GNS representation

$\mathcal{A}$ unital C^*-algebra.

ω state on $\mathcal{A}$.

$$\mathcal{J}_\omega := \{ A \in \mathcal{A} \mid \omega(B^*A) = 0 \ \ \forall B \in \mathcal{A} \}$$

- $\mathcal{J}_\omega$ is a <u>left ideal</u> $(\mathcal{A}\mathcal{J}_\omega \subseteq \mathcal{J}_\omega)$ because

$$0 \leq \omega((AJ)^*(AJ)) = \omega(J^*A^*AJ) \leq \omega(J^*J)\,\|A^*A\|$$
$$(J^*A^*AJ \leq \|A^*A\|\, J^*J)$$
$$= \|A\|^2 \omega(J^*J) = 0$$
$$\forall J \in \mathcal{J}_\omega \ \ \forall A \in \mathcal{A}$$

- $\mathcal{J}_\omega$ is <u>closed in $\mathcal{A}$</u> because

$$\mathcal{J}_\omega \ni J_n \xrightarrow{\|\ \|} J \Rightarrow \omega(B^*J) = \lim_{n\to\infty} \omega(B^*J_n) = 0 \Rightarrow J \in \mathcal{J}_\omega.$$

Then we can consider $\mathcal{A}/\mathcal{J}_\omega$, the space of equivalent classes

$$\Psi_A := \{ A + J \mid J \in \mathcal{J}_\omega \}$$

equipped with the natural operations
$$\begin{cases} \Psi_A + \Psi_B = \Psi_{A+B} \\ \lambda \Psi_A = \Psi_{\lambda A} \\ J \in \mathcal{J}_\omega \Rightarrow \Psi_A + \Psi_J = \Psi_A \\ \quad \text{i.e. } \Psi_J = 0 \text{ in } \mathcal{A}/\mathcal{J}_\omega \end{cases}$$

and with the scalar product $\langle \Psi_A, \Psi_B \rangle := \omega(A^*B)$

(it is well-defined because $\omega((A+J_1)^*(B+J_2)) = \dots = \omega(A^*B)$ $\forall A, B \in \mathcal{A}$ $\forall J_1, J_2 \in \mathcal{J}_\omega$; it is positive definite because $0 = \langle \Psi_A, \Psi_A \rangle = \omega(A^*A) \Rightarrow A \in \mathcal{J}_\omega \Rightarrow \Psi_A = 0$.)

The completion of $\mathcal{A}/\mathcal{J}_\omega$ w.r.t. $\langle , \rangle$ is the Hilbert space $\mathcal{H}_\omega$.

Now construct a map $\mathcal{A} \ni A \longmapsto \pi_\omega(A)$ where $\pi_\omega(A)$ acts on $\mathcal{H}_\omega$.

On each $\psi_B \in \mathcal{A}/\mathcal{J}_\omega$ (dense in $\mathcal{H}_\omega$) $\quad \pi_\omega(A)\psi_B := \psi_{AB}$.

Well defined: if $C = B + J$, $J \in \mathcal{J}_\omega$, then

$$\psi_{AC} = \{ AC + J' \mid J' \in \mathcal{J}_\omega \} = \{ AB + \underbrace{J + J' + \overbrace{AJ}^{\in \mathcal{J}_\omega \text{ left ideal}}}_{\in \mathcal{J}_\omega} \mid J' \in \mathcal{J}_\omega \}$$
$$= \psi_{AB}.$$

Linearity: $\pi_\omega(A)(\lambda\psi_B + \mu\psi_C) = \pi_\omega(A)\psi_{\lambda B + \mu C}$

$$= \psi_{\lambda AB + \mu AC} = \lambda \psi_{AB} + \mu \psi_{AC}$$
$$= \lambda \pi_\omega(A)\psi_B + \mu \pi_\omega(A)\psi_C .$$

Multiplicativity: $\pi_\omega(A_1)\pi_\omega(A_2)\psi_B = \psi_{A_1 A_2 B} = \pi_\omega(A_1 A_2)\psi_B$

ie $\pi_\omega(A_1)\pi_\omega(A_2) = \pi_\omega(A_1 A_2)$.

Boundedness: $\| \pi_\omega(A)\psi_B \|^2 = \langle \psi_{AB}, \psi_{AB} \rangle = \omega((AB)^*(AB))$

$$= \omega(B^* A^* A B) \le \|A^* A\| \, \omega(B^* B) = \|A\|^2 \|\psi_B\|^2$$
$$\Rightarrow \|\pi_\omega(A)\| \le \|A\| \text{ on } \mathcal{A}/\mathcal{J}_\omega$$

Adjoint: $\langle \pi_\omega(A^*)\psi_B, \psi_C \rangle = \langle \psi_{A^*B}, \psi_C \rangle = \omega(B^* A C)$

$$= \langle \psi_B, \psi_{AC} \rangle = \langle \psi_B, \pi_\omega(A)\psi_C \rangle$$
$$\Rightarrow (\pi_\omega(A))^* = \pi_\omega(A^*).$$

Thus π_ω extends to a representation $\mathcal{A} \xrightarrow{\pi_\omega} \mathcal{B}(\mathcal{H}_\omega)$

There is a cyclic vector for π_ω in $\mathcal{H}_\omega$: $\Omega_\omega := \psi_{\mathbb{1}}$

indeed each $\psi_A \in \mathcal{A}/\mathcal{I}_\omega$ is obtained as $\psi_A = \pi_\omega(A)\psi_{\mathbb{1}}$

therefore $\{\pi_\omega(A)\psi_{\mathbb{1}} \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.

Realisation of the state ω in this representation:

$$\underline{\langle \Omega_\omega , \pi_\omega(A)\Omega_\omega \rangle} = \langle \psi_{\mathbb{1}} , \pi_\omega(A)\psi_{\mathbb{1}} \rangle$$

$$= \langle \psi_{\mathbb{1}} , \psi_A \rangle = \omega(\mathbb{1}^* A) = \underline{\omega(A)} .$$

Uniqueness of this construction up to unitary equivalence:

if $(\mathcal{H}, \pi, \Omega)$ is another cyclic representation of $\mathcal{A}$ such that

$$\omega(A) = \langle \Omega_\omega , \pi_\omega(A)\Omega_\omega \rangle_{\mathcal{H}_\omega} = \langle \Omega , \pi(A)\Omega \rangle_{\mathcal{H}} \quad \underline{\forall A \in \mathcal{A}}$$

then the two representations are unitarily equivalent, i.e.

$$\exists\, U : \mathcal{H} \xrightarrow{\cong} \mathcal{H}_\omega , \text{ unitary},$$

$$\text{such that} \quad \left|\begin{array}{l} \pi_\omega(A) = U\pi(A)U^{-1} \quad \forall A \in \mathcal{A} \\ \Omega_\omega = U\Omega . \end{array}\right.$$

U is defined by $\quad U^{-1}\pi_\omega(A)\Omega_\omega := \pi(A)\Omega \quad \forall A \in \mathcal{A}$.

norm preserving: $\|\pi(A)\Omega\|^2 = \langle \Omega , \pi(A^*A)\Omega \rangle = \omega(A^*A) =$
$= \langle \Omega_\omega , \pi_\omega(A^*A)\Omega_\omega \rangle = \|\pi_\omega(A)\Omega_\omega\|^2$

then extend by density.