; \mathbb{Q}(j)).$$

where the right hand side denotes Beilinson's motivic rational cohomology groups. In general, when -1 is not a sum of squares in $\mathcal{O}(S)$, one has only a direct sum decomposition

$$[S^i, {\mathbb{G}_m}^{\wedge j}]_{\mathbb{P}^1} \otimes \mathbb{Q} \cong H^{j-i}(k; \mathbb{Q}(j)) \oplus A_{i,j}(S)$$

This clearly implies Theorem 1.1.5. It follows from results in Section 6.2 below that these groups are independent of j. We propose to set

$$H^n_W(S; \mathbb{Q}) := A_{-n,j}(S)$$

and to call these groups the *rational Witt cohomology groups* of S. We conjecture in fact that these groups are just isomorphic to the Nisnevich cohomology groups with coefficients in unramified Witt groups[5] tensored with $\mathbb{Q}$

$$H^n_{Nis}(S; \underline{W} \otimes \mathbb{Q}).$$

5.3 A formula for the homotopy sheaves of $\mathbb{G}_m$-loop spectra

Given any presheaf of abelian groups F on $\mathcal{V}$ we define another presheaf of abelian groups by

$$X \mapsto \ker(F(X \times \mathbb{G}_m) \to F(X))$$

Iterating this construction we define F_{-i} for $i \geq 0$. Clearly, if F is a sheaf, so is F_{-1}.

For any S^1 spectrum E we write $E^{(\mathbb{G}_m)}$ for the internal Hom object. Thus the functor $\mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1}(k), E \mapsto E^{(\mathbb{G}_m)}$ is the right adjoint to the functor $F \mapsto F \wedge (\mathbb{G}_m)$.

Lemma 5.3.1 *For any $\mathbb{A}^1$-local S^1-spectrum E, the canonical morphisms*

$$\pi_n(E^{(\mathbb{G}_m)}) \to \pi_n(E)_{-1}$$

are isomorphisms for all integers n.

Indeed, using the (simplicial) Postnikov tower of E, we easily reduce to the case $E = HM$ for $M \in \Pi(k)$. Using Corollary 4.2.7. 3) (and a base change argument) we end up with proving that over any field, one has

$$\pi_n(HM^{(\mathbb{G}_m)}) = H^{-n}(\mathbb{G}_m; M) = \begin{cases} 0 \text{ for } n \neq 0 \\ M_{-1}(k) \text{ for } n = 0. \end{cases}$$

The value for $n = 0$ is clear. To prove vanishing for $n \neq 0$, the idea is to use the facts that $\mathbb{G}_m[1] = \mathbb{P}^1$ and $\mathbb{P}^1$ has Nisnevich cohomological dimensional 1. Thus $\mathbb{G}_m$ is 0-dimensional from this point of view.

[5] see Section 6.3 for the definition of unramified Witt groups

Remark 5.3.2 The homotopy t-structure in $\mathcal{SH}(k)$. One can define an $\mathbb{A}^1$-t-structure on $\mathcal{SH}(k)$ by defining a $\mathbb{G}_m$-S^1-spectrum E to be non-negative if and only if it is stably $\mathbb{A}^1$-isomorphic to a $\mathbb{G}_m$-S^1-spectrum F whose n-th term E_n is $n-1$-connected in $\mathcal{SH}^{S^1}(k)$. One can describe the associated heart as the category $\Pi_*(k)$ obtained from $\Pi^{S^1}(k)$ by inverting $\mathbb{G}_m \in \Pi^{S^1}(k)$; more precisely objects in $\Pi_*(k)$ are collections (M_*, σ_*) where for each integer $n \in \mathbb{Z}$, M_n is a strictly $\mathbb{A}^1$-invariant sheaf and σ_n is an isomorphism of sheaves $(M_n)_{-1} \cong M_{n-1}$.

Thus we get canonical functors

$$\pi_n^{\mathbb{A}^1}(-)_* : \mathcal{SH}(k) \to \Pi_*(k)$$

In concrete terms, the functor $\pi_0(-)_* : \mathcal{SH}(k) \to \Pi_*(k)$ maps a $\mathbb{P}^1$-spectrum E to the object $\pi_0(E)_*$ whose n-th term $\pi_0(E)_n$ is the strictly invariant sheaf associated to the presheaf (previously defined)

$$U \mapsto \tilde{\pi}_0(E)_n(U) \cong [[U], E \wedge (\mathbb{G}_m)^n]_{\mathbb{P}^1}$$

A consequence of Theorem 4.2.5 is that for all non-negative $\mathbb{P}^1$-spectra E (e.g. $[X]$ for $X \in \mathcal{V}$), all $i < 0$ and all $n \in \mathbb{Z}$

$$\pi_i^{\mathbb{A}^1}(E)_n(k) = [S^i, E \wedge (\mathbb{G}_m)^{\wedge n}]_{\mathbb{P}^1} = 0$$

This follows from what we have seen in Section 5.1. Thus the homotopy objects $\pi_i^{\mathbb{A}^1}(E)_* \in \Pi_*$ vanish for $i < 0$ and E non-negative: in fact

$$E \text{ non-negative } \Leftrightarrow \pi_i^{\mathbb{A}^1}(E)_* = 0 \text{ for each } i < 0$$

Residues appear. Let C be a smooth curve over k. Let $i : S \to C$ denote a finite étale (closed) subscheme. This is just a finite set of closed points. We let $U_S := C - S$ denote the open complement. The purity triangle reads

$$[U_S] \to [C] \to (Th(\nu_i))$$

where ν_i is a (trivial) line bundle over S. By choosing a generator $\pi_x \in \mathfrak{m}_x \subseteq$ for each closed point x, the triangle reads

$$[U_S] \to [C] \to \vee_{x \in S}(\mathbb{P}^1) \wedge [Spec(\kappa(x)]$$

Since $\mathbb{P}^1 = \mathbb{G}_m[1]$, for any $M \in \Pi^{S^1}(k)$ this gives a long exact sequence

$$\cdots \to H^i_{Nis}(U_S; M) \to H^i_{Nis}(C; M) \to \oplus_{x\in S} H^{i-1}_{Nis}((\mathbb{G}_m)_{\kappa(x)}; M) \to \cdots$$

Passing to the colimit as S increases, we obtain a sum over the set $C^{(1)}$ of closed points, and using $H^0_{Nis}((\mathbb{G}_m)_{\kappa(x)}; M) = M_{-1}(\kappa(x))$, we obtain a not so long exact sequence:

$$0 \to M(C) \to M(F) \stackrel{\{\partial^\pi_x\}}{\to} \oplus_{x\in C^{(1)}} M_{-1}(\kappa(x)) \to H^1_{Nis}(C; M) \to 0$$

where F is the function field of C and where $\partial^\pi_x : M(F) \to M_{-1}(\kappa(x))$ is a morphism called the *residue* at x associated to π_x.

For instance let us denote by $\underline{K}^M_n$, for each integer $n \in \mathbb{N}$, the sheaf of unramified n-th Milnor K-theory. Its values on a smooth k-scheme X is the kernel of a canonical morphism

$$K^M_n(k(X)) \stackrel{\{\partial^\pi_x\}}{\to} \oplus_{x\in X^{(1)}} K^M_{n-1}(\kappa(x))$$

Indeed, one can prove that the sheaf $(\underline{K}^M_n)_{-1}$ is isomorphic to $\underline{K}^M_{n-1}$ and thus these sheaves of unramified Milnor K-theory together form an object in $\Pi_*(k)$. It can be shown to be isomorphic to both $\pi_{0,*}(H_\mu\mathbb{Z})$ and $\pi_{0,*}(M\mathbb{G}l)$, where $H_\mu\mathbb{Z}$ denotes the motivic cohomology spectrum and $M\mathbb{G}l$ the algebraic cobordism spectrum [29].

6 $\pi_0(S^0)$ and Milnor-Witt K theory of fields.

Let $\mathcal{X}$ be a simplicial sheaf of sets on $\mathcal{V}_{Nis}$. To simplify our notations, we will still denote by $[\mathcal{X}]$ (resp. $(\mathcal{X})$ if $\mathcal{X}$ is pointed) the $\mathbb{P}^1$-spectrum $\sigma_{\mathbb{G}_m}([\mathcal{X}])$ (resp. $\sigma_{\mathbb{G}_m}((\mathcal{X}))$) when no confusion can arise. In fact in the case of $\mathbb{G}_m$, $\mathbb{P}^1$ we will very often use the same symbol, $\mathbb{G}_m$, $\mathbb{P}^1$ to denote the $\mathbb{P}^1$-spectra $(\mathbb{G}_m)$, $(\mathbb{P}^1)$.

We have already observed in 5.3.2 that for $X \in \mathcal{V}$ the homotopy objects

$$\pi^{\mathbb{A}^1}_i([X] \wedge (\mathbb{G}_m)^{\wedge n})_*$$

are 0 for all $i < 0$, all $n \in \mathbb{Z}$.

Our aim now is to study the next step in the case X is a smash-power of the multiplicative group. Our main result below will be an explicit computation of the object

$$\pi_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n})$$

for $n \in \mathbb{Z}$. In particular, we will get a computation of

$$\pi_0^{\mathbb{A}^1}(S^0)_0(k) = [S^0, S^0]_{\mathbb{P}^1}.$$

6.1 Some obvious maps and less obvious relations

The material in this section is joint work with M.J. Hopkins.

We try to understand the graded associative ring

$$\oplus_{n\in\mathbb{Z}}[S^0, {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1}$$

whose product is induced by the smash-product.

First note that any $u \in k^\times = \mathbb{G}_m(k)$, which we view as a pointed morphism $S^0 \to \mathbb{G}_m$, defines a morphism (in $\mathcal{SH}(k)$)

$$[u] : S^0 = [Spec(k)] \to \mathbb{G}_m$$

which we see as an element of degree 1 in that ring. In addition there is the $\mathbb{P}^1$-stable Hopf map

$$\eta : (\mathbb{G}_m) \to S^0$$

or equivalently $\eta : S^0 \to (\mathbb{G}_m)^{-1}$ which we see as an element of degree -1 in the ring. It is induced by the classical Hopf map

$$\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$$

and the observations that up to $\mathbb{A}^1$-homotopy, $(\mathbb{A}^2 - \{0\}) \cong (\mathbb{G}_m \wedge \mathbb{G}_m)[1]$ (use the purity triangle for the closed immersion $\{0\} \to \mathbb{A}^2$) and that $\mathbb{P}^1 = \mathbb{G}_m[1]$.

We first mention another definition of η (see [23]):

Lemma 6.1.1 *The product $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ on $\mathbb{G}_m$ is equal in $\mathcal{SH}(k)$ to the morphism*

$$\langle Id_{\mathbb{G}_m}, Id_{\mathbb{G}_m}, \eta\rangle : (\mathbb{G}_m) \vee (\mathbb{G}_m) \vee (\mathbb{G}_m)^{\wedge 2} = (\mathbb{G}_m \times \mathbb{G}_m) \to (\mathbb{G}_m)$$

This will be very convenient for our purposes, as it will imply most of the less "obvious" relations amongst the previous elements.

Lemma 6.1.2 *The following relations hold*

1. $[u][1-u] = 0$ *(Hu-Kriz [8]))*

2. $[uv] = [u] + [v] + \eta[u][v]$

3. $\eta \cdot [u] = [u] \cdot \eta$

4. $\eta^2[-1] + 2\eta = 0$

The idea behind the proof of (1) is to show that the composite $\mathbb{G}_m - \{1\} \to \mathbb{G}_m \times \mathbb{G}_m \to {\mathbb{G}_m}^{\wedge 2}$ is null, where the first map is the closed immersion $x \to (x, 1-x)$. Drawing a picture of the real points "suggests" a proof. Statement (2) follows at once from the previous lemma, as does (3).

For $u \in k^\times$, we define a degree 0 element by

$$\langle u \rangle := 1 + \eta[u] \in [\mathbb{G}_m, \mathbb{G}_m]_{\mathbb{P}^1}.$$

Lemma 6.1.3 *In the group* $[\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{P}^1}$ *we have the equality*

$$\langle u \rangle = (f_u : \mathbb{P}^1 \to \mathbb{P}^1),$$

where $f_u : \mathbb{P}^1 \to \mathbb{P}^1$ *is the morphism* $[x_0, x_1] \mapsto [x_0, ux_1]$.

Thus the equality (4) is equivalent to

$$\eta h = \eta(\langle -1 \rangle + 1) = 0,$$

where h denotes the hyperbolic plane $< -1 > +1$, or equivalently to $\eta(- < -1 >) = \eta$. This now follows from the commutativity of the product on $\mathbb{G}_m$ together with the fact that the transposition

$$\epsilon : \mathbb{G}_m \wedge \mathbb{G}_m \cong \mathbb{G}_m \wedge \mathbb{G}_m$$

is multiplication by $- < -1 >$ (see [23]).

Remark 6.1.4 The graded ring

$$\oplus_{n \in \mathbb{Z}} [S^0, {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1}$$

is clearly ϵ-commutative.

Definition 6.1.5 *The Milnor-Witt K-theory $K_*^{MW}(k)$ of a field k is the graded associative ring generated by symbols $[u]$ of degree $+1$ and η of degree -1 subject to the relations (1)-(4) above.*

Remark 6.1.6 (See also [17, 23])

(a) Recall that the *Milnor K-theory* [15] of k is defined as the graded ring quotient of the tensor algebra $Tens_{\mathbb{Z}}(k^\times)$ of the abelian group $k^\times$ by the Steinberg relations (1). Thus there is a canonical isomorphism of rings

$$K_*^{MW}(k)/\eta = K_*^M(k)$$

(b) The correspondence $u \mapsto \langle u \rangle \in K_0^{MW}(k)$ induces an isomorphism

$$GW(k) \to K_0^{MW}(k)$$

from the Grothendieck-Witt ring to $K_0^{MW}(k)$, in which $\langle u \rangle$ corresponds to the quadratic form of rank one $u.X^2$. Thus h corresponds to the hyperbolic plane and relation (4) defines canonical morphisms $W(k) \to K_i^{MW}(k)$ for $i < 0$, from the Witt ring of k, $W(k) := GW(k)/h$. These are isomorphisms.

(c) The ring $K_*^{MW}(k)[\eta^{-1}]$ obtained by inverting η is thus isomorphic the ring of Laurent polynomials $W(k)[\eta, \eta^{-1}]$ with coefficient in the Witt ring of k. Note thus that $K_*^{MW}(k)$ is always non-zero in both positive and negative degrees.

By construction we thus have a canonical morphism for each $n \in \mathbb{Z}$

$$K_n^{MW}(k) \to [S^0, {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1}$$

6.2 The main theorem

Theorem 6.2.1 *[20] If k is a perfect field (with $\operatorname{char} k \neq 2$) then the homomorphism*

$$K_*^{MW}(k) \xrightarrow{\cong} [S^0, {\mathbb{G}_m}^{\wedge *}]_{\mathbb{P}^1}$$

is an isomorphism of graded rings.

As a by-product we observe that we proved:

Theorem 6.2.2 *[20] If k is a perfect field (with char$k \neq 2$) then the homomorphism*

$$GW(k) \to [S^0, S^0]_{\mathbb{P}^1}$$

is an isomorphism of rings.

This fits very well with the author's work on the Adams spectral sequence based on mod 2 motivic cohomology [16, 18].

Remark 6.2.3 The proof of Theorem 6.2.1 which will be sketched in Section 6.3 below uses among other things a recent result of Arason-Elman [1]. We observe that their paper relies on the proof of the Milnor Conjecture on quadratic forms asserting that

$$K_*^M(F)/2 \to \oplus_n I^n/I^{n+1},$$

is an isomorphism. Nevertheless, the author is convinced that Theorem 6.2.1 could be proven in a very much simpler way. Indeed, we do have a programme to achieve this: first prove all the analogue of the well known properties for Milnor K-theory and the Witt groups for the Milnor-Witt K-theory (for instance existence of transfers) and rewrite Rost's paper [27] on Chow groups with coefficients, replacing Milnor K-theory with Milnor-Witt K-theory. This should lead to an elementary proof of 6.2.1.

6.3 A sketched proof

We now outline the proof of Theorem 6.2.1 in several steps.

Step 1 *Construction of a strictly $\mathbb{A}^1$-invariant sheaf of* unramified Witt groups

$$\underline{W} \in \Pi^{S^1}(k).$$

We have known for a long time how to define a contravariant Witt group functor on smooth k-schemes $(\mathcal{V})^{op} \to \mathcal{A}b, X \mapsto W(X)$ (see [2] for instance). We claim now that its sheafification $\underline{W}$ in the Nisnevich topology is a strictly $\mathbb{A}^1$-invariant sheaf. To do this we use a result by Hornbostel [6] establishing that the above presheaf is represented by an S^1-spectrum $KW \in Sp^{S^1}(\mathcal{V}_{Nis})$ in the sense that

$$W(X) \cong [X_+, KW].$$

It then follows from 4.2.5 that the associated sheaf $\underline{W}$ is strictly $\mathbb{A}^1$-invariant.

Remark 6.3.1 I. Panin has a direct proof of this fact inspired by Voevodsky's approach using transfer in Witt groups.

Step 2 *Construction of a strictly* $\mathbb{A}^1$*-invariant sheaf of* unramified powers of the fundamental ideals

$$\underline{I}^n \in \Pi^{S^1}(k)$$

Recall that the *fundamental ideal* of the Witt ring $W(X)$ of a scheme X (in which 2 is invertible) is the Kernel $I(X)$ of the mod 2 rank homomorphism $W(X) \to H^0_{Zar}(X;\mathbb{Z}/2)$. For any integer $n \in \mathbb{N}$, let $I^n(X)$ denote the n-th power of this ideal. Let $\underline{I}^n$ denote the sheafification of $X \mapsto I^n(X)$.

Lemma 6.3.2 *For $n \geq 0$ the sheaf $\underline{I}^n$ is strictly $\mathbb{A}^1$-invariant.*

We argue by induction on n. There is, by construction, an exact sequence of sheaves.

$$0 \to \underline{I}^n \to \underline{I}^{n-1} \to \underline{I}^{n-1}/\underline{I}^n \to 0,$$

and $\underline{I}^{n-1}/\underline{I}^n$ is the sheaf associated to a cycle module in the sense of Rost, which is known to be strictly $\mathbb{A}^1$-invariant by Rost [27].

Remark 6.3.3 If you don't care about being elementary, you may directly use the Milnor conjecture

$$\underline{K}^M_{n-1}/2 \cong \underline{I}^{n-1}/\underline{I}^n.$$

and apply the known fact that unramified mod 2 Milnor K-theory is a strictly $\mathbb{A}^1$-invariant sheaf.

Step 3 *Construction of a strictly* $\mathbb{A}^1$*-invariant sheaf of* unramified J-groups $\underline{J}^n$.

For any integer $n \in \mathbb{Z}$ let us denote by $\underline{J}^n := \underline{I}^n \times_{\underline{I}^n/\underline{I}^{n+1}} \underline{K}^M_n$ the obvious fibre product[6]. We thus have a short exact sequence of sheaves in the Nisnevich topology

$$0 \to \underline{I}^{n+1} \to \underline{J}^n \to \underline{K}^M_n \to 0$$

[6]this fibre product was considered over fields in [3]

so that $\underline{J}^n \in \Pi(k)$ is a strictly $\mathbb{A}^1$-invariant sheaf. Moreover, it is not hard to define a canonical isomorphism

$$(\underline{J}^n)_{-1} \cong \underline{J}^{n-1}$$

thus defining an object $\underline{J}^* \in \Pi_*(k)$. We also have natural pairings

$$\underline{J}^m \times \underline{J}^n \to \underline{J}^{m+n}$$

which define a product on $\underline{J}^*$. Next we define an element $\theta \in \underline{J}^1(\mathbb{G}_m)$. We let

$$\theta : \mathbb{G}_m \to \underline{J}^1 = \underline{I} \times_{\underline{I}/\underline{I}^2} \underline{K}_1^M$$

denote the map induced by the correspondence

$$u \longmapsto (\langle\langle u\rangle\rangle, u).$$

where we denote by $\langle\langle u\rangle\rangle = \langle 1, -u\rangle \in I(K)$ the Pfister form, over a field K associated to the unit $u \in K^\times$. Using the above pairing, this induces a morphism, say in $\mathcal{SH}_s^{S^1}(k)$,

$$\mathbb{G}_m{}^{\wedge j} \to H(\underline{J}^j).$$

which finally induces a morphism of $\mathbb{G}_m$-S^1-spectra, which we view in $\mathcal{SH}(k)$ (because $\underline{J}^* \in \Pi_*(k)$),

$$S^0 \to H\underline{J}_*$$

and thus, taking $\pi_0(-)_*$, a morphism in $\Pi_*(k)$

$$\pi_0^{\mathbb{A}^1}(S^0) \to \underline{J}_*$$

Step 4 *Construction of a strictly $\mathbb{A}^1$-invariant sheaf of* unramified Milnor-Witt K-groups

For any finite type field extension F of k, the composition

$$K_*^{MW}(F) \to [S_F^0, \mathbb{G}_m{}^{\wedge *}] \to \underline{J}^*(F) = J^*(F)$$

of the morphism defined in section 6.1 and of the morphism obtained in Step 3 is easily checked to be the only morphism of graded rings which maps $[u] \in K_1^{MW}(F)$ to $(\langle\langle u\rangle\rangle, u) \in J^1(F)$ and which maps η to $1 \in J^{-1}(F) = W(F)$ (observe that multiplication by η on $J^*(F)$ is the composite $J^n(F) \to I^n(F) \to J^{n-1}(F)$).

Theorem 6.3.4 *[17] The morphism*

$$K_*^{MW}(F) \xrightarrow{\cong} J^*(F)$$

is an isomorphism.

Step 5 We then show that our canonical map

$$\phi_F : K_*^{MW}(F) \to [S_F^0, \mathbb{G}_m{}^{\wedge n}] = \pi_0^{\mathbb{A}^1}(\mathbb{G}_m{}^{\wedge n})(F)$$

is induced (using Theorem 6.3.4) by a morphism in $\Pi_*(k)$

$$\Phi : \underline{K}_*^{MW} := \underline{J}^* \to \pi_0^{\mathbb{A}^1}(S^0)_*$$

To do this we observe that it suffices to show that the morphism ϕ_F does commute with residues homomorphisms because for any $M \in \Pi^*(k)$ and any irreducible smooth k-scheme X with function field $k(X)$, the group $M(X)$ is just the kernel of

$$M(k(X)) \xrightarrow{\{\partial_x^\pi\}} \oplus_{x \in X^{(1)}} M_{-1}(\kappa(x)).$$

Here we use the geometric interpretation of residues given above. Thus Φ provides a canonical section in $\Pi_*(k)$ of $\pi_0^{\mathbb{A}^1}(S^0)_* \to \underline{J}^*$. We also observe that Φ does preserve the product.

Step 6 We conclude by observing that the composition in $\mathcal{SH}(k)$

$$S^0 \to H(\underline{J}^*) \xrightarrow{H(\Phi)} H(\pi_0^{\mathbb{A}^1}(S^0)_*)$$

is the obvious morphism, thus proving that the composition

$$\pi_0^{\mathbb{A}^1}(S^0)_* \to \underline{J}^* \xrightarrow{\Phi} \pi_0^{\mathbb{A}^1}(S^0)_*$$

is the identity as well.

6.4 Some consequences

We observe that the previous proof gives more than Theorem 6.2.1.

Corollary 6.4.1 *In $\Pi^*(k)$ we have a canonical identification*

$$\pi_0^{\mathbb{A}^1}(S^0)_* = \underline{K}_*^{MW}$$

By a careful analysis of the above argument, we get in fact computations of groups of morphisms in $\mathcal{SH}^{S^1}(k)$:

Corollary 6.4.2 *For all $i \geq 0$ and all $j \geq 1$*

$$[{\mathbb{G}_m}^{\wedge i}, {\mathbb{G}_m}^{\wedge j}] \cong (\underline{J}^j)_{-i}(k) \cong \underline{J}^{j-i}(k) \cong [S^0, (\mathbb{G}_m)^{j-i}]_{\mathbb{P}^1}.$$

This suggest the following cancellation conjecture in $\mathcal{SH}^{S^1}(k)$:

Conjecture 6.4.3 *For any two S^1-spectra E and F, and any $n \geq 1$ the morphisms*

$$[E \wedge (\mathbb{G}_m), F \wedge (\mathbb{G}_m)] \to [E \wedge (\mathbb{G}_m)^{\wedge n}, F \wedge (\mathbb{G}_m)^{\wedge n}] \to [\sigma_{\mathbb{G}_m}(E), \sigma_{\mathbb{G}_m}(F)]_{\mathbb{P}^1}$$

are isomorphisms.

Remark 6.4.4 If $j < 0$ then $\underline{J}^j = \underline{W}$, and thus

$$[{\mathbb{G}_m}^{\wedge i}, \mathbb{G}_m] \cong W(k) \text{ for } i > 1.$$

Similarly

$$\underline{J}^0 = \underline{W} \times_{\underline{\mathbb{Z}/2}} \mathbb{Z} = \underline{GW}$$

so that

$$[{\mathbb{G}_m}^{\wedge i}, {\mathbb{G}_m}^{\wedge i}] \cong GW(k) \text{ for } i \geq 1$$

(recall that $[S^0, S^0] = \mathbb{Z}$.)

Remark 6.1.6 easily implies

Corollary 6.4.5 *1) For any integer $n \in \mathbb{Z}$*

$$[S^0, {\mathbb{G}_m}^{\wedge n} \wedge C(\eta)]_{\mathbb{P}^1} \cong [S^0, M\mathbb{G}l \wedge {\mathbb{G}_m}^{\wedge n}]_{\mathbb{P}^1} \cong M\mathbb{G}l^{n,n}(Spec(k)) \cong K_n^M(k),$$

where $M\mathbb{G}l$ is the motivic Thom $\mathbb{P}^1$-spectrum [29].

2) For any integer $n \in \mathbb{N}$,

$$[S^0, {\mathbb{G}_m}^{\wedge(-n)}]_{\mathbb{P}^1} \cong [S^0, S^0[\eta^{-1}]]_{\mathbb{P}^1} \cong W(k)$$

In particular, $S^0[\eta^{-1}]$ is *never* contractible, although $M\mathbb{G}l \wedge \eta \simeq 0$. Thus η is not nilpotent.

In much the same way, using one the result in [17] we see that for $n \geq 1$

$$[S^0, {\mathbb{G}_m}^{\wedge n} \wedge C(h)] \cong [S^0, {\mathbb{G}_m}^{\wedge n} \wedge C(h)]_{\mathbb{P}^1} \cong I^n(k)$$

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